# A BALANCED DIPLOMACY TOURNAMENT

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ABSTRACT. We present a design for a seven game tournament of the 7-player board game *Diplomacy*, in which each player plays each country one time and each pair of players shares a border either 4 or 5 times. It is impossible for each pair of players to share a border the same number of times in such a tournament, and so the tournament presented is the most "balanced" possible in this sense. A similarly balanced tournament can be constructed for a generalized version of the game involving an arbitrary number of countries. We also present an infinite family of graphs that cannot be balanced.

*Keywords:* graph, Latin square, tournament

In this paper we examine a notion of balance, or fairness, related to tournaments of multiplayer games. Mathematical features of games have been studied in many places, notably in [2]. The aspect that we wish to balance is how often any two players oppose each other in the game. As such, the games we are concerned with in this paper are ones in which there is a set of possible positions, each of which is assigned to a unique player, along with information about which positions are adjacent (or, in the language of our motivating example, share a border). Thus an $n$-player game $\mathcal{G}$ of this type can be modeled by a simple graph $G$ on $n$ vertices, where an edge $\{i, j\}$ in $G$ indicates that positions $i$ and $j$ are adjacent in $\mathcal{G}$.

Our motivating example is the game *Diplomacy*, a 7-player board game in which each player is assigned one of the countries Austria, England, France, Germany, Italy, Russia, and Turkey [1]. We will number these countries alphabetically 1 through 7, so that we may describe their borders

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with the (symmetric) $7 \times 7$ adjacency matrix

$$B_D := \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

For example, the 1s in the $(1,4)$ and $(4,1)$ positions mean that Austria and Germany share a border in the game, while the 0s in the $(1,2)$ and $(2,1)$ positions mean that Austria and England do not. The fact that no country borders itself accounts for the 0s along the main diagonal.

The game play in *Diplomacy* is noteworthy because it involves a negotiation phase. This feature of the game is not relevant to the current work, but has been studied from various computational perspectives, including in [5] and [7].

Defining $B_D$ is not completely intuitive geographically. For example, although Austria and Turkey are separated by several countries geographically, the separating countries are not controlled by players in *Diplomacy*, and so we consider Austria and Turkey to share a border in the game. This matrix may be represented by the graph in Figure 1. With a nod toward the work of this paper, we note that there are $\binom{7}{2} = 21$ pairs of distinct countries, but only 13 pairs share a border in the game. Thus if two countries are chosen at random, the probability that they share a border is 13/21, the number of edges divided by the number of pairs.

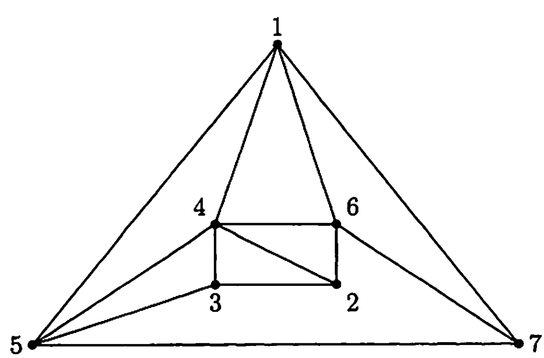



FIGURE 1. A graphical depiction of the countries and borders in *Diplomacy*.

The relevance of shared borders in *Diplomacy* arises from the fact that a player can most easily attack his or her direct neighbors. Thus it may be advantageous (or disadvantageous) to share a border with a particular player, or to play a position that has many (or few) borders. In a certain sense, *Diplomacy* lacks symmetry. For example, suppose that an extraordinarily good player has chosen to play Austria. Then it is to one's advantage to avoid the good player, and so to play either England or France. Another instance of the game's asymmetry is that Austria, which is forced to fight on multiple fronts, often loses earlier in the game than England, which has fewer neighbors.

We begin by describing a seven game tournament of *Diplomacy* which is as "balanced" (that is, which recovers the symmetry whose absence was described above) as possible. To support this notion of fairness, each player ought to play each country exactly once. This can be represented by a $7 \times 7$ Latin square in which the rows represent the different games of the tournament, the columns represent the different countries in the game, and the entry $p$ in row $r$ and column $c$ means that in the $r$th game of the tournament, player $p$ is assigned to the country $c$.

As will be made precise below, the "balance" of a tournament is characterized by choosing a Latin square so that every pair of distinct players faces each other across a border approximately the same number of times. More precisely, let $n_{ij}$ be the number of times that player $i$ and player $j$ play bordering countries. Thus, certainly, for all $i$ and $j$, $0 \leq n_{ij} \leq 7$ in *Diplomacy*. Our goal is to have all of $\{n_{ij} : i \neq j\}$ be approximately equal, and to explore the potential for such "balance" in other games more generally.

## 1. Tournaments

**Definition 1.1.** A *Latin square* is an $n \times n$ array filled by $n$ symbols so that each row and column contains exactly one copy of each symbol.

For our purposes, the $n$ symbols in a Latin square will be the integers $1, 2, \ldots, n$, representing the different players in a game.

**Definition 1.2.** An $n \times n$ Latin square is *reduced* if its first row and first column are both written in increasing order.

Consider the reduced Latin square

$$L_D := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 6 & 5 & 4 & 7 & 3 \\ 3 & 7 & 2 & 6 & 1 & 5 & 4 \\ 4 & 3 & 5 & 7 & 6 & 1 & 2 \\ 5 & 6 & 4 & 3 & 7 & 2 & 1 \\ 6 & 4 & 7 & 1 & 2 & 3 & 5 \\ 7 & 5 & 1 & 2 & 3 & 4 & 6 \end{bmatrix}.$$

As described above, the entry of row $r$ and column $c$ in a Latin square will name the player assigned to country $c$ in game $r$ of the tournament. The first condition that $L_D$ be reduced, that the first row is

$$1\ 2\ 3\ 4\ 5\ 6\ 7,$$

allows us to abuse notation somewhat and refer to the player occupying country $c$ in the first game by the name "$c$." The second condition that $L_D$ be reduced, that the first column is the transpose of the first row, means that we order the last six games of the tournament so that the first country (Austria, in the case of *Diplomacy*) is played successively by players $2, 3, 4, 5, 6$, and 7. For the purpose of finding a balanced tournament, it is obviously sufficient to look at reduced Latin squares.

We now formalize the notion of a tournament for an arbitrary game in which special importance is given to when two players share a border (for example, that might be the only time that these players can interact with each other), and these borders are defined by the game itself. Throughout this paper, we will assume that our games consist of a set of positions and information about which positions border which other positions. Thus our games are equivalent to finite simple graphs.

**Definition 1.3.** A *tournament* for an $n$-player game $\mathcal{G}$ is a pair of $n \times n$ matrices $(B, L)$, where $B$ is a symmetric $\{0,1\}$-matrix with 0s along the main diagonal, giving the border relationships in $\mathcal{G}$, and $L$ is a reduced Latin square. That is, $B$ is the adjacency matrix of the graph associated to $\mathcal{G}$. The number of 1s above the diagonal in $B$ will be called the number of *borders* in $\mathcal{G}$, and will be denoted $b(\mathcal{B})$. Thus $b(\mathcal{B})$ is the number of edges in the graph. Set $N = \binom{n}{2}$, the total number of supradiagonal entries. Then $p(\mathcal{G}) = b(\mathcal{G})/N$ is the number of edges divided by the number of pairs of players. When no confusion will arise, we set $b = b(\mathcal{G})$ and $p = p(\mathcal{G})$.

One way to visualize the entire tournament is to draw seven copies of the graph in Figure 1. The graph as labeled above corresponds to the first row of $L_D$ and the first game of the tournament; the original graph with the labels permuted as $1 \to 2, 2 \to 1, 3 \to 6, 4 \to 5, 5 \to 4, 6 \to 7, 7 \to 3$ corresponds to the second row of $L_D$ and the second game of the tournament, and so on.

Throughout our discussions, we will only consider one game $\mathcal{G}$ at a time, and thus we may discuss tournaments without specifically naming $\mathcal{G}$.

We now define the statistics that will determine whether a tournament is balanced.

**Definition 1.4.** Fix a tournament $(B, L)$ for an $n$-player game $\mathcal{G}$, and let $1 \le i \ne j \le n$. Let $n_{ij}$ count the number of times that players $i$ and $j$ share a border throughout the tournament $(B, L)$. That is, suppose that in row $r$ of $L$, the value $i$ appears in column $c_{r,i}$ and the value $j$ appears in column $c_{r,j}$. Then

$$n_{ij} = \sum_{r=1}^{n} \begin{cases} 1 & \text{if } c_{r,i} \text{ is adjacent to } c_{r,j}, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

To each tournament $(B, L)$, we may associate two extremal values:

$$u(B, L) := \max_{i<j} n_{ij} \tag{2}$$

and

$$\ell(B, L) := \min_{i<j} n_{ij}. \tag{3}$$

Looking at $u(B, L)$ and $\ell(B, L)$ together will give an indication of how well the tournament has been balanced; that is, whether each pair of players face each other approximately the same number of times.

## 2. A BALANCED TOURNAMENT FOR *Diplomacy*

We now take a moment to consider the example of *Diplomacy*, and whether a tournament $(B, L)$ can be constructed for the game in which the values of $u(B, L)$ and $\ell(B, L)$ are equal, or nearly equal.

*Diplomacy* is a 7-player game, and the game has 13 edges. Thus, in any *Diplomacy* tournament,

$$\sum_{i<j} n_{ij} = 13 \cdot 7 = 91.$$

If all 21 values of $n_{ij}$ with $i < j$ were equal to some $k \in \mathbb{Z}$, then we would need $21k$ to equal 91. However, $21 \nmid 91$, so it is impossible for all of the these values to be equal. It is thus impossible for $u(B, L)$ and $\ell(B, L)$ to coincide in any *Diplomacy* tournament $(B, L)$. Instead, because $4 < 91/21 < 5$, we see that the most uniform conceivable tournament $(B, L)$ would have

$u(B,L) = 5$ and $\ell(B,L) = 4$. In fact, the tournament

$$(B_D, L_D) = \left( \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 6 & 5 & 4 & 7 & 3 \\ 3 & 7 & 2 & 6 & 1 & 5 & 4 \\ 4 & 3 & 5 & 7 & 6 & 1 & 2 \\ 5 & 6 & 4 & 3 & 7 & 2 & 1 \\ 6 & 4 & 7 & 1 & 2 & 3 & 5 \\ 7 & 5 & 1 & 2 & 3 & 4 & 6 \end{bmatrix} \right),$$

which was presented already, is balanced in the sense that direct calculation shows that indeed $u(B_D, L_D) = 5$ and $\ell(B_D, L_D) = 4$.

Given that $u(B,L)$ and $\ell(B,L)$ can never coincide in *Diplomacy*, the tournament $(B_D, L_D)$ is as "balanced" as possible, in the sense that each pair of players shares a common border approximately the same number of times throughout the course of the tournament.

The number of reduced $7 \times 7$ Latin squares (that is, the number of options for $L$ in a *Diplomacy* tournament) is 16942080. To find one in which $u(B,L) - \ell(B,L)$ was minimal, we used the simplest imaginable brute force technique, using a computer code from Brendan McKay [6]. For each $7 \times 7$ Latin square $L$, we have $u(B,L) \in \{5,6,7\}$ and $\ell(B,L) \in \{0,1,2,3,4\}$. The frequency distribution of these pairs is

| $u(B,L) =$ | 5 | 6 | 7 |
|---|---|---|---|
| $\ell(B,L) = 0$ | 0 | 72772 | 303548 |
| 1 | 574 | 1373134 | 3378652 |
| 2 | 19068 | 4187862 | 5158218 |
| 3 | 65170 | 1672472 | 708488 |
| 4 | 1408 | 714 | 0 |

The crux of this calculation is that $\#\{L : u(B,L) = 5 \text{ and } \ell(B,L) = 4\} > 0$. This shows that there are 1408 reduced Latin squares for which $u(B,L) - \ell(B,L)$ is minimal. The Latin square $L_D$ is one of these 1408 possibilities.

## 3. Infinite families of games having balanced tournaments

Throughout this section, suppose we have an $n$-player game $\mathcal{G}$ with an $n \times n$ adjacency matrix $B$. Each reduced $n \times n$ Latin square $L$ produces a tournament $(B,L)$ for $\mathcal{G}$. Let $u = u(B,L)$ and $\ell = \ell(B,L)$, as defined in equations (2) and (3). Set $b = b(\mathcal{G})$ and $p = p(\mathcal{G})$, as in Definition 1.3.

**Proposition 3.1.** *Given a tournament $(B,L)$ on an $n$-player game $\mathcal{G}$, we have*

$$\ell \le pn \le u. \tag{4}$$

*Proof.* Note that

$$\ell = \min n_{ij} \le \text{avg } n_{ij} \le \max n_{ij} = u,$$

and

$$\text{avg } n_{ij} = \frac{bn}{\binom{n}{2}} = pn.$$

□

**Corollary 3.2.** *In any n-player game $\mathcal{G}$,*

$$\ell \le \lfloor pn \rfloor \quad \textit{and} \quad u \ge \lceil pn \rceil.$$

*Proof.* This follows from Proposition 3.1 and the fact that $u, \ell \in \mathbb{Z}$. □

In the example of a *Diplomacy* tournament in Section 2, we had $b = 13$ and $N = 21$, so that $p = 13/21$, $pn = (13/21)7 = 13/3$ and the inequalities of Corollary 3.2 become

$$\ell \le 4 \text{ and } u \ge 5.$$

**Definition 3.3.** A tournament $(B, L)$ is *balanced* if $\ell(B, L) = \lfloor pn \rfloor$ and $u(B, L) = \lceil pn \rceil$.

Note that the Definition 3.3 has the following implications.

- If $pn = \frac{b}{N}n = \frac{2b}{n-1} \in \mathbb{Z}$, then "balanced" means that $\ell = u = n_{ij}$ for all $i < j$.
- If $pn \notin \mathbb{Z}$ (that is, if $n - 1 \nmid 2b$), then "balanced" means that $u - \ell = 1$ and that some $n_{ij}$ are equal to $\ell$ while all others are equal to $u$.

An example of this latter case is the balanced *Diplomacy* tournament, above, in which there were seven pairs of distinct players $i < j$ for which $n_{ij} = 5 = u$, and fourteen pairs of distinct players $i < j$ for which $n_{ij} = 4 = \ell$.

In the following subsections, we show that a balanced tournament is always possible in several game situations: when its complementary game is balanced, when the number of edges in the game is very small or very large, and when the game's graph is an odd cycle consisting of at least three elements.

### 3.1. Complementary games.

**Definition 3.4.** The complement of a graph is the graph having the same vertex set and having exactly those edges which were not present in the original graph. Correspondingly, the complement $B^c$ of an adjacency matrix $B$ is achieved by replacing every non-diagonal entry $x$ by $1 - x$.

**Theorem 3.5.** *Consider the n-player game $\mathcal{G}$ and the complementary game $\mathcal{G}^c$. If $L$ is an $n \times n$ Latin square such that the tournament for $\mathcal{G}$ is balanced, then the tournament for $\mathcal{G}^c$ is balanced as well.*

*Proof.* Let $n_{ij}$ be defined for the tournament $(B, L)$ as in equation (1), and let $n^c_{ij}$ be the corresponding values for the tournament $(B^c, L)$. It follows from the definition of these values that

$$n^c_{ij} = n - n_{ij}$$

for all $i < j$. Thus

$$u(\mathcal{G}^c) = n - \ell(\mathcal{G})$$

and

$$\ell(\mathcal{G}^c) = n - u(\mathcal{G}).$$

Because

$$\begin{aligned} u(\mathcal{G}^c) - \ell(\mathcal{G}^c) &= n - \ell(\mathcal{G}) - n + u(\mathcal{G}) \\ &= u(\mathcal{G}) - \ell(\mathcal{G}), \end{aligned} \tag{5}$$

the tournament for $\mathcal{G}^c$ is balanced if and only if the tournament for $\mathcal{G}$ is balanced. □

3.2. **Games with very few (or many) borders.** We now show that there is a balanced tournament when the number of edges is sufficiently small compared to the number of vertices. By taking complements, it will then follow that there is also a balanced tournament when the number of edges is symmetrically close to $\binom{n}{2}$.

**Theorem 3.6.** *Let $\mathcal{G}$ be an n-player game with $b(\mathcal{G}) \le \log_4 n$. Then there exists a balanced tournament for $\mathcal{G}$.*

*Proof.* Take any graph having $n$ vertices and $b$ edges, where $b \ge 1$ and $n \ge 4^b$. Then $0 < pn = \frac{2b}{n-1} \le \frac{2b}{2^{2b}-1} < 1$, so $(\ell, u) = (0, 1)$. Therefore we must find a Latin square describing a tournament in which no pair of players share a border more than once. There are at most $2b$ non-isolated vertices in any graph with $b$ edges. Say that there are $k+1$ such vertices, and label these, in any way, by $2^0, 2^1, 2^2, 2^3, \ldots, 2^k$, where $k \le 2b-1$. Assign all unused labels from $\{1, \ldots, n\}$ to the isolated vertices. Let the rows of the Latin square $L$, determining the tournament, be given by the cyclic permutations $123\ldots n$, $234\ldots n1$, $345\ldots n12$, and so on, with last row $n12\ldots(n-1)$. If two players ever meet, then their labels always differ by $2^i - 2^j \bmod n$ for a particular pair $i$ and $j$. But at most one pair of adjacent vertices can have labels differing by this value when $n > 2\max|2^i - 2^j| = 2(2^{2b-1} - 2^0) = 4^b - 2$, so no pair of players can be adjacent more than once. □

**Corollary 3.7.** *Let $\mathcal{G}$ be an n-player game with $b(\mathcal{G}) \ge \binom{n}{2} - \log_4 n$. Then the graph associated to $\mathcal{G}$ is connected, and there exists a balanced tournament for $\mathcal{G}$.*

*Proof.* The graph for $\mathcal{G}$ is isomorphic to a subgraph of the complete graph $K_n$ formed by removing at most $\log_4 n$ edges. Since $K_n$ has edge connectivity $n-1$, the resulting graph is connected. The rest of the proof follows from Theorems 3.5 and 3.6. □

Note that we do not have any opinion concerning whether $\log_4 n$ can be replaced by a larger quantity.

**3.3. Cyclic Diplomacy.** We now look at adjacency matrices whose associated graphs are cycles, and the family of games that they define.

**Definition 3.8.** Consider an $r$-by-$r$ array of integers. We will consider the upper-left entry in this array to be given the index $(1,1)$, the entry to its immediate right is indexed $(1,2)$, the lower-right entry in this array to be indexed $(r,r)$, and all other entries given the obvious indices.

**Definition 3.9.** In an $r$-by-$r$ array of integers, the *d-diamond* is the set of entries having the indices

$$\{(x,y) : y \in \{d+1-x, x+d, 2r-d+1-x, x-d\}\} \cap ([1,r] \times [1,r]).$$

**Example 3.10.** In the following 7-by-7 array, the 5-diamond has been written in boldface and traced with dashed lines.

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | **5** | **6** | 7 |
| 2 | 4 | 1 | **6** | 3 | 7 | **5** |
| 3 | 1 | **5** | 2 | 7 | 4 | **6** |
| 4 | **6** | 2 | 7 | 1 | **5** | 3 |
| **5** | 3 | 7 | 1 | **6** | 2 | 4 |
| **6** | 7 | 4 | **5** | 2 | 3 | 1 |
| 7 | **5** | **6** | 3 | 4 | 1 | 2 |

**Example 3.11.** The $r$-diamond of an $r$-by-$r$ array is the set of $r$ entries along the diagonal having indices $\{(r,1),(r-1,2),\ldots,(1,r)\}$.

Note that a $d$-diamond has cardinality

$$\begin{cases} 2r & \text{if } d < r, \text{ and} \\ r & \text{if } d = r. \end{cases}$$

**Definition 3.12.** The *content* of the $d$-diamond of an $r$-by-$r$ array is the string obtained by reading the contents of the $d$-diamond, starting with the entry indexed by $(d,1)$ and reading in clockwise order.

**Example 3.13.** Reading clockwise around the 5-diamond of the 7-by-7 array in Example 3.10, starting with the entry indexed by $(5,1)$, we obtain

(56565)(65)(65656)(56),

where the parentheses have been included to clarify the four sides of the diamond. The content of this diamond is simply

$$56565656565656.$$

**Lemma 3.14.** *For any distinct $d$ and $d'$ having the same parity, the $d$- and $d'$-diamonds of an array are disjoint.*

*Proof.* Suppose that the $d$- and $d'$-diamonds have nonempty intersection, and that $d > d'$, with $d$ and $d'$ having the same parity. Then an ordered pair in their intersection will force one of the following equalities:

$$\begin{aligned} d+1-x &= x+d', \\ d+1-x &= x-d', \\ 2r-d+1-x &= x+d', \text{ or} \\ 2r-d+1-x &= x-d'. \end{aligned}$$

However, in each of these cases, the fact that $d$ and $d'$ have the same parity and that $\{x, r, d, d'\} \subset \mathbb{Z}$ causes a contradiction. □

**Corollary 3.15.** *Fix a positive integer $r$. Knowing the content of all of the $d$-diamonds for positive $d \in \{r, r-2, r-4, \ldots\}$ completely determines an $r$-by-$r$ array.*

*Proof.* Consider $(x, y) \in ([1, r] \times [1, r])$. Note that $x + y \in [2, 2r]$, and so exactly one of the integers $x + y - 1$ and $2r - (x + y - 1)$ is in the interval $[1, r]$. Moreover, exactly one of the integers $x - y$ and $y - x$ is in the interval $[1, r]$.

Suppose, for the moment that $r$ is odd. If $x \pm y$ is even, then choose $d \in \{x + y - 1, 2r - (x + y - 1)\} \cap [1, r]$. If $x \pm y$ is odd, then choose $d \in \{x - y, y - x\} \cap [1, r]$. Either way, $d$ is odd, and $(x, y)$ is in the $d$-rectangle of the array.

Now suppose that $r$ is even. If $x \pm y$ is even, then choose $d \in \{x - y, y - x\} \cap [1, r]$. If $x \pm y$ is odd, then choose $d \in \{x+y-1, 2r-(x+y-1)\} \cap [1, r]$. Either way, $d$ is even, and $(x, y)$ is in the $d$-rectangle of the array.

Thus every entry of the array is in some $d$-rectangle, where $d$ and $r$ have the same parity. The remainder of the proof follows from Lemma 3.14. □

We will now define a particular array $A_r$ when $r$ is odd, using Corollary 3.15.

**Definition 3.16.** Let $r$ be an odd positive integer, and let $A_r$ be the $r$-by-$r$ array whose $d$-diamond has content

$$d(d+1)d(d+1)\cdots d(d+1)$$

for all $d \in \{r-2, r-4, r-6, \ldots\}$, and whose $r$-diamond has content $rr \cdots r$.

**Example 3.17.** The 7-by-7 array in Example 3.10 is $A_7$.

**Lemma 3.18.** *Let $r$ be an odd positive integer. The array $A_r$ is a Latin square.*

*Proof.* We will prove that each row consists of distinct integers. The proof that each column consists of distinct integers is entirely analogous.

Consider the entries in row $i$ of the array $A_r$. These are of the forms $\{d+1-i, d+i, 2r-d+1-i, i-d\}$, for $d \in \{r, r-2, r-4, \ldots\}$. We know that $r$ is odd, so $d$ is odd as well. We want to show that the $r$ entries in this row are distinct. If any two were to be equal, then surely they would have the same parity. Thus we must have either $d+1-i = 2r-d'+1-i$ for some $d, d'$, or we must have $d+i = i-d'$. Thus either $2r = d+d'$, or $d = -d'$. The former case implies that $d = d' = r$, while the latter is impossible. But in the first case, we would really be talking about a single entry, not two identical entries, so again there is no repetition within any row of the array. □

For $n \geq 3$, consider a $n$-player game of *Diplomacy* whose graph of adjacencies is a cycle, and call this *Cyclic Diplomacy.* For $m \geq 3$, let $C_m$ be the cycle on $m$ vertices, and label its vertices $1, 2, \ldots, m$ in order around the cycle. For example, the graph $C_7$ is given in Figure 2. Let $B(C_m)$ be the adjacency matrix for the graph $C_m$.

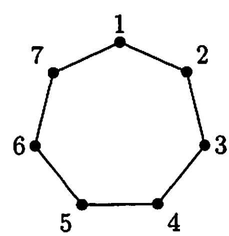



FIGURE 2. The graph $C_7$ describing 7-player Cyclic Diplomacy.

**Example 3.19.** The adjacency matrix for $C_7$ is

$$B(C_7) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

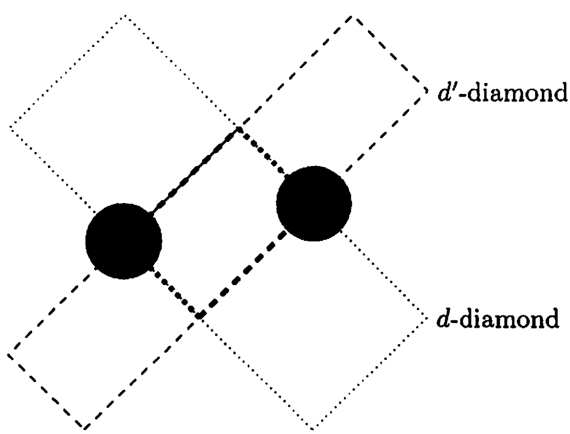
**Theorem 3.20.** *Let $r > 1$ be an odd integer. The $r$-player Cyclic Diplomacy tournament defined by $(B(C_r), A_r)$ is balanced.*

*Proof.* What we must determine is how many times the integers $a, b \in [1, r]$ might be adjacent in the rows of the array $A_r$. Because we are playing Cyclic Diplomacy, $a$ and $b$ are "adjacent" if they appear in the entries indexed $(x, y)$ and $(x, y+1)$, where $y+1$ is considered modulo $r$.

There are $r^2$ total adjacencies that occur in a tournament of Cyclic Diplomacy with $r$ players, and there are $\binom{r}{2}$ pairs of the players. Thus the average number of meetings of any two players would be $r^2/\binom{r}{2} = 2r/(r-1)$. Because $r \geq 3$, we have $2 < 2r/(r-1) \leq 3$. Thus, we want to show that any two integers are adjacent either 2 or 3 times in the array $A_r$.

Due to the construction of $A_r$, where all the values of $a$ lie along the $(2\lceil \frac{a}{2} \rceil - 1)$-diamond, we just need to understand how a $d$- and a $d'$-diamond might have adjacent entries (where we assume for the duration of this proof that $d$ and $d'$ are both odd). If $a$ and $b$ appear on the same $d$-diamond, then we can assume, without loss of generality, that $(a, b) = (d, d+1)$. Then $a$ and $b$ are adjacent precisely two times: in the top and the bottom rows of $A_r$, except in the case when $(r+1)/2 \in \{d, d+1\}$, in which case they are also adjacent around row $(r+1)/2$ of the array, for a total of three times.

Now suppose that $a$ appears on the $d$-diamond and $b$ appears on the $d'$-diamond, with $d < d'$. Ignore, for the moment, any adjacencies that occur around the array. The four possible remaining sites for adjacencies between $a$ and $b$ are marked in the following figure.

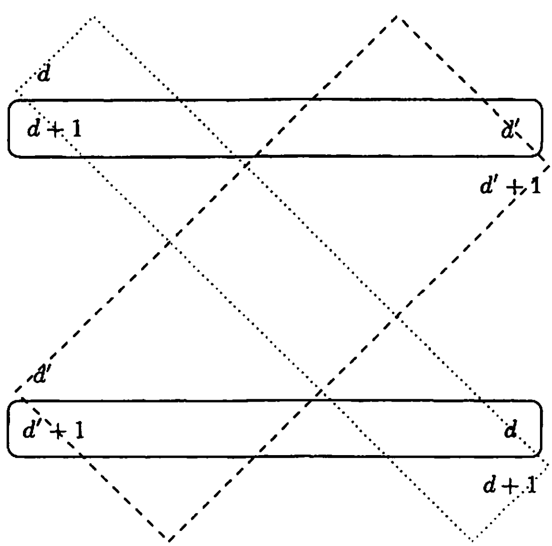



Note that the rectangle drawn in dark lines has sides of length $d$ in the southwest-northeast direction, and sides of length $r - d'$ in the northwest-southeast direction. The fact that $d$, $d'$, and $r$ have the same parity is enough to show that the four squares comprising the regions of these adjacencies fall into two categories: the north- and south-most (lighter) have the same layout, while the west- and east-most (darker) have the same layout, and these two layouts are different. The two layouts, each appearing twice, are:

$$\begin{matrix} \alpha & d'+1 \\ d' & \beta \end{matrix} \quad \text{and} \quad \begin{matrix} \alpha & d' \\ d'+1 & \beta \end{matrix},$$

where $\{\alpha, \beta\} = \{d, d+1\}$, depending on the precise values of $d$ and $d'$. Then, since $a \in \{d, d+1\}$ and $b \in \{d', d'+1\}$, the values $a$ and $b$ are adjacent exactly two times in these regions.

It remains to consider when $a$ and $b$ might be adjacent around the array. For this to happen, the corner of the $d$-diamond along the left side of the array must differ in height by 1 unit from the corner of the $d'$-diamond along the right side of the array; that is, $d = r - d' \pm 1$. Of course, this means that the right corner of the $d$-diamond and the left corner of the $d'$-diamond also have heights differing by 1 unit. In the case when $d = r - d' - 1$, this gives the following scenario (the case when $d = r - d' + 1$ is analogous).




The adjacencies caused by these rectangles that go around the array have been circled. Note that in one, the values $d+1$ and $d'$ are adjacent, while in the other, the values $d'+1$ and $d$ are adjacent. The degenerate case $d' = r$, where the $d'$-diamond is just a diagonal of the array, causes 1 and $r$ to be adjacent around the array one time, and 2 and $r$ to be adjacent around the array one time. Therefore, a given pair of values $\{a, b\}$ becomes adjacent at most one extra time by considering adjacencies around the boundary of the array.

Thus each pair of values is adjacent either 2 or 3 times, and so the given tournament is balanced. □

**Example 3.21.** The array $A_7$ of Example 3.10 gives a balanced tournament for 7-player Cyclic Diplomacy, the adjacency graph of which depicted in Figure 2.

## 4. Examples of games that cannot be balanced

One might conjecture that given any border arrangement $B$ of a game, there always exists a Latin square $L(B)$ for which the tournament $(B, L(B))$ is balanced. However, as the next example shows, this is not the case.

**Example 4.1.** There is no balanced tournament for the graph of four vertices and two disjoint edges:

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Here $pn = (2/6)4 = 4/3$, so a balanced tournament requires $(\ell, u) = (\lfloor 4/3 \rfloor, \lceil 4/3 \rceil) = (1, 2)$. However there are only four reduced $4 \times 4$ Latin squares, and it is easy to check that none of them has this property. (Incidentally, the two "most balanced" of them, in which $u - \ell$ is minimal, both have $(\ell, u) = (0, 2)$.)

In fact, Example 4.1 is an instance of a more general class of graphs; namely, the disjoint union of two cliques $K_r \cup K_s$, where the *clique* $K_t$ is the complete graph on $t$ vertices. There is a useful parity lemma that holds for all of these graphs, and explains properties of the graph $K_2 \cup K_2$ that appeared in Example 4.1.

**Lemma 4.2.** *For any distinct players $i$ and $j$ in the graph $K_r \cup K_s$, the number $n_{i,j}$ has the same parity as $r + s$.*

*Proof.* Partition the $r + s$ games of the tournament by which portion of the graph $K_r \cup K_s$ contains player $i$.

Consider the $r$ games during which player $i$ occupies a position in $K_r$. Suppose that $j$ is also in this portion of the graph during $r - t$ of those games; that is, $i$ and $j$ play each other in $K_r$ exactly $r - t$ times. During the $s$ games in which player $i$ occupies a position in $K_s$, player $j$ can also be in $K_s$ for only $s - t$ of these. This is because player $j$ has already occupied $t$ of the $K_s$ positions while $i$ was in $K_r$.

Therefore $i$ and $j$ appear in the same component of the graph — that is, face each other in a game — exactly $(r - t) + (s - t) = r + s - 2t$ times. □

Since the matrix in Example 4.1 has $pn = \left(\frac{2}{6}\right) 4 = 4/3$, Lemma 4.2 immediately implies that every tournament must have $u - \ell \geq 2$. Actually, the values $(\ell, u) = (0, 2)$ do occur here.

The graph for *Diplomacy* was connected. This makes sense because the goal of the game is domination of all of the seven countries and one cannot dominate a country if one cannot reach it. Thus, a weaker conjecture that would still generalize the *Diplomacy* result is that given any adjacency matrix $B$ corresponding to a *connected* graph, there exists a Latin square square $L(B)$ so that the tournament $(B, L(B))$ is balanced. Note that Example 4.1 does not contradict this conjecture, since the graph for $B_1$ is obviously disconnected.

However, this conjecture, too, is false, since the complement of any disconnected graph is connected and retains the difference of $u - \ell$ as mentioned in Proposition 3.1. The complement of $K_2 \cup K_2$ is $C_4$, and $(\ell(C_4), u(C_4)) = (2, 4)$.

One might hope that it is easier to achieve a balanced tournament when $pn$ is not integer, but our next example has both a connected graph and $pn \notin \mathbb{Z}$.

**Example 4.3.** There is no balanced tournament for the adjacency matrix

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

The (connected) graph with adjacency matrix $B_2$ is given in Figure 3. We also verify this with a computer, which determines that the tournament $(B_2, L)$ that is as close to being balanced as possible has $(\ell(B_2, L), u(B_2, L))$ equal to $(2, 4)$, instead of the desired values $(\lfloor (7/15)6 \rfloor, \lceil (7/15)6 \rceil) = (2, 3)$.

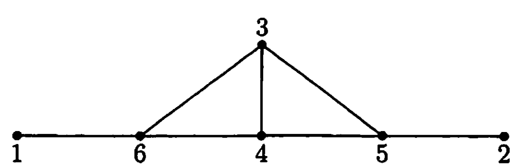



FIGURE 3. The graph with adjacency matrix $B_2$.

Intuitively, it is not surprising that the examples of this section cannot be balanced, because the strategic asymmetries of these tournaments are highly pronounced.

## 5. An infinite family of graphs that cannot be balanced

The last two examples were graphs that do not have a balanced tournament, in fact in both examples the smallest value of $u - \ell$ is 2. Let us refer to this minimum difference for the game $\mathcal{G}$ as

$$\Delta = \Delta(\mathcal{G}) = \inf\{u(T) - \ell(T) : T \text{ is a tournament for } \mathcal{G}\},$$

and use this difference to measure the deviation from being balanced. For example, $\Delta = 2$ in Example 4.3. For all graphs on at most 7 vertices, we found $\Delta \leq 2$. However, by relying on the theory of symmetric balanced incomplete block designs, we can find an infinite collection of graphs for which $\Delta \geq 4$. In fact it seems likely that $\Delta = 4$ for all of these, but to establish equality would require us to find a suitable Latin square for each graph.

**Theorem 5.1.** *There exist infinitely many graphs of the form $K_r \cup K_s$ for which $\Delta \geq 4$.*

**Corollary 5.2.** *There exist infinitely many connected graphs for which $\Delta \geq 4$.*

*Proof of Corollary 5.2.* Identity (5) implies that

$$\Delta(\mathcal{G}) = \Delta(\mathcal{G}^c)$$

for any game $\mathcal{G}$. In particular, $\Delta(K_r \cup K_s) = \Delta((K_r \cup K_s)^c)$, and there are infinitely many such graphs with $\Delta \geq 4$, by Theorem 5.1. Note that $(K_r \cup K_s)^c = K_{r,s}$, the complete bipartite graph, which is connected. □

In order to prove Theorem 5.1, we will use a theorem of Bruck, Chowla, and Ryser about symmetric balanced incomplete block designs ("SBIBDs"). An elementary treatment of SBIBDs appears in [8, pp. 456–463]. That work describes the proof of the easier, $\nu$ even, case of Theorem 5.3. The difficult proof of the $\nu$ odd case appears in [4].

**Theorem 5.3** ([3, 4])**.** *Let $v$, $k$, and $\lambda$ be positive integers with $k < v$ and $\lambda(v-1) = r(r-1)$.*

(a) *If $v$ is even and there exists a $(v,k,\lambda)$ SBIBD, then $k - \lambda$ is a perfect square.*

(b) *If $v$ is odd and there exists a $(v,k,\lambda)$ SBIBD, then the equation*

$$x^2 = (k-\lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2$$

*has a solution for some nonnegative integers $x$, $y$, and $z$, not all 0.*

*Proof of Theorem 5.1.* We interpret graphs and their tournament based on a Latin square as an SBIBD. In this setting, the number of vertices $r + s$ is denoted $v$. The smaller order clique ($K_r$, without loss of generality), is considered a block of order $r$, and we set $k = r$. In the Latin square, each vertex $i$ appears $r$ times in the block $K_r$. If we have perfect balance, then each pair of vertices $\{i, j\}$ appear together in a $K_r$ block $\lambda$ times, where balance requires that

$$\lambda \frac{v(v-1)}{2} = v \frac{r(r-1)}{2}.$$

This simplifies to

$$\lambda(v-1) = \lambda(r+s-1) = r(r-1).$$

If these parameters admit an SBIBD, then we have a balanced tournament. This occurs because each pair $\{i, j\}$ appears together $\lambda$ times in $K_r$, and therefore we have $2(r-\lambda)$ times when exactly one appears. This leaves $s+r-\lambda-2(r-\lambda) = s-r+\lambda$ times when neither is present in $K_r$, meaning that both $i$ and $j$ are in $K_s$. Thus $n_{i,j} = s-r+2\lambda$. When the SBIBD exists, every pair produces this value. Thus $u = \ell = s-r+2\lambda$, and $\Delta = 0$.

Suppose that no SBIBD exists for certain parameters. Then the average value of $n_{i,j}$ remains $s-r+2\lambda$, but not every pair produces this value. Then there will be some pair $\{i, j\}$ with $n_{i,j} > s-r+2\lambda$. By Lemma 4.2, we must have $n_{i,j} \geq s-r+2\lambda+2$. Similarly, there is a pair $\{i', j'\}$ with $n_{i',j'} \leq s-r+2\lambda-2$. Therefore, for any such graph $K_r \cup K_s$, we have $\Delta \geq 4$.

To show that there are infinitely many parameters that do not admit an SBIBD, we use Theorem 5.3. For each positive integer $t$, consider the graph

$$K_{8t-1} \cup K_{64t^2-32t+4}$$

with $v = 64t^2 - 24t + 3$, $\lambda = 1$, and $k = 8t - 1$. Because

$$\lambda(v-1) = 1(64t^2 - 24t + 2) = (8t-1)(8t-2) = k(k-1),$$

existence of an SBIBD requires a nonnegative integer solution to the equation

$$x^2 = (8t-2)y^2 - z^2.$$

A minimum solution cannot have all $\{x, y, z\}$ be even. Reading this equation modulo 8 yields $x^2 \equiv 6y^2 - z^2$. Choosing any one of $\{x, y, z\}$ to be odd leaves an impossible modulo 8 congruence for the other two variables. Thus no solution exists, and no SBIBD exists. □

We illustrate Theorem 5.1 and Corollary 5.2 with an example.

**Example 5.4.** Consider $K_8 \cup K_{21}$. Then $\nu = 8 + 21$, $\lambda = 2$, and $k = 8$. Existence of an SBIBD requires a nonnegative, not identically 0, integer solution to $x^2 = 6y^2 + 2z^2$. Suppose that there is such a solution $(x_0, y_0, z_0)$, chosen so that $\gcd(x_0, y_0, z_0) = 1$. The value of $x_0$ is necessarily even, so write $x_0 = 2x_1$, yielding $2x_1^2 = 3y_0^2 + z_0^2$. Note that by construction, $y_0$ and $z_0$ may not both be even. Taking the equation modulo 2 yields that $y_0$ and $z_0$ must have the same parity, so they must both be odd. But then, taking the equation modulo 8 would yield a contradiction. Thus, by Theorem 5.3(b), we have $\Delta \geq 4$. To show equality, we must construct a demonstrative $29 \times 29$ Latin square. Read all player labels modulo 29, and assign row $i$ to place the eight players $\{i, i+1, i+4, i+6, i+17, i+19, i+25, i+26\}$ in $K_8$. This gives 29 pairs $\{i, i+12\}$ with $n_{i,i+12} = 15$, and another 29 pairs $\{i, i+14\}$ with $n_{i,i+14} = 15$. Likewise, we have 29 pairs $\{i, i+4\}$ with $n_{i,i+4} = 19$, and another 29 pairs $\{i, i+13\}$ with $n_{i,i+13} = 19$. All other pairs, of which there are 290 of them, give $n_{i,j} = 17$. Consequently, $u - \ell = 19 - 15$, and so $\Delta = 4$.

The same Latin square applied to the connected complementary graph $K_{8,21}$ also obtains $\Delta = u - \ell = 4$.

## 6. Further directions

We have not yet discussed $2k$-player Cyclic Diplomacy. If $n = 2k = 4$, then the graph is $C_4$, which is the complement of $K_2 \cup K_2$ in Example 4.1. Thus $(\ell, u) = (2, 4)$ and $\Delta = 2$.

On the other hand, when $n = 2k = 6$, we have $pn = (6/15)6$. This means that a balanced tournament $(B, L)$ for 6-player Cyclic Diplomacy would need to have $(\ell(B, L), u(B, L)) = (2, 3)$. The tournament

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 6 & 3 & 1 & 4 \\ 3 & 6 & 1 & 5 & 4 & 2 \\ 4 & 3 & 5 & 2 & 6 & 1 \\ 5 & 1 & 4 & 6 & 2 & 3 \\ 6 & 4 & 2 & 1 & 3 & 5 \end{bmatrix}$$

has $(\ell(B, L), u(B, L)) = (2, 3)$ and thus produces a balanced tournament. Similarly, the following Latin square produces a balanced tournament for 8-player Cyclic Diplomacy:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\ 3 & 4 & 6 & 2 & 8 & 7 & 1 & 5 \\ 4 & 8 & 5 & 1 & 7 & 3 & 2 & 6 \\ 5 & 7 & 1 & 8 & 2 & 4 & 6 & 3 \\ 6 & 5 & 2 & 7 & 4 & 8 & 3 & 1 \\ 7 & 3 & 8 & 6 & 1 & 2 & 5 & 4 \\ 8 & 6 & 7 & 5 & 3 & 1 & 4 & 2 \end{bmatrix}.$$

Theorem 3.20 and the two Latin squares above show that $\Delta$ is equal to 0 for all $n$-player games of Cyclic Diplomacy when

$$n \in \{2k + 1 : k \in \mathbb{Z}\} \cup \{6, 8\}.$$

On the other hand, we have shown above that the error measure $\Delta = 0$ in 4-player Cyclic Diplomacy. An argument very similar to the proof of Theorem 3.20 produces, for each cyclic graph of even length (that is, each game of Cyclic Diplomacy with an even number of players), a tournament for which $\Delta \leq 2$. However, given that $\Delta$ is actual equal to 0 for the 6- and 8-player games, it seems quite possible that there are balanced tournaments for $2k$-player Cyclic Diplomacy for larger values of $k$. Proving that, for $n \neq 4$, the $n$-player game of Cyclic Diplomacy can be balanced (that is, a tournament can be produced with $\Delta = 0$) would be most satisfying.

Another avenue for future research would be to work with adjacency matrices arising from more general graphs. One first step in this direction might be to try to determine the error measure $\Delta$ for circulant graphs.

Theorem 5.1, which produces infinitely many graphs with $\Delta \geq 4$, leaves two open problems. The first is to determine whether the inequality in the theorem could be improved to a statement of equality. The second, more general, open problem is to determine if any graphs exist with $\Delta \geq 5$. Our experience up to this point gives us no reason to suspect that this may occur.


## 7. Acknowledgements

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