# A note on $K_r$-free graphs with chromatic number at least $r$

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**Abstract**

We provide a new proof of a result of Hanson and Toft classifying the maximum-size $K_r$-free graphs on $n$ vertices with chromatic number at least $r$.

## 1 Introduction

Fix integers $n \geq 1$ and $r \geq 3$, and let $[n] = \{1, 2, \ldots, n\}$. Turán's classical theorem [6] states that the maximum size of a $K_r$-free graph on $n$ vertices is achieved only by the so-called Turán graph, $T_{r-1}(n)$, the $(r-1)$-partite graph on $n$ vertices in which each part has $\lfloor n/(r-1) \rfloor$ or $\lceil n/(r-1) \rceil$ vertices. In this work we consider the related question: what is the largest size of a $K_r$-free graph $G$ on $n$ vertices with chromatic number $\chi(G) \geq r$? To that end, we define

$$\mathcal{G}_{n,r} := \{G : V(G) = [n],\ G \not\supseteq K_r,\ \text{and}\ \chi(G) \geq r\}.$$

We then let $g(n,r) := \max_{G \in \mathcal{G}_{n,r}} e(G)$, where $e(G)$ denotes the size of the graph $G$, and let $\mathcal{G}^*_{n,r} := \{G \in \mathcal{G}_{n,r} : e(G) = g(n,r)\}$. That is, $\mathcal{G}_{n,r}$ is the family of $K_r$-free graphs on $[n]$ with chromatic number at least $r$, while $g(n,r)$ and $\mathcal{G}^*_{n,r}$ are the size and family, respectively, of maximum-size such graphs. We note that each graph $G \in \mathcal{G}^*_{n,r}$ is $r$-saturated, i.e., the addition of any missing edge creates a copy of $K_r$.

The first to consider $g(n,r)$ were Erdős and Gallai, and, independently, Andrásfai ([2]), who determined $g(n,3)$ and $\mathcal{G}^*_{n,3}$ for all $n \geq 5$. To state their results we introduce further definitions and notation. First, for a graph $H$ on $[n]$, we say a graph $G$ is a *blow-up* of $H$ if there exist positive integers $k_1, \ldots, k_n$ and a partition $V_1 \cup V_2 \cup \cdots \cup V_n$ of $V(G)$ such that $|V_i| = k_i$ for $1 \le i \le n$ and $E(G) = \{xy : x \in V_i, y \in V_j, ij \in E(H)\}$, and we write $G = H(k_1, \ldots, k_n)$. As a blow-up of $H$ has the same chromatic and clique numbers as $H$, candidates for $\mathcal{G}^*_{n,3}$ are blow-ups of the cycle $C_5$ (taking $V(C_5) = [5]$ and $E(C_5) = \{12, 23, 34, 45, 51\}$) of maximum size. Let $\mathcal{H}_n$ denote the family of maximum-size blow-ups of a $C_5$ on $[n]$; routine maximization arguments show each graph $H \in \mathcal{H}_n$ has the form $H = C_5(1, 1, a, b, c)$, where $\{a + c, b + 1\} = \{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil\}$, and has size $\lfloor \frac{(n-1)^2}{4} \rfloor + 1$.

**Theorem 1** (Erdős and Gallai, Andrásfai [2]). *For $n \ge 5$, $g(n, 3) = \lfloor \frac{(n-1)^2}{4} \rfloor + 1$ and $\mathcal{G}^*_{n,3} = \mathcal{H}_n$.*

For a simple construction of $K_r$-free $r$-chromatic graphs for all $r \ge 3$, let $G \vee H$ denote the *join* of two vertex-disjoint graphs $G$ and $H$, formed from the union of $G$ and $H$ by adding all edges between $V(G)$ and $V(H)$. We note that $\chi(G \vee H) = \chi(G) + \chi(H)$. Furthermore, to simplify notation, we allow for $H$ to be the null graph on 0 vertices (and with 0 edges), in which case we take $\chi(H) = 0$ and $G \vee H = G$ for all graphs $G$. Next, for $r \ge 3$ and $n \ge r + 2$, let $\mathcal{F}_{n,r}$ be the family of maximum-size graphs $G$ on $[n]$ of the form $G = H \vee T_{r-3}(n-k)$, where $5 \le k \le n - (r-3)$ and $H \in \mathcal{H}_k$. We note that $\mathcal{F}_{n,3} = \mathcal{H}_n$, and that $\mathcal{F}_{n,r} \subseteq \mathcal{G}_{n,r}$. A result[1] of Hanson and Toft [3], and, later and independently, Kang and Pikhurko [4], asserts that this is the correct construction.

**Theorem 2** (Hanson and Toft [3], Kang and Pikhurko [4]). *For $r \ge 3$ and $n \ge r + 2$, $\mathcal{G}^*_{n,r} = \mathcal{F}_{n,r}$.*

The condition $n \ge r + 2$ is necessary: a graph $G$ on $n \le r + 1$ vertices with $\chi(G) \ge r$ has at least $r$ singleton color classes under a proper coloring, and hence must contain a copy of $K_r$. From Theorem 2, determining $g(n, r)$ is a straightforward task.

**Corollary 1.** *For $r \ge 3$,*

*(i) if $r + 2 \le n \le 2(r-1)$, then $g(n, r) = e(T_{r-1}(n)) - 2$, and*

*(ii) if $n > 2(r-1)$, then $g(n, r) = e(T_{r-1}(n)) - \lfloor \frac{n}{r-1} \rfloor + 1$.*

The existence of a constant $C$ such that $g(n, r) \le e(T_{r-1}(n)) - \frac{n}{r-1} + C$ for fixed $r \ge 3$ and $n$ sufficiently large was first established by Simonovits [5], while (ii) was first shown by Brouwer [1]. Our aim is to give a new

[1] In [3], the authors determine the graphs which are 'maximal' with respect to the number of edges; our statement is equivalent to theirs.

proof of Theorem 2 through a novel swapping procedure, which avoids the inductive approach taken in [4] and many of the case analyses presented in [3].

# 2 The proofs

For $k \geq 5$, let $F_3(k) \in \mathcal{H}_k$ be formed by subdividing an edge in a copy of the complete bipartite graph $T_2(k-1)$ on $[k-1]$ with the vertex $v^* = k$, so that $F_3(k) = C_5(1,1,1,\lfloor\frac{k-1}{2}\rfloor - 1, \lceil\frac{k-1}{2}\rceil - 1)$. Next, for $r \geq 3$, $n \geq r+2$ and $5 \leq k \leq n-(r-3)$, let $F_r(n,k)$ denote the graph formed from the join of $F_3(k)$ and a copy of $T_{r-3}(n-k)$ on $[k+1,\ldots,n]$, noting $F_r(n,k) \in \mathcal{G}_{n,r}$ holds. Letting $X_1, X_2$ and $X_3, \ldots, X_{r-1}$ denote the parts of $F_3(k) - v^*$ and of the copy of $T_{r-3}(n-k)$, respectively, we may assume

$$|X_1| \leq |X_2| \leq |X_1|+1 \quad \text{and} \quad |X_3| \leq |X_4| \leq \cdots \leq |X_{r-1}| \leq |X_3|+1.$$

Since every graph in $\mathcal{H}_k$ has the same size, for some $k$ we must have $F_r(n,k) \in \mathcal{F}_{n,r}$, but rather than identify which such $k$ maximize $e(F_r(n,k))$, we will only need the following short claim:

**Claim 1.** *If $r \geq 4$, $n \geq r+2$, and $F_r(n,k) \in \mathcal{F}_{n,r}$, then*

*(a) if $k \geq 6$, then $|X_3| \geq |X_2|$, and*

*(b) there exists an integer $k'$ for which $F_r(n,k') \in \mathcal{F}_{n,r}$ and $|X_{r-1}| \leq |X_1|+1$.*

*Proof.* If $k \geq 6$, then $|X_2| \geq 3$, so suppose $|X_2| > |X_3|$: we then form a copy of $F_r(n,k-1)$ by deleting a vertex in $X_2$ nonadjacent to $v^*$ and duplicating a vertex in $X_3$. This procedure first removes $|X_1| + (n-k)$ edges and then adds $(k-1)+(n-k-|X_3|) = |X_1|+|X_2|+(n-k)-|X_3|$ edges, increasing the size by $|X_2| - |X_3| > 0$ edges and contradicting the maximality of $e(F_r(n,k))$. Similarly, if $|X_{r-1}| \geq |X_1|+2$, the analogous procedure yields $e(F_r(n,k+1)) \geq e(F_r(n,k))$, and $k'$'s existence follows from finiteness. □

## 2.1 Proof of Theorem 2

Let $\mathcal{G}'_{n,r} = \{G \in \mathcal{G}^*_{n,r} : \exists v \in [n] \text{ such that } \chi(G-v) = r-1\}$.

**Lemma 1.** *For all $n \geq r+2 \geq 5$, if $\mathcal{G}'_{n,r} \neq \emptyset$, then $\mathcal{G}'_{n,r} = \mathcal{F}_{n,r}$.*

*Proof.* Let $G \in \mathcal{G}'_{n,r}$, and choose a vertex $v$ satisfying $\chi(G-v) = r-1$ so that $v$ has minimum degree among all such vertices. Let $X_1, X_2, \ldots, X_{r-1}$ be the color classes of $G-v$ under an $(r-1)$-coloring: then $N_G(v) \cap$

$X_i \neq \emptyset$ holds for all $i$, so for some pair of distinct classes $(X_i, X_j)$ we have $e(X_i, X_j) < |X_i| \cdot |X_j|$. We relabel if necessary so that $(X_1, X_2)$ is such a pair satisfying both $|X_1| \leq |X_2|$ and $|X_1| \geq \min\{|X_i|, |X_j|\}$ for all other such pairs $(X_i, X_j)$. We then let $K = \{v\} \cup X_1 \cup X_2$, $k = |K|$, and $L = [n] \setminus K$, noting $\chi(G[K]) = 3$ and $\chi(G[L]) = r - 3$.

Since $G$ is $r$-saturated, there exist vertices $x_i \in X_i$ so that $x_1x_2 \notin E(i)$ and $v, x_1, x_2, \ldots, x_{r-1}$ form a copy of $K_r$ in the graph $G + x_1x_2$. We claim that $k \geq 5$: otherwise, $\chi(G[K]) = 3$ and $x_1x_2 \notin E(G)$ together imply $X_1 = \{x_1\}$, $X_2 = \{x_2, y\}$ for some vertex $y$, and $G[\{v, x_1, y\}] \cong K_3$. Since $G$ is $K_r$-free, $yx_j \notin E(G)$ holds for some $j > 2$, implying $1 = |X_1| \geq \min\{|X_2|, |X_j|\}$ and so $X_j = \{x_j\}$. But then $\chi(G[K \cup X_j]) = \chi(G[\{v\} \cup \{x_1, x_2\} \cup \{y, x_j\}]) \leq 3$, a contradiction.

We next construct a graph $G'$ from $G$ through a sequence of edge 'swaps' as follows. For $i \in \{1, 2\}$ and each vertex $y \in N_G(v) \cap (X_i \setminus \{x_i\})$, $yx_j \notin E(G)$ holds for at least one index $j \in [r-1] \setminus \{i\}$: we delete the edge $yv$ and add one such edge $yx_j$. Let $G'$ denote the graph formed following these swaps, observing that $e(G') = e(G) = g(n, r)$, $N_{G'}(v) \cap K = \{x_1, x_2\}$, and $G'[\{v\} \cup L] = G[\{v\} \cup L]$.

Since $\chi(G'[L]) = \chi(G[L]) = r-3$, $e(G'[L]) \leq e(T_{r-3}(n-k))$ follows. As $k \geq 5$, we also have $e(G'[K]) \leq |X_1||X_2| + 1 \leq \lfloor (k-1)^2/4 \rfloor + 1 = e(F_3(k))$. Thus,

$$g(n, r) = e(G') \leq e(G'[K] \vee G'[L]) \leq e(F_3(k) \vee T_{r-3}(n-k)) = e(F_r(n, k)),$$

implying equality throughout. This yields the containment $\mathcal{F}_{n,r} \subseteq \mathcal{G}'_{n,r}$, and that the graph $G'$ satisfies $G' = G'[K] \vee G'[L]$, $G'[L] \cong T_{r-3}(n-k)$, and $G'[K] \cong F_3(k)$ (viewing $v$ as subdividing the edge $x_1x_2$ in a copy of $T_2(k-1)$ on $X_1 \cup X_2$), so $G' \cong F_r(n, k)$.

It remains, then, to argue that $G \in \mathcal{F}_{n,r}$. Suppose first that $G \neq G[K] \vee G[L]$, implying we swapped an edge $yv \in E(G)$ for an edge $yx_j \in E(G')$, $j > 2$, where $y \in X_i \setminus \{x_i\}$ for some $i \in \{1, 2\}$. We claim $yx_{3-i} \in E(G)$ and so $G[\{v, y, x_{3-i}\}] \cong K_3$, which follows as $yx_{3-i} \in E(G')$ and $N_{G'}(y) = (N_G(y) \setminus \{v\}) \cup \{x_j\}$. Now, $N_G(v) \cap L = L = N_G(x_{3-i}) \cap L$ and $N_G(y) \cap L = L \setminus \{x_j\}$, yielding $X_j = \{x_j\}$ as $G$ is $K_r$-free. Thus, $x_j \in U$ and, by Claim 1(a), $k = 5$ and so $|X_1| = |X_2| = 2$. By our choice of $v \in U$, $|L \cup \{y, x_1, x_2\}| \leq d_G(v) \leq d_G(x_j) \leq (|L|-1) + (k-1) = |L| + 3$, so equality holds throughout and $N_G(v) = L \cup \{y, x_1, x_2\}$. Writing $X_{3-i} \setminus \{x_{3-i}\} = \{z\}$, $\chi(G[K \cup X_j]) = \chi(G[\{x_1, x_2\} \cup \{v, z\} \cup \{y, x_j\}]) \leq 3$ follows, a contradiction showing that $G = G[K] \vee G[L]$.

Since $G[L] = G'[L] \cong T_{r-3}(n-k)$, $G[K]$ is therefore triangle-free, and since $G[(X_1 \cup X_2) \setminus \{x_1, x_2\}] = G'[(X_1 \cup X_2) \setminus \{x_1, x_2\}]$ is a complete bipartite graph, $N_G(v) \cap X_i = \{x_i\}$ follows for some $i \in \{1, 2\}$. If $N_G(v) \cap X_{3-i} = X_{3-i}$, then $N_G(x_i) \cap X_{3-i} = \emptyset$, implying $\chi(G[K]) = \chi(G[(\{v\} \cup X_i) \cup$

$(X_{3-i} \cup \{x_i\})] = 2$, a contradiction. Thus, $G[K]$ must be a blow-up of $C_5$ with (non-empty) parts $\{v\}, \{x_i\}, X_{3-i} \setminus N_G(v), X_i \setminus \{x_i\}$, and $X_{3-i} \cap N_G(v)$, and $G \in \mathcal{F}_{n,r}$ follows. □

By Lemma 1, it suffices to show $\mathcal{G}^*_{n,r} \subseteq \mathcal{G}'_{n,r}$, which we do through a simple application of Zykov's symmetrization [7]. Given nonadjacent vertices $x, y$ in a graph $G$, we symmetrize $x$ to $y$ by deleting all edges incident with $x$ and then adding all edges between $x$ and $N_G(y)$. It is well-known that symmetrization in a $K_r$-free graph produces a $K_r$-free graph.

Let $G \in \mathcal{G}^*_{n,r}$, and let $Z \subseteq [n]$ with $G[Z] \cong K_{r-1}$. Label the vertices in $[n] \setminus Z$ as $v_1, \dots, v_{n-r+1}$ arbitrarily, and let $G_0 = G$. For $1 \le i \le n-r+1$, we form a graph $G_i$ from $G_{i-1}$ by symmetrizing $v_i$ to a nonadjacent $z = z(i) \in Z$, which exists as $G_{i-1}$ is $K_r$-free inductively. Let $j$ be the maximum index so that $G_0, \dots, G_j \in \mathcal{G}^*_{n,r}$, noting that $\chi(G_{n-r+1}) = r-1$ and so $j \le n-r$. We show by induction that $G_j, G_{j-1}, \dots, G_0 = G$ all lie in $\mathcal{G}'_{n,r}$.

To see that $G_j \in \mathcal{G}'_{n,r}$, observe that $G_{j+1} \notin \mathcal{G}^*_{n,r}$ implies $\chi(G_{j+1}) = r-1$ or $e(G_{j+1}) \neq e(G_j)$. In the former case, $\chi(G_j - v_{j+1}) = \chi(G_{j+1} - v_{j+1}) = r-1$. In the latter case, letting $\{x, y\} = \{v_{j+1}, z(j+1)\}$ with $d_{G_j}(x) < d_{G_j}(y)$, symmetrizing $x$ to $y$ in $G_j$ produces a $K_r$-free graph $G'$ with $e(G') > e(G_j) = g(n,r)$, and so $\chi(G_j - x) = \chi(G') = r-1$.

Now, suppose $1 \le i \le j$ and $G_i \in \mathcal{G}'_{n,r} = \mathcal{F}_{n,r}$, so $G_i = H \vee T$, where, for some $k$, $T \cong T_{r-3}(n-k)$ and $H \cong H'$ for some $H' \in \mathcal{H}_k$. Letting $K^- = V(H) \setminus \{v_i\}$ and $L^- = [n] \setminus (K \cup \{v_i\})$, as $\chi(G_{i-1} - v_i) = \chi(G_i) = r$ it follows that $G_{i-1}[L^-]$ is a complete $(r-3)$-partite graph with parts $W_1, \dots, W_{r-3}$, and $G_{i-1}[K^-]$ is a blow-up of $C_5$ with parts $Y_1, \dots, Y_5$, where, without loss of generality, $Y_1 = \{y_1\}$ and $Y_2 = \{y_2\}$ for some vertices $y_1, y_2$. Now, if $G_{i-1}[K^- \cup \{v_i\}]$ contains a triangle, then $N_{G_{i-1}}(v_i) \cap W_\ell = \emptyset$ for some $\ell$, yielding $\chi(G_i[L^- \cup \{v_i\}]) = r-3$ and, say, $\chi(G_{i-1} - y_1) = r-1$. Otherwise, $N_{G_{i-1}}(v_i) \cap K^- \subseteq Y_a \cup Y_{a+2}$ for some $a$ (with addition performed modulo 5): let $Y'_{a+1} = Y_{a+1} \cup \{v_i\}$ and $Y'_b = Y_b$ for $b \neq a+1$. Then $G_{i-1}[K^- \cup \{v_i\}]$ is contained in the blow-up of $C_5$ with parts $Y'_a$, and $\min\{|Y'_1|, |Y'_2|\} = 1$ implies $\min\{\chi(G_{i-1} - y_1), \chi(G - y_2)\} = r-1$. We conclude that $G_{i-1} \in \mathcal{G}'_{n,r}$, completing the proof. □

## 2.2 Proof of Corollary 1

Recalling the discussion at the start of this section, select an integer $k$ for which $F_r(n,k) \in \mathcal{F}_{n,r} = \mathcal{G}^*_{n,r}$, and let $v^*$ and $X_1, \dots, X_{r-1}$ as given above. Suppose that $r+2 \le n \le 2(r-1)$: then Claim 1 implies that $|X_3| = 1$, $|X_1| = |X_2| = 2$, and $|X_j| \le 2$ for $j > 3$. Letting $X_3 = \{x_3\}$, by deleting the edge $v^*x_3$ and adding $v^*$ to $X_3$, we obtain a subgraph of $T_{r-1}(n)$ missing exactly 3 edges: one between $X_1$ and $X_2$ and two connecting $v^*$ to $X_1 \cup X_2$, yielding (i).

If $n \geq 2(r-1)+1$, then by Claim 1 we may also assume that $|X_1| \leq |X_2| \leq \cdots \leq |X_{r-1}| \leq |X_1|+1$. Deleting the sole edge connecting $v^*$ to $X_1$ and then adding $v^*$ to $X_1$ produces a subgraph of $T_{r-1}(n)$ missing only $1+(|X_2|-1) = \lfloor \frac{n}{r-1} \rfloor$ edges, and (ii) follows. □

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