# On the $E_3$-cordiality of some Graphs*

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**Abstract** The definition of $E_k$-cordial graphs is advanced by Cahit and Yilmaz[1]. Based on [1],a graph $G$ is said to be $E_3$-cordial if it is possible to label the edges with the numbers from the set {0, 1, 2} in such a way that, at each vertex $v$, the sum of the labels on the edges incident with $v$ modulo 3 satisfies the inequalities $|v(i) - v(j)| \leq 1$ and $|e(i) - e(j)| \leq 1$, where $v(s)$ and $e(t)$ are, respectively, the number of vertices labeled with $s$ and the number of edges labeled with $t$.In [1]-[3], authors discussed the $E_3$-cordiality of $P_n(n \geq 3)$; stars $S_n$, $|Sn| = n+1$ ;$K_n(n \geq 3)$, $C_n(n \geq 3)$, the one point union of any number of copies of $K_n$ and $K_m \bigodot K_m$. In this paper, we give the $E_3$-cordiality of $W_n, P_m \times P_n, K_{m,\ n}$ and trees .
**Keyword** edge-cordial labellings, $E_3$-cordial graph, $P_m \times P_n$, $K_{m,\ n}$, trees
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## 1 Introduction

Edge-graceful graphs and cordial graphs have been attracting the attention of graphs theorists for a long time[2]-[7].In [4] the authors have adopted cordial labeling of graphs [5],[6] to edge-cordial graphs. Cahit and Yilmaz generalized edge-cordial labellings to $E_k$ -cordial labelings of graphs[1],and they hoped that a study of $E_k$ -cordial labelings of graphs may give us a better understanding of edge-graceful graphs.Based on the definition of $E_k$-cordial graphs,we give the detailed definition of $E_3$-cordial graphs.

**Definition 1** Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$.$f$ is an edge labeling of $G$, such that $f(e) \in \{0,1,2\}, e \in E(G)$, and the induced vertex labeling is given as $f(v) = \sum_{x_i \in N(v)} f(vx_i)\ (mod\ 3) \in \{0,1,2\}$,where $v \in V(G), \{vx_i\} \in E(G)$.If the following two conditions are satisfied for $i,\ j = 0,1,2, i \neq j$:

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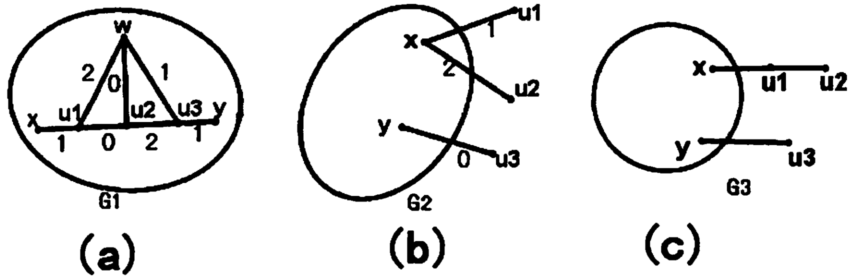



Figure 1: The graphs described in Lemma 1,Lemma 2 and Lemma 3

(1) $|v(i) - v(j)| \leq 1$;
(2) $|e(i) - e(j)| \leq 1$;
where $v(i)$ and $e(j)$ are, respectively, the number of vertices labeled with $i$ and the number of edges labeled with $j$.
$G$ is said to be $E_3$-cordial, and $f$ is an $E_3$-cordial labeling.

Cahit and Yilmaz[1] proved the following graphs are $E_3$-cordial: $P_n(n \geq 3)$; stars $S_n$, $|Sn| = n+1$ if and only if $n \not\equiv 1$ $(mod\ 3)$;$K_n(n \geq 3)$, $C_n(n \geq 3)$, etc. Bapat and Limaye[3] provide $E_3$-cordial labelings for $K_n(n \geq 3)$ and $K_m \bigodot K_m$ ; the one point union of any number of copies of $K_n$ where $n \equiv$ 0 or 2 $(mod\ 3)$.

In this paper, we mainly give the proof for $W_n, P_m \times P_n, K_{m,\ n}$ and trees to be $E_3$-cordial.In the following discussion,we consider finite,undirected simple graphs.$G^* - G$ is a graph with vertex set $V(G^* - G) = V(G^*) - V(G) + V(G \cap G^*)$ and edge set $E(G^* - G) = E(G^*) - E(G)$.The definitions of $W_n, P_m \times P_n, K_{m,\ n}$ , trees and other undefined terms can be found in [8].

# 2 Preliminaries

**Lemma 1** Let $G$ be $E_3$-cordial, $w \in V(G)$, $xy \in E(G)$, then $G_1$(see Figure 1(a)) constructed by adding three new vertices $u_1$, $u_2$, $u_3$ to the edge $xy$ and joining $w$ to $u_i$, is $E_3$-cordial for $i = 1,\ 2,\ 3$.

**Proof.** Let $f$ be an $E_3$-cordial labeling of $G$ and $f(x) = a$, $f(y) = b$, according to Definition 1,$f(xy) \in \{0,1,2\}$. Suppose that $f(xy) = 1$ in $G$, let $f(wu_1) = 2$, $f(wu_2) = 0$, $f(wu_3) = 1$ in $G_1$, then $f(u_1) = 0$, $f(u_2) = 2$, $f(u_3) = 1$ .One can easily check that $v(0) = v(1) = v(2)$ and $e(0) = e(1) = e(2)$ in $G_1 - G$ . Since $G$ is $E_3$-cordial, $G_1$ is $E_3$-cordial. When $f(xy) = 0$ or $f(xy) = 2$, the Lemma can be proved with the same method.

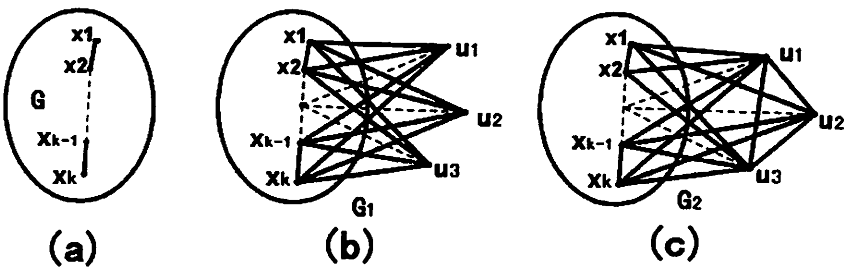



Figure 2: The graphs $G$, $G_1$ and $G_2$ in Lemma 4

**Lemma 2** Let $G$ be $E_3$-cordial, $\{x,\ y\} \subset V(G)$, then $G_2$(see Figure 1(b)) constructed by adding three new vertices $u_1,\ u_2,\ u_3$ to $G$ and joining the vertex $x$ to $u_1$ and $u_2$,joining $y$ and $u_3$,is $E_3$-cordial.

**Proof.** Let $f$ be an $E_3$-cordial labeling of $G$ and $f(x) = a,\ f(y) = b$, we can prove it by giving the labels 1,2 and 0 to the edges $xu_1,\ xu_2$, and $yu_3$ respectively. It can be clearly seen that when $x = y$, the conclusion is still true.

Here the transform from $G$ to $G_2$ in Lemma 2 is called $\alpha$-transform, simply recorded as $\alpha$-Tra, and the transform from $G_2$ to $G$ in Lemma 2 is called the inversion of $\alpha$-Tra.

**Lemma 3** Let $G$ be $E_3$-cordial, $\{x,\ y\} \subset V(G)$, then $G_3$, as shown in Figure 1(c), constructed by adding three new vertices $u_1,\ u_2,\ u_3$ to $G$ and joining the vertex $x$ and $u_1$ , $u_1$ and $u_2$, $y$ and $u_3$, is $E_3$-cordial.

**Proof.** This maybe refer to 9 cases with the different labellings of $x$ and $y$. Without loss of generality, we assume that $f(x) = 0,\ f(y) = 2$. In this case, just label the edges $xu_1,\ u_1u_2,\ yu_3$ with the numbers 0, ,2 ,1, the question is solved. Other cases could be proved in the same way, where the most important thing is ensuring $e(i) = e(j)$ and $v(i) = v(j)$ in $G_3 - G$, $i, j = 0,\ 1,\ 2$.

The transform from $G$ to $G_3$ in Lemma 3 is called $\beta$-transform, simply recorded as $\beta$-Tra, and the transform from $G_3$ to $G$ in Lemma 3 is called the inversion of $\beta$-Tra.

**Lemma 4** Let $G$ be $E_3$-cordial, $\{x_1,\ x_2,\ ...,\ x_k\} \subset V(G)$(as shown in Figure 2(a)), then the new graphs $G_1$((as shown in Figure 2(b))) and $G_2$((as shown in Figure 2(c)))are $E_3$-cordial,where $V(G_1) = V(G)\bigcup\{u_1,\ u_2,\ u_3\}$, $E(G_1) = E(G)\bigcup\{u_ix_j\}$ for $i = 1,\ 2,\ 3;\ j = 1, ...,\ k$,and $V(G_2) = V(G_1)$,$E(G_2) = E(G_1)\bigcup\{u_1u_2,\ u_3u_2, u_1u_3\}$.

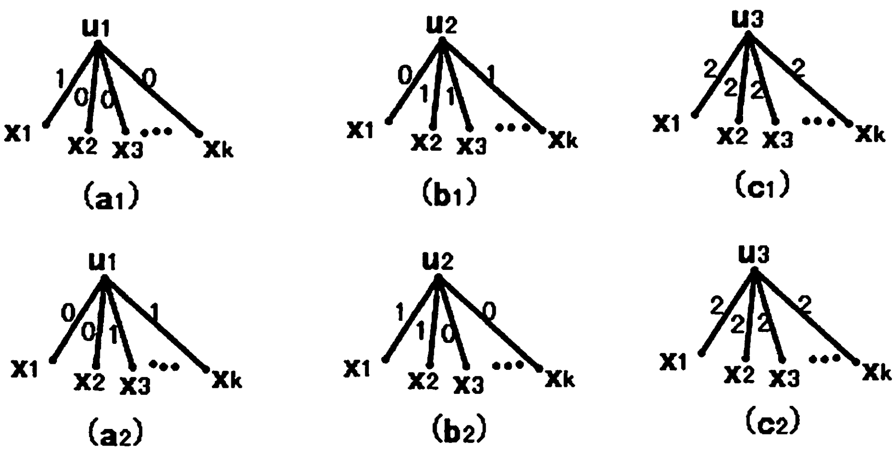



Figure 3: Illustrating Case 1 and Case 3 of Lemma 4

**Proof.** Firstly, we prove the $E_3$-cordiality of $G_1$, there are three cases to be discussed.

**Case 1.** $k \equiv 0\ (mod\, 3)$. Label the new edges $u_i x_j$ as Figure $3(a_1)(b_1)(c_1)$ shows, then $f(u_1) = 1$, $f(u_2) = 2$, $f(u_3) = 0$, while the labels of $x_j$ stay the same as they are in $G$, the number of edges $u_i x_j$ labeled $i$ is $\frac{k}{3}$, $i = 1,\ 2,\ 3$; $j = 1, .. , k$. This satisfies $|v(i) - v(j)| \leq 1$ , $|e(i) - e(j)| \leq 1$, $i,\ j = 0, 1, 2$, and of course, $G_1$ is $E_3$-cordial.

**Case 2.** $k \equiv 1\ (mod\ 3)$. Just exchange the labellings of $u_1 x_j$ and $u_2 x_j$,while keeping the other edge labellings staying the same as they are in Case 1. Thus, one can easily check the $E_3$-cordiality of $G_1$.

**Case 3,** $k \equiv 2\ (mod\ 3)$. The edge labellings of $u_i x_j$, are shown in Figure $3(a_2)(b_2)(c_2)$. Similarly, we get the $E_3$-cordiality of $G_1$.
Secondly, we consider $G_2$, let $f(u_1 u_2) = 0$, $f(u_3 u_2) = 2$, $f(u_1 u_3) = 1$, , while other edges are labeled the same as they are in $G_1$, then $\{f(u_1),\ f(u_2),\ f(u_3)\} = \{0,\ 1,\ 2\}$. According to the above discussion and Definition 1, $G_2$ is $E_3$-cordial.

**Lemma 5** Let $G$ be $E_3$-cordial, $\{x_1,\ x_2,\ ...,\ x_n\} \subset V(G)$, then $G_1$(as shown in Figure 4(a)) is $E_3$-cordial.

**Proof.** We distinguish the following two cases.

**Case 1.** Assume that $n$ is odd. Label $G_1 - G$ as shown in Figure 4(b), consider the labels of the adding edges from $G$ to $G_1$, that is ,the edges of $G_1 - G$. $e(0) = 2n - 1$, $e(1) = n + n - 1 = e(2) = 2n - 1 = e(0)$. It is easy

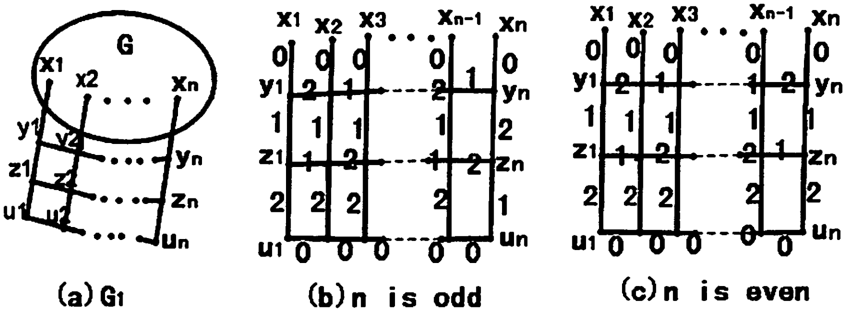



Figure 4: Illustrating Lemma 5

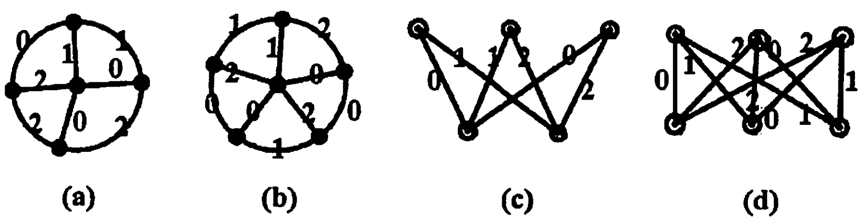



Figure 5: The $E_3-$ cordial labellings of $W_5$, $W_6$, $K_{2,3}$ and $K_{3,3}$

to check that $v(0) = v(1) = v(2) = 1$ on every vertical line of $G_1 - G$, so that $v(0) = v(1) = v(2)$ in $G_1 - G$.Hence $G_1$ is $E_3$-cordial.

Case 2. When $n$ is even. See the edge labellings of $G_1 - G$ in Figure 4(c),we get the $E_3$-cordiality of $G_1$.

# 3 Main Results

**Theorem 1** (1) $Wn(|Wn| = n)$ is $E_3$-cordial when $n \geq 4$.

(2) $K_{m,n}$ is $E_3$-cordial except $K_{1,3k+1}$ and $K_{3k+1,1}$ for $n, m, k \geq 1$.

**Proof.** (1) Since $W_4 = K_4$ is $E_3$-cordial[1][2], and we can simply get the $E_3$-cordial labellings of $W_5$(as shown in Figure 5(a)) and $W_6$(as shown in Figure 5(b)), by Lemma 1, $W_{3k+1}$, $W_{3k+2}$, $W_{3k}$ is $E_3$-cordial. Of course, $Wn$ is $E_3$-cordial when $n \geq 4$.

(2) If $m = 1$ or $n = 1$, $K_{1,n} = K_{n,1} = S_n$ is $E_3$-cordial[1][8] if and only if $n \not\equiv 1$ $(mod\ 3)$, i.e. $K_{1,3k+1}$ is not $E_3$-cordial, while $K_{1,3k} = K_{3k,1}$ and $K_{1,3k+2} = K_{3k+2,1}$ are $E_3$-cordial. Since $K_{2,1} = K_{1,2} = S_2$, $K_{2,2} =$

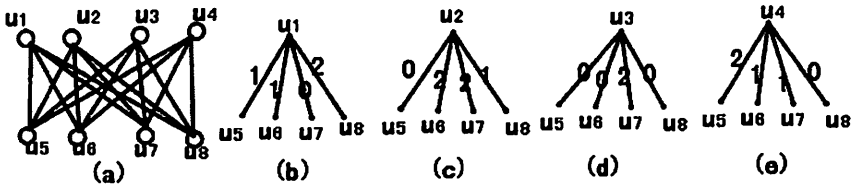



Figure 6: The $E_3-$ cordial labellings of $K_{4,4}$

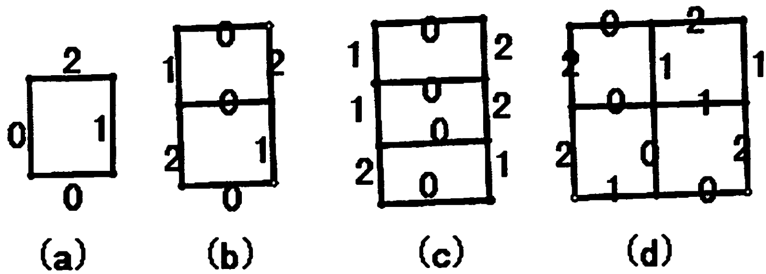
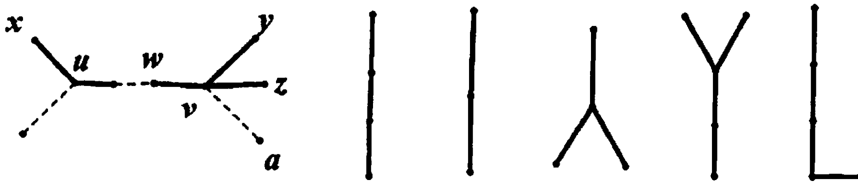



Figure 7: A step in the proof of Theorem 2

$C_4$ are $E_3$-cordial[1][3], $K_{2,3}$ labeled as shown in Figure 5(c) is $E_3$-cordial. Based on Lemma 4, $K_{2,3k+1}$, $K_{2,3k+2}$, $K_{2,3k}$ are $E_3$-cordial, so $K_{2,n}$ is $E_3$-cordial,then $K_{n,2} = K_{2,n}$ is $E_3$-cordial . Go on using Lemma 4 , $K_{n,3k+2}$ is $E_3$-cordial($n \geq 1$); $K_{3,3}$ labeled as shown in Figure 5(d) is $E_3$-cordial, and $K_{1,3}$ , $K_{2,3}$ are all $E_3$-cordial, so in the same way , by Lemma 4 $K_{n,3k}$ is $E_3$-cordial; $K_{4,1} = K_{1,4}$ is not $E_3$-cordial, $K_{4,2} = K_{2,4}$, $K_{4,3} = K_{3,4}$ are $E_3$-cordial, $K_{4,4}$ (see Figure 6(a)), label the edges of the graph (a) as the graphs (b), (c), (d),(e) of Figure 6 shows, it is easily to chéck that $K_{4,4}$ is $E_3$-cordial. By Lemma 4, $K_{4,3k+2}$, $K_{4,3k}$,$K_{4,3k+1}$ are $E_3$-cordial.,so $K_{n,4} = K_{4,n}$ is $E_3$-cordial.Still using Lemma 4, $K_{n,3k+1}$ is $E_3$-cordial. As stated above,$K_{n,3k+2}$,$K_{n,3k}$,$K_{n,3k+1}$ are all $E_3$-cordial. We draw the conclusion that $K_{m,n}$ is $E_3$-cordial except $K_{1,3k+1}$ and $K_{3k+1,1}$ for $n, m, k \geq 1$.

**Theorem 2** $P_m \times P_n$( $m \geq 2$, $n \geq 2$) is $E_3$-cordial.

**Proof.** Firstly, we prove that $P_2 \times P_n$ is $E_3$-cordial. Label the edges of $P_2 \times P_2$, $P_2 \times P_3$, $P_2 \times P_4$ as shown in Figure 7(a), (b), (c), we can easily check that they are $E_3$-cordial. By Lemma 5, $P_2 \times P_5$, $P_2 \times P_6$, $P_2 \times P_7$ are also $E_3$-cordial. Go on using Lemma 5, we get the $E_3$-cordiality of $P_2 \times P_{3k+2}$, $P_2 \times P_{3k}$, $P_2 \times P_{3k+1}$,that is,$P_2 \times P_n$ is $E_3$-cordial. Secondly, we give the proof to the $E_3$-cordiality of $P_3 \times P_n$ and $P_4 \times P_n$.Since $P_3 \times P_1 = P_3$, $P_3 \times P_2 = P_2 \times P_3$, $P_3 \times P_3$ can be labeled as shown in Figure 7(d), applying Lemma 5, $P_3 \times P_4$, $P_3 \times P_5$, $P_3 \times P_6$ are $E_3$-cordial. Go on




(a) (b) (c) (d) (e) (f)

Figure 8: Illustrating Theorem 3

using Lemma 5, $P_3 \times P_n$ is $E_3$-cordial. As for $P_4 \times P_n$, the proof can use the same method. Now applying Lemma 5 to $P_2 \times P_n$, $P_3 \times P_n$, $P_4 \times P_n$ respectively, we can get the $E_3$-cordiality of $P_{3k+2} \times P_n$, $P_{3k} \times P_n$, $P_{3k+1} \times P_n$ The desired result $P_m \times P_n(\ m \geq 2,\ n \geq 2)$ is $E_3$-cordial would be gotten.

**Theorem 3** Let $T$ be a tree, $|T| = n+1$ ,$d(T) \geq 3$,then $T$ is $E_3$-cordial.

**Proof.** Since $d(T) \geq 3$, $T$ is not $S_n$.

Find two endpoints $x$ and $y$ of $T$, between which the distance is the largest in $T$. Let $v$ be the neighbor vertex of $y$,as shown in Figure 8(a), consider the degree of $v$, obviously, $d(v) \geq 2$. If $d(v) = 2$, let $w$ be the other neighbor of $v$,applying the inversion of $\beta$–Tra on the edges $ux$, $wv$, $vy$, thus $T$ is reduced; If $d(v) = 3$, another neighbor of $v$ is $z$, using the inversion of $\alpha$–Tra on the edges $ux$, $yv$, $vz$, the graph $T$ can be reduced; If $d(v) = 4$, let the fourth neighbor of $v$ is $a$, as shown in Figure 8(a), using the inversion of $\alpha$–Tra on the edges $va$, $yv$, $vz$, $T$ is reduced. In a word ,we can reduce the graph $T$ by using these inverse transforms . If $d(v) \geq 4$, we can firstly reduce $T$ by using the inversion of $\alpha$–Tra $[\frac{d(v)}{3}]$-1 times, $d(v)$ would be 2 or 3.

Go on the above steps on the new reduced graphs, $T$ will be reduced again and again. In fact, if $|T| = n+1 \neq 3k+2$, that is, $n \neq 3k+1$, $T$ will be reduced to be $S_3$ or a path $P_2$ or $P_3$, as shown in Figure 8(b)(c)(d). If $|T| = n+1 = 3k+2$, there is a particular situation should be considered, that is ,the final graph is $S_4$, as we know $S_4$ is not $E_3$-cordial. In this situation, we should consider the above step, and change the way of the above inverse transform in case of the appearance of $S_4$. At last, $T$ will be reduced to $P_5$ or the graph whose edge labellings are shown in Figure 8(e). All in all, it is obviously to see that the reduced graphs shown in Figure 8(b)(c)(d)(e)(f) are all $E_3$-cordial. Now starting from these reduced graphs, constantly using $\alpha$-Tra or $\beta$-Tra , we can construct $T$. And at each stage , the constructed graph is $E_3$-cordial (by Lemma 2 or Lemma 3). So $T$ (tree) is E3-cordial.

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