On Monotonicity of Some Sequences Related to Hyperfibonacci Numbers and Hyperlucas Numbers[1]

Feng-Zhen Zhao
Department of Mathematics,
Shanghai University,
Shanghai 200444, China.
{E-mail: fengzhenzhao@shu.edu.cn }

Chun Wang
School of Mathematical Sciences,
Dalian University of Technology,
Dalian 116024, China.
{E-mail: wangchunmath@gmail.com}

Abstract

In this paper, we mainly discuss the monotonicity of some sequences related to the hyperfibonacci sequences $\{F_n^{[r]}\}_{n\geq 0}$ and the hyperlucas sequences $\{L_n^{[r]}\}_{n\geq 0}$, where r is a positive integer. We prove that $\{\sqrt[n]{F_n^{[1]}}\}_{n\geq 1}$ and $\{\sqrt[n]{F_n^{[2]}}\}_{n\geq 1}$ are unimodal and $\{\sqrt[n]{L_n^{[1]}}\}_{n\geq 1}$, $\{\sqrt[n]{F_{n+1}^{[1]}/F_n^{[1]}}\}_{n\geq 1}$ and $\{\sqrt[n]{L_{n+1}^{[1]}/L_n^{[1]}}\}_{n\geq 2}$ are decreasing. Furthermore, we discuss the monotonicity of the sequences of $\{\sqrt[n+1]{F_{n+1}^{[1]}}/\sqrt[n]{F_n^{[1]}}\}_{n\geq 1}$ and $\{\sqrt[n+1]{L_{n+1}^{[1]}}/\sqrt[n]{L_n^{[1]}}\}_{n\geq 1}$.

Key words. hyperfibonacci numbers, hyperlucas numbers, log-convexity, log-concavity, monotonicity.

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1 Introduction

Let r be a positive integer. The hyperfibonacci sequences $\{F_n^{[r]}\}_{n\geq 0}$ and the hyperlucas sequences $\{L_n^{[r]}\}_{n\geq 0}$ are defined by (see [5])

$$F_n^{[r]} = \sum_{k=0}^{n} F_k^{[r-1]}, \quad L_n^{[r]} = \sum_{k=0}^{n} L_k^{[r-1]}, \quad r \geq 2,$$

$$F_n^{[1]} = \sum_{k=0}^{n} F_k, \quad L_n^{[1]} = \sum_{k=0}^{n} L_k,$$

where $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ denote the Fibonacci sequence and the Lucas sequence, respectively. The Binet form of F_n and L_n is

$$F_n = \frac{\alpha^n - (-1)^n \alpha^{-n}}{\sqrt{5}}, \quad L_n = \alpha^n + (-1)^n \alpha^{-n},$$

where $\alpha = (1+\sqrt{5})/2$. It is well known that $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ satisfy the following recurrence relation

$$W_{n+1} = W_n + W_{n-1}, \quad n \geq 1. \tag{1.1}$$

Some values of $\{F_n^{[1]}\}_{n\geq 0}$, $\{F_n^{[2]}\}_{n\geq 0}$ and $\{L_n^{[1]}\}_{n\geq 0}$ are as follows:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$F_n^{[1]}$	0	1	2	4	7	12	20	33	54	88	143	232	376
$F_n^{[2]}$	0	1	3	7	14	26	46	79	133	221	364	596	973
$L_n^{[1]}$	2	3	6	10	17	28	46	75	122	198	321	520	842

The sequences $\{F_n^{[r]}\}_{n\geq 0}$ and $\{L_n^{[r]}\}_{n\geq 0}$ are sequences A000071 and A0001610 in Sloane's Encyclopedia [13]. In fact, $\{F_n^{[r]}\}_{n\geq 0}$ and $\{L_n^{[r]}\}_{n\geq 0}$ are the convolutions of $\{F_n^{[r-1]}\}_{n\geq 0}$ and $\{1\}_{n\geq 0}$, $\{L_n^{[r-1]}\}_{n\geq 0}$ and $\{1\}_{n\geq 0}$, respectively. For some properties of $\{F_n^{[r]}\}_{n\geq 0}$ and $\{L_n^{[r]}\}_{n\geq 0}$, see [3,5,11,13].

Sun [15] posed a series of conjectures on monotonicity of sequences of the types $\{\sqrt[n]{z_n}\}$ and $\{\sqrt[n+1]{z_{n+1}}/\sqrt[n]{z_n}\}$, where $\{z_n\}_{n\geq 0}$ is a combinatorial sequence of positive integers. Many conjectures of [15] have been confirmed. See for instance [4, 9, 15–17]. In this paper, we mainly discuss the monotonicity of some sequences related to $\{F_n^{[r]}\}_{n\geq 0}$ and $\{L_n^{[r]}\}_{n\geq 0}$. Now we recall some definitions involved in this paper.

Definition 1.1 *Let $\{z_n\}_{n\geq 0}$ be a sequence of positive real numbers. We say that $\{z_n\}_{n\geq 0}$ is log-concave (or log-convex) if $z_n^2 \geq z_{n-1}z_{n+1}$ (or $z_n^2 \leq z_{n-1}z_{n+1}$) for all $n \geq 1$.*

Definition 1.2 *Let $\{z_n\}_{n\geq 0}$ be a sequence of positive real numbers. If $z_0 \leq z_1 \leq \cdots \leq z_{m-1} \leq z_m \geq z_{m+1} \geq \cdots$ for some m, $\{z_n\}_{n\geq 0}$ is called unimodal, and m is called a mode of the sequence.*

Log-concavity (log-convexity) and unimodality of combinatorial sequences play an important role in many subjects such as quantum physics, white noise theory, probability, and mathematical biology and they are instrumental in obtaining the growth rate of a sequence. In particular, log-concavity and log-convexity of sequences are also fertile sources of inequalities. For some applications of log-concavity (log-convexity) of combinatorial sequences, see for instance [1,2,6–10,12,14]. In the next section, we prove that $\{\sqrt[n]{F_n^{[1]}}\}_{n\geq 1}$ and $\{\sqrt[n]{F_n^{[2]}}\}_{n\geq 1}$ are unimodal, $\{\sqrt[n]{L_n^{[1]}}\}_{n\geq 1}$ is decreasing, and $\{\sqrt[n]{F_{n+1}^{[1]}/F_n^{[1]}}\}_{n\geq 1}$ and $\{\sqrt[n]{L_{n+1}^{[1]}/L_n^{[1]}}\}_{n\geq 2}$ are decreasing. Furthermore, we discuss the monotonicity of the sequences $\{\sqrt[n+1]{F_{n+1}^{[1]}}/\sqrt[n]{F_n^{[1]}}\}_{n\geq 1}$ and $\{\sqrt[n+1]{L_{n+1}^{[1]}}/\sqrt[n]{L_n^{[1]}}\}_{n\geq 1}$.

2 The Monotonicity of Some Sequences Related to Hyperfibonacci Numbers and Hyperlucas Numbers

In this section, we state and prove the main results of this paper. We first prove a lemma.

Lemma 2.1 *Let $\{z_n\}_{n\geq 0}$ be a positive sequence of real numbers defined by the following recurrence relation*

$$z_n = c_1 z_{n-1} - c_2 z_{n-2} - \cdots - c_k z_{n-k}, \quad n \geq k, \tag{2.1}$$

where $k \geq 2$, $c_1 > 0$, $c_j \geq 0$ $(2 \leq j \leq k)$. If $\{z_0, z_1, z_2, \cdots, z_k, z_{k+1}\}$ is log-concave (log-convex), the sequence $\{z_n\}_{n\geq 0}$ is log-concave (log-convex).

Proof. Assume that $\{z_0, z_1, z_2, \cdots, z_k, z_{k+1}\}$ is log-concave. For $n \geq 0$, let $x_n = z_{n+1}/z_n$. It follows from (2.1) that

$$x_n = c_1 - \frac{c_2}{x_{n-1}} - \frac{c_3}{x_{n-1}x_{n-2}} - \cdots - \frac{c_k}{x_{n-1}x_{n-2}\cdots x_{n-k+1}}, \quad n \geq k. \tag{2.2}$$

We need prove that $\{x_n\}_{n\geq 0}$ is decreasing in order to verify the log-concavity of $\{z_n\}_{n\geq 0}$. We prove by induction that $\{x_n\}_{n\geq 0}$ is decreasing. In fact, since $\{z_0, z_1, z_2, \cdots, z_k, z_{k+1}\}$ is log-concave, $\{x_0, x_1, x_2, \cdots, x_k\}$ is decreasing. For $n \geq k$, assume that $x_n \leq x_{n-1} \leq \cdots \leq x_{n-k}$. It follows from (2.2) that

$$\begin{aligned} x_n - x_{n+1} &= c_2\left(\frac{1}{x_n} - \frac{1}{x_{n-1}}\right) + \frac{c_3}{x_{n-1}}\left(\frac{1}{x_n} - \frac{1}{x_{n-2}}\right) + \cdots \\ &\quad + \frac{c_k}{x_{n-1}x_{n-2}\cdots x_{n-k+2}}\left(\frac{1}{x_n} - \frac{1}{x_{n-k+1}}\right). \end{aligned}$$

By the assumption, we have $x_n - x_{n+1} \geq 0$. Hence $\{x_n\}_{n\geq 0}$ is decreasing and $\{z_n\}_{n\geq 0}$ is log-concave. ∎

Lemma 2.2 [16] *Let $\{z_n\}_{n\geq 0}$ be a sequence of positive numbers. Assume that $\{z_n\}_{n\geq N}$ is log-concave and $\sqrt[N]{z_N} > \sqrt[N+1]{z_{N+1}}$ for some $N \geq 1$. Then $\{\sqrt[n]{z_n}\}_{n\geq N}$ is strictly decreasing.*

Theorem 2.1 *For the hyperfibonacci sequences $\{F_n^{[1]}\}_{n\geq 0}$, $\{F_n^{[2]}\}_{n\geq 1}$, and the hyperlucas sequence $\{L_n^{[1]}\}_{n\geq 0}$, we have*

(i) $\{\sqrt[n]{F_n^{[1]}}\}_{n\geq 1}$ and $\{\sqrt[n]{F_n^{[2]}}\}_{n\geq 1}$ are unimodal;

(ii) $\{\sqrt[n]{L_n^{[1]}}\}_{n\geq 1}$ is decreasing.

Proof. (i) Using (1.1), we get

$$F_n^{[1]} = F_{n+2} - 1, \quad L_n^{[1]} = L_{n+2} - 1. \tag{2.3}$$

For $n \geq 1$, it is clear that

$$\begin{aligned} \sqrt[n]{F_n^{[1]}} \geq \sqrt[n+1]{F_{n+1}^{[1]}} \quad &\Leftrightarrow \quad (F_{n+2} - 1)^{n+1} \geq (F_{n+3} - 1)^n \\ &\Leftrightarrow \quad (n+1)\ln(F_{n+2} - 1) - n\ln(F_{n+3} - 1) \geq 0. \end{aligned}$$

For $n \geq 1$, let $f(n) = (n+1)\ln(F_{n+2} - 1) - n\ln(F_{n+3} - 1)$. Clearly, $f(1) = -\ln 2$. For $n \geq 1$, we have

$$\begin{aligned} f(2n) - f(2n+1) &= (2n+1)\ln(F_{2n+2} - 1) - 2n\ln(F_{2n+3} - 1) \\ &\quad -(2n+2)\ln(F_{2n+3} - 1) + (2n+1)\ln(F_{2n+4} - 1) \\ &= (2n+1)\ln\frac{(F_{2n+2} - 1)(F_{2n+4} - 1)}{(F_{2n+3} - 1)^2} \\ &= (2n+1)\ln\frac{L_{4n+6} - 5F_{2n+2} - 5F_{2n+4} + 2}{L_{4n+6} - 10F_{2n+3} + 7} \end{aligned}$$

and

$$\begin{aligned} f(2n+1) - f(2n+2) &= (2n+2)\ln\frac{(F_{2n+3} - 1)(F_{2n+5} - 1)}{(F_{2n+4} - 1)^2} \\ &= (2n+2)\ln\frac{L_{4n+8} - 5F_{2n+3} - 5F_{2n+5} + 8}{L_{4n+8} - 10F_{2n+4} + 3}. \end{aligned}$$

For $n \geq 0$, it follows from (1.1) that

$$\begin{aligned} -5F_{2n+2} - 5F_{2n+4} + 10F_{2n+3} - 5 &= -5F_{2n} - 5 \\ &< 0, \\ -5F_{2n+3} - 5F_{2n+5} + 10F_{2n+4} + 5 &= -5F_{2n+1} + 5 \\ &< 0, \quad (n \geq 1). \end{aligned}$$

Then

$$\begin{aligned} 0 &< \frac{L_{4n+6} - 5F_{2n+2} - 5F_{2n+4} + 2}{L_{4n+6} - 10F_{2n+3} + 7} < 1, \\ 0 &< \frac{L_{4n+8} - 5F_{2n+3} - 5F_{n+5} + 8}{L_{4n+8} - 10F_{2n+4} + 3} < 1, \quad (n \geq 1). \end{aligned}$$

For $n \geq 1$, we have $f(2n) - f(2n+1) < 0$ and $f(2n+1) - f(2n+2) < 0$. Hence $\{f(n)\}_{n\geq 2}$ is strictly increasing. We observe that

$$\begin{aligned} f(n) &= 2\ln\alpha - \ln\sqrt{5} + (n+1)\ln\left[1 - \frac{(-1)^n}{\alpha^{2n+4}} - \frac{\sqrt{5}}{\alpha^{n+2}}\right] \\ &\quad -n\ln\left[1 + \frac{(-1)^n}{\alpha^{2n+6}} - \frac{\sqrt{5}}{\alpha^{n+3}}\right] \end{aligned}$$

and

$$\lim_{n\to\infty} f(n) = 2\ln\alpha - \ln\sqrt{5} > 0.$$

On the other hand,

$$f(1) = -\ln 2, \quad f(2) = -\ln 2,$$

$$f(3) = 8\ln 2 - 3\ln 7 < 0,$$

$$f(4) = 5\ln 7 - 8\ln 2 - 4\ln 3 < 0.$$

Then there exists a positive integer N such that $f(n) < 0$ $(n < N)$ and $f(n) > 0$ $(n \geq N)$. This means that $\{\sqrt[n]{F_n^{[1]}}\}_{n\geq 1}$ is unimodal.

It follows from (2.3) that

$$F_n^{[2]} = F_{n+4} - n - 3.$$

For $n \geq 1$, it is evident that

$$\begin{aligned} \sqrt[n]{F_n^{[2]}} \geq \sqrt[n+1]{F_{n+1}^{[2]}} &\Leftrightarrow (F_{n+4} - n - 3)^{n+1} \geq (F_{n+5} - n - 4)^n \\ &\Leftrightarrow (n+1)\ln(F_{n+4} - n - 3) - n\ln(F_{n+5} - n - 4) \geq 0. \end{aligned}$$

For $n \geq 1$, set $g(n) = (n+1)\ln(F_{n+4} - n - 3) - n\ln(F_{n+5} - n - 4)$. Evidently, $g(1) = -\ln 3$. For $n \geq 1$, we have

$$\begin{aligned} &g(2n) - g(2n+1) \\ &= (2n+1)\ln\frac{(F_{2n+4} - 2n - 3)(F_{2n+6} - 2n - 5)}{(F_{2n+5} - 2n - 4)^2} \\ &= (2n+1)\ln\frac{L_{4n+10} - 3 + X_1}{L_{4n+10} + 2 + Y_1} \end{aligned}$$

and

$$
\begin{aligned}
& g(2n+1) - g(2n+2) \\
&= (2n+2)\ln\frac{(F_{2n+5}-2n-4)(F_{2n+7}-2n-6)}{(F_{2n+6}-2n-5)^2} \\
&= (2n+2)\ln\frac{L_{4n+12}+3+X_2}{L_{4n+12}-2+Y_2},
\end{aligned}
$$

where

$$
\begin{aligned}
X_1 &= -5(2n+5)F_{2n+4} - 5(2n+3)F_{2n+6} + 5(2n+3)(2n+5), \\
Y_1 &= -20(n+2)F_{2n+5} + 20(n+2)^2, \\
X_2 &= -5(2n+6)F_{2n+5} - 5(2n+4)F_{2n+7} + 20(n+2)(n+3), \\
Y_2 &= -10(2n+5)F_{2n+6} + 5(2n+5)^2.
\end{aligned}
$$

Applying (1.1), we obtain

$$
\begin{aligned}
X_1 - Y_1 &= -10nF_{2n+2} - 5F_{2n+2} + 10F_{2n+1} - 5 \\
&\leq -10F_{2n+2} - 5F_{2n+2} + 10F_{2n+1} \\
&= -10F_{2n} - 5F_{2n+2} \\
&< 0
\end{aligned}
$$

and

$$
\begin{aligned}
X_2 - Y_2 &= -10nF_{2n+3} + 10F_{2n+2} - 5 \\
&< -5.
\end{aligned}
$$

Then

$$
\begin{aligned}
0 < \ln\frac{L_{4n+10}-3+X_1}{L_{4n+10}+2+Y_1} < 1, \\
0 < \ln\frac{L_{4n+12}+3+X_2}{L_{4n+12}-2+Y_2} < 1,
\end{aligned}
$$

and $g(2n)-g(2n+1)<0$ and $g(2n+1)-g(2n+2)<0$ for $n\geq 1$. Hence $\{g(n)\}_{n\geq 1}$ is strictly increasing. We note that

$$\begin{aligned} g(n) &= 4\ln\alpha-\ln\sqrt{5}+(n+1)\ln\left[1-\frac{(-1)^n}{\alpha^{2n+8}}+\frac{(n+3)\sqrt{5}}{\alpha^{n+4}}\right] \\ &\quad -n\ln\left[1+\frac{(-1)^n}{\alpha^{2n+10}}+\frac{(n+4)\sqrt{5}}{\alpha^{n+5}}\right], \\ \lim_{n\to\infty} g(n) &= 4\ln\alpha-\ln\sqrt{5}>0, \end{aligned}$$

and

$$g(1)=-\ln 3<g(2)=3\ln 3-2\ln 7<g(3)=\ln 7-3\ln 2<0.$$

Then there exists a positive integer M such that $g(n)<0$ $(n<M)$ and $g(n)>0$ $(n\geq M)$. This implies that $\{\sqrt[n]{F_n^{[2]}}\}_{n\geq 1}$ is unimodal.

(ii) By (1.1) and (2.3), we can verify that $\{L_n^{[1]}\}_{n\geq 0}$ satisfies

$$W_{n+1}=2W_n-W_{n-2}. \tag{2.4}$$

On the other hand, $\{L_3^{[1]},L_4^{[1]},L_5^{[1]},L_6^{[1]}\}$ is log-concave. It follows from Lemma 2.1 that $\{L_n^{[1]}\}_{n\geq 3}$ is log-concave. We note that $\sqrt[3]{L_3^{[1]}}>\sqrt[4]{L_4^{[1]}}$. By applying Lemma 2.2, we prove that $\{\sqrt[n]{L_n^{[1]}}\}_{n\geq 3}$ is decreasing. On the other hand, we can verify that $L_1^{[1]}>\sqrt{L_2^{[1]}}>\sqrt[3]{L_3^{[1]}}$. Hence $\{\sqrt[n]{L_n^{[1]}}\}_{n\geq 1}$ is decreasing. ■

Theorem 2.2 *For the hyperfibonacci sequence $\{F_n^{[1]}\}_{n\geq 0}$ and the hyperlucas sequence $\{L_n^{[1]}\}_{n\geq 0}$, the sequences $\{\sqrt[n]{F_{n+1}^{[1]}/F_n^{[1]}}\}_{n\geq 1}$ and $\{\sqrt[n]{L_{n+1}^{[1]}/L_n^{[1]}}\}_{n\geq 2}$ are decreasing.*

Proof. For $n\geq 1$, put $x_n=F_{n+1}^{[1]}/F_n^{[1]}$. It is obvious that $\{F_1^{[1]},F_2^{[1]},F_3^{[1]},F_4^{[1]},F_5^{[1]}\}$ is log-concave. On the other hand, the sequence $\{F_n^{[1]}\}_{n\geq 1}$ satisfies the recurrence (2.4). It follows from Lemma 2.1 that $\{F_n^{[1]}\}_{n\geq 1}$ is log-concave. Then $\{x_n\}_{n\geq 1}$ is decreasing.

For $n \geq 1$, it is obvious that

$$\sqrt[n]{x_n} \geq \sqrt[n+1]{x_{n+1}} \quad \Leftrightarrow \quad x_n^{n+1} \geq x_{n+1}^n$$
$$\Leftrightarrow \quad (n+1)\ln x_n - n\ln x_{n+1} \geq 0.$$

For $n \geq 1$,

$$\begin{aligned}(n+1)\ln x_n - n\ln x_{n+1} &= \ln x_n + n\ln\frac{x_n}{x_{n+1}} \\ &= \ln x_n + n\ln\left(1+\frac{x_n - x_{n+1}}{x_{n+1}}\right).\end{aligned}$$

Since $\ln(1+x) \geq x/(1+x)$ for $x \geq 0$ and $x_n > 1$, we have

$$\begin{aligned}(n+1)\ln x_n - n\ln x_{n+1} &> \frac{x_n - x_{n+1}}{x_n} \\ &\geq 0.\end{aligned}$$

Then the sequence $\{\sqrt[n]{F_{n+1}^{[1]}/F_n^{[1]}}\}_{n\geq 1}$ is decreasing. Using a similar method it can be shown that the sequence $\{\sqrt[n]{L_{n+1}^{[1]}/L_n^{[1]}}\}_{n\geq 2}$ is also decreasing. ■

From the proofs of Theorems 2.1–2.2, we also show that $\{F_n^{[1]}\}_{n\geq 1}$ and $\{L_n^{[1]}\}_{n\geq 3}$ are log-concave by Lemma 2.1. In fact, their log-concavity can be proved by the monotonicity of their quotient sequences (see [18]).

Theorem 2.3 *For the hyperfibonacci sequence $\{F_n^{[1]}\}_{n\geq 0}$ and the hyperlucas sequence $\{L_n^{[1]}\}_{n\geq 0}$, the sequences $\{\sqrt[n]{nF_n^{[1]}}\}_{n\geq 5}$ and $\{\sqrt[n]{nL_n^{[1]}}\}_{n\geq 3}$ are decreasing.*

Proof. It is well known that

$$\begin{aligned}\sqrt[n]{nF_n^{[1]}} > \sqrt[n+1]{(n+1)F_{n+1}^{[1]}} \quad &\Leftrightarrow \quad (nF_n^{[1]})^{n+1} > \left[(n+1)F_{n+1}^{[1]}\right]^n \\ &\Leftrightarrow \quad (n+1)\ln n - n\ln(n+1) + (n+1)\ln F_n^{[1]} \\ &\qquad -n\ln F_{n+1}^{[1]} \\ &> \quad 0.\end{aligned}$$

For $n \geq 1$, let

$$\begin{aligned} h(n) &= (n+1)\ln(nF_n^{[1]}) - n\ln\left[(n+1)F_{n+1}^{[1]}\right], \\ \widetilde{h(n)} &= (n+1)\ln n - n\ln(n+1). \end{aligned}$$

Clearly,

$$h(n) = \widetilde{h(n)} + (n+1)\ln F_n^{[1]} - n\ln F_{n+1}^{[1]}.$$

For $n \geq 1$, we obtain

$$\begin{aligned} \widetilde{h(2n)} - \widetilde{h(2n+1)} &= (2n+1)\ln(4n^2+4n) - 2(2n+1)\ln(2n+1) \\ &< 0, \\ \widetilde{h(2n+1)} - \widetilde{h(2n+2)} &= (2n+2)\ln(4n^2+8n+3) - 2(2n+2)\ln(2n+2) \\ &< 0. \end{aligned}$$

Then $\{\widetilde{h(n)}\}_{n\geq 1}$ is increasing. Since $\{(n+1)\ln F_n^{[1]} - n\ln F_{n+1}^{[1]}\}$ is also increasing, $\{h(n)\}_{n\geq 1}$ is increasing. We can verify that $h(5) > 0$. Then $h(n) > 0$ $(n \geq 5)$. Hence $\{\sqrt[n]{nF_n^{[1]}}\}_{n\geq 5}$ is decreasing.

As $\{\sqrt[n]{L_n^{[1]}}\}_{n\geq 1}$ and $\{\sqrt[n]{n}\}_{n\geq 3}$ are decreasing, $\{\sqrt[n]{nL_n^{[1]}}\}_{n\geq 3}$ is decreasing. ■

Theorem 2.4 *For the hyperfibonacci sequence $\{F_n^{[1]}\}_{n\geq 0}$ and the hyperlucas sequence $\{L_n^{[1]}\}_{n\geq 0}$, $\{\sqrt[n+1]{F_{n+1}^{[1]}}/\sqrt[n]{F_n^{[1]}}\}_{n\geq 6}$ and $\{\sqrt[n+1]{L_{n+1}^{[1]}}/\sqrt[n]{L_n^{[1]}}\}_{n\geq 6}$ are increasing.*

Proof. For $n \geq 2$,

$$\begin{aligned} &\sqrt[n+1]{F_{n+1}^{[1]}}/\sqrt[n]{F_n^{[1]}} \geq \sqrt[n]{F_n^{[1]}}/\sqrt[n-1]{F_{n-1}^{[1]}} \\ \Leftrightarrow\ & \frac{\ln F_{n+1}^{[1]}}{n+1} + \frac{\ln F_{n-1}^{[1]}}{n-1} - \frac{2\ln F_n^{[1]}}{n} > 0 \\ \Leftrightarrow\ & (n^2-n)\ln F_{n+1}^{[1]} + (n^2+n)\ln F_{n-1}^{[1]} - 2(n^2-1)\ln F_n^{[1]} > 0, \\ &\sqrt[n+1]{L_{n+1}^{[1]}}/\sqrt[n]{L_n^{[1]}} \geq \sqrt[n]{L_n^{[1]}}/\sqrt[n-1]{L_{n-1}^{[1]}} \\ \Leftrightarrow\ & (n^2-n)\ln L_{n+1}^{[1]} + (n^2+n)\ln L_{n-1}^{[1]} - 2(n^2-1)\ln L_n^{[1]} > 0. \end{aligned}$$

For $n \geq 2$, let

$$\begin{aligned}\widetilde{f(n)} &= (n^2-n)\ln F_{n+1}^{[1]} + (n^2+n)\ln F_{n-1}^{[1]} - 2(n^2-1)\ln F_n^{[1]},\\ \widetilde{g(n)} &= (n^2-n)\ln L_{n+1}^{[1]} + (n^2+n)\ln L_{n-1}^{[1]} - 2(n^2-1)\ln L_n^{[1]}.\end{aligned}$$

We observe that

$$\widetilde{f(n)} - \widetilde{f(n+1)} = n(n+1)\ln \frac{F_{n-1}^{[1]}(F_{n+1}^{[1]})^3}{F_{n+2}^{[1]}(F_n^{[1]})^3},$$

$$\widetilde{g(n)} - \widetilde{g(n+1)} = n(n+1)\ln \frac{L_{n-1}^{[1]}(L_{n+1}^{[1]})^3}{L_{n+2}^{[1]}(L_n^{[1]})^3}.$$

For $n \geq 2$, let

$$\begin{aligned}S(n) &= F_{n-1}^{[1]}(F_{n+1}^{[1]})^3 - F_{n+2}^{[1]}(F_n^{[1]})^3,\\ T(n) &= L_{n-1}^{[1]}(L_{n+1}^{[1]})^3 - L_{n+2}^{[1]}(L_n^{[1]})^3.\end{aligned}$$

We can prove that

$$(F_n^{[1]})^2 - F_{n-1}^{[1]}F_{n+1}^{[1]} = (-1)^{n+1} + F_{n-1}, \tag{2.5}$$

$$(L_n^{[1]})^2 - L_{n-1}^{[1]}L_{n+1}^{[1]} = 5(-1)^n + L_{n-1}. \tag{2.6}$$

By using (2.5) and (1.1), we have

$$\begin{aligned}S(2n) &= F_{2n-1}^{[1]}(F_{2n}+1)F_{2n+1}^{[1]} - (F_{2n-1}-1)F_{2n+2}^{[1]}F_{2n}^{[1]}\\ &= (F_{2n+1}-1)(F_{2n+3}-1) + F_{2n}(F_{2n+1}-1)(F_{2n+3}-1)\\ &\quad +(F_{2n+2}-1)(F_{2n+4}-1) - F_{2n-1}(F_{2n+2}-1)(F_{2n+4}-1)\\ &= (F_{2n+1}-1)(F_{2n+3}-1) + F_{2n}(F_{2n+1}-1)(F_{2n+3}-1)\\ &\quad +(F_{2n+2}-1)(F_{2n+4}-1) - F_{2n-1}(F_{2n+2}-1)(F_{2n+3}-1)\\ &\quad -F_{2n-1}(F_{2n+2}-1)F_{2n+2}\\ &= (F_{2n+1}-1)(F_{2n+3}-1) + (F_{2n+2}-1)(F_{2n+4}-1)\\ &\quad -F_{2n-1}(F_{2n+2}-1)F_{2n+2} - (F_{2n-2}+1)(F_{2n+3}-1)\\ &= F_{2n+1}F_{2n+3} + F_{2n+2}F_{2n+4} + F_{2n-1}F_{2n+2} - F_{2n+3} - F_{2n+5}\\ &\quad +2 - F_{2n-2}F_{2n+3} - (F_{2n+3} - 1 - F_{2n-2}) - F_{2n-1}F_{2n+2}^2.\end{aligned}$$

Due to

$$\begin{aligned}
&F_{2n+1}F_{2n+3}+F_{2n+2}F_{2n+4}+F_{2n-1}F_{2n+2}-F_{2n-2}F_{2n+3}\\
&=\frac{L_{4n+4}+L_{4n+6}+15}{5},\\
&F_{2n+2}^2\\
&=\frac{L_{4n+4}-2}{5},
\end{aligned}$$

and (1.1), we get

$$\begin{aligned}
S(2n) &= \frac{L_{4n+4}+L_{4n+6}-F_{2n-1}L_{4n+4}}{5}+\frac{2F_{2n-1}}{5}-2F_{2n-1}\\
&\quad -2F_{2n+3}-F_{2n+2}-F_{2n+4}+6\\
&< \frac{L_{4n+4}+L_{4n+6}-F_{2n-1}L_{4n+4}}{5}.
\end{aligned}$$

For $n \geq 3$, $F_{2n-1} \geq 5$. Then we have

$$\begin{aligned}
S(2n) &< \frac{L_{4n+6}-4L_{4n+4}}{5}\\
&= \frac{-L_{4n+2}-L_{4n+4}}{5}\\
&< 0.
\end{aligned}$$

For $n \geq 1$, using the similar method, we have

$$\begin{aligned}
S(2n+1) &= F_{2n}^{[1]}(F_{2n+1}-1)F_{2n+2}^{[1]}-F_{2n+1}^{[1]}(F_{2n}+1)F_{2n+3}^{[1]}\\
&= [(F_{2n+2}-1)(F_{2n+1}-1)\\
&\quad -(F_{2n+3}-1)(F_{2n}+1)](F_{2n+4}-1)\\
&\quad -(F_{2n+3}-1)(F_{2n}+1)F_{2n+3}\\
&= (3-F_{2n+2}-F_{2n+1}-F_{2n+3}+F_{2n})(F_{2n+4}-1)\\
&\quad -(F_{2n+3}-1)(F_{2n}+1)F_{2n+3}\\
&= (3-2F_{2n+1}-F_{2n+3})(F_{2n+4}-1)\\
&\quad -(F_{2n+3}-1)(F_{2n}+1)F_{2n+3}\\
&< 0.
\end{aligned}$$

Then, for $n \geq 6$, $S(n) < 0$ and $\{\widetilde{f(n)}\}_{n\geq 6}$ is decreasing. We note that

$$\lim_{n\to\infty} \widetilde{f(n)} = 4\ln\alpha - \ln 5 > 0.$$

Then $\widetilde{f(n)} > 0$ $(n \geq 6)$ and $\{\sqrt[n+1]{F_{n+1}^{[1]}} / \sqrt[n]{F_n^{[1]}}\}_{n\geq 6}$ is increasing.

By (2.6) and (1.1), we derive

$$\begin{aligned}
T(2n) &= L_{2n-1}^{[1]}(L_{2n}-5)L_{2n+1}^{[1]} - L_{2n}^{[1]}(L_{2n-1}+5)L_{2n+2}^{[1]} \\
&= -5(L_{2n-1}^{[1]}L_{2n+1}^{[1]} + L_{2n}^{[1]}L_{2n+2}^{[1]}) + (5-L_{2n-2})(L_{2n+3}-1) \\
&\quad -L_{2n-1}L_{2n+2}(L_{2n+2}-1) \\
&< 0
\end{aligned}$$

and

$$\begin{aligned}
T(2n+1) &= 5(L_{2n+2}-1)(L_{2n+4}-1) + 5(L_{2n+3}-1)(L_{2n+5}-1) \\
&\quad -(5+L_{2n-1})(L_{2n+4}-1) - L_{2n}(L_{2n+3}-1)L_{2n+3} \\
&= 5(L_{2n+2}L_{2n+4} + L_{2n+3}L_{2n+5}) + L_{2n}L_{2n+3} - L_{2n-1}L_{2n+4} \\
&\quad -5(L_{2n+4}-1) + L_{2n-1} - 5(L_{2n+2}+L_{2n+4}-1) \\
&\quad -5(L_{2n+3}+L_{2n+5}-1) - L_{2n}L_{2n+3}^2 \\
&< 5(L_{2n+2}L_{2n+4} + L_{2n+3}L_{2n+5}) + L_{2n}L_{2n+3} - L_{2n-1}L_{2n+4} \\
&\quad -L_{2n}(L_{4n+6}-2) - 5L_{2n+2} \\
&= 5(L_{4n+6}+L_{4n+8}) + 15 - L_{2n}(L_{4n+6}-2) - 5L_{2n+2} \\
&< 5(L_{4n+6}+L_{4n+8}) - L_{2n}L_{4n+6}.
\end{aligned}$$

We note that $T(7) < 0$. For $n \geq 4$,

$$\begin{aligned}
T(2n+1) &< 5(L_{4n+6}+L_{4n+8}) - 29L_{4n+6} \\
&< 0.
\end{aligned}$$

Hence $\widetilde{g(n)} > 0$ $(n \geq 6)$ and $\{\sqrt[n+1]{L_{n+1}^{[1]}} / \sqrt[n]{L_n^{[1]}}\}_{n\geq 6}$ is increasing. ∎

We have discussed the monotonicity of some sequences related to hyperfibonacci numbers and hyperlucas numbers. Now we consider the generalized hyperfibonacci sequences $\{U_n^{[r]}\}_{n\geq 0}$ and the generalized hyperlucas sequences $\{V_n^{[r]}\}_{n\geq 0}$, where r is a positive integer. Let $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ be the generalized Fibonacci and Lucas sequence, respectively. The Binet forms of U_n and V_n are

$$U_n = \frac{\tau^n - (-1)^n\tau^{-n}}{\sqrt{\Delta}}, \quad V_n = \tau^n + (-1)^n\tau^{-n},$$

where $\Delta = p^2 + 4, \tau = (\sqrt{\Delta} + p)/2$, and $p \geq 1$ is an integer. It is well known that $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ satisfy

$$W_{n+1} = pW_n + W_{n-1}, \quad n \geq 1.$$

$\{U_n^{[r]}\}_{n\geq 0}$ and $\{V_n^{[r]}\}_{n\geq 0}$ are defined as follows:

$$U_n^{[r]} = \sum_{k=0}^{n} U_k^{[r-1]}, \quad V_n^{[r]} = \sum_{k=0}^{n} V_k^{[r-1]},$$

where $U_n^{[0]} = U_n$ and $V_n^{[0]} = V_n$. Some properties of $\{U_n^{[r]}\}_{n\geq 0}$ and $\{V_n^{[r]}\}_{n\geq 0}$ are discussed in [3, 11, 18]. We discuss the monotonicity of some sequences involving generalized hyperfibonacci numbers and generalized hyperlucas numbers. For example, we can prove that

Theorem 2.5 *For $p \geq 2$, the sequence $\{\sqrt[n]{U_n^{[1]}}\}_{n\geq 1}$ is increasing. For $p \geq 1$, the sequence $\{\sqrt[n]{V_n^{[1]}}\}_{n\geq 1}$ is decreasing.*

Proof. (i) Let $A(n) = (n+1)\ln U_n^{[1]} - n\ln U_{n+1}^{[1]}$. In order to prove that $\{\sqrt[n]{U_n^{[1]}}\}_{n\geq 1}$ is increasing, we need verify that $A(n) < 0$ for $n \geq 1$. Clearly,

$$\begin{aligned} A(2n) - A(2n+1) &= (2n+1)\ln\frac{U_{2n}^{[1]}U_{2n+2}^{[1]}}{(U_{2n+1}^{[1]})^2}, \\ A(2n+1) - A(2n+2) &= (2n+2)\ln\frac{U_{2n+1}^{[1]}U_{2n+3}^{[1]}}{(U_{2n+2}^{[1]})^2}. \end{aligned}$$

By using (see [18])

$$U_n^{[1]} = \frac{U_n + U_{n+1} - 1}{p},$$

we have

$$\begin{aligned}
&A(2n) - A(2n+1) \\
&= (2n+1)\ln\frac{(U_{2n}+U_{2n+1}-1)(U_{2n+2}+U_{2n+3}-1)}{(U_{2n+1}+U_{2n+2}-1)^2}, \\
&A(2n+1) - A(2n+2) \\
&= (2n+2)\ln\frac{(U_{2n+1}+U_{2n+2}-1)(U_{2n+3}+U_{2n+4}-1)}{(U_{2n+2}+U_{2n+3}-1)^2}.
\end{aligned}$$

It follows from

$$U_{2n}U_{2n+2} = \frac{V_{4n+2}-V_2}{\Delta}, \quad U_{2n}U_{2n+3} = \frac{V_{4n+3}-V_3}{\Delta},$$
$$U_{2n+1}U_{2n+2} = \frac{V_{4n+3}+V_1}{\Delta}, \quad U_{2n+1}U_{2n+3} = \frac{V_{4n+4}+V_2}{\Delta},$$
$$U_{2n+1}U_{2n+4} = \frac{V_{4n+5}-V_3}{\Delta}, \quad U_{2n+2}U_{2n+3} = \frac{V_{4n+5}+V_1}{\Delta},$$
$$U_{2n+1}^2 = \frac{V_{4n+2}+2}{\Delta} \quad \text{and} \quad U_{2n+2}^2 = \frac{V_{4n+4}-2}{\Delta}$$

that

$$\begin{aligned}
A(2n) - A(2n+1) &= (2n+1)\ln\frac{V_{4n+2}+2V_{4n+3}+V_{4n+4}+X_3}{V_{4n+2}+2V_{4n+3}+V_{4n+4}+Y_3} \\
&< 0, \\
A(2n+1) - A(2n+2) &= (2n+2)\ln\frac{V_{4n+4}+2V_{4n+5}+V_{4n+6}+Y_4}{V_{4n+4}+2V_{4n+5}+V_{4n+6}+Y_4} \\
&< 0,
\end{aligned}$$

where

$$\begin{aligned}
X_3 &= -pV_2 - \Delta(U_{2n}+U_{2n+1}+U_{2n+2}+U_{2n+3}) + \Delta, \\
Y_3 &= 2p - 2\Delta(U_{2n+1}+U_{2n+2}) + \Delta, \\
X_4 &= pV_2 - \Delta(U_{2n+1}+U_{2n+2}+U_{2n+3}+U_{2n+4}) + \Delta, \\
Y_4 &= -2p - 2\Delta(U_{2n+2}+U_{2n+3}) + \Delta.
\end{aligned}$$

Then the sequence $\{A(n)\}_{n\geq 1}$ is increasing. We note that

$$\begin{aligned}
&\lim_{n\to\infty} A(n) \\
&= \lim_{n\to\infty} \Bigg\{ -\ln\sqrt{\Delta} - \ln p + (1+n)\ln\left[1+\tau - \frac{(-1)^n}{\tau^{2n}} + \frac{(-1)^n}{\tau^{2n+1}}\right] \\
&\quad -n\ln\left[1+\tau+\frac{(-1)^n}{\tau^{2n+2}} - \frac{(-1)^n}{\tau^{2n+3}}\right]\Bigg\} \\
&= -\ln\sqrt{\Delta} - \ln p + \ln(1+\tau) < 0.
\end{aligned}$$

Thus $A(n) < 0$ for $n \geq 1$ and the sequence $\{\sqrt[n]{U_n^{[1]}}\}_{n\geq 1}$ is increasing.

(ii) Some initial values of $\{V_n^{[1]}\}_{n\geq 3}$ are

$$\begin{aligned}
&V_1^{[1]} = p+2, \quad V_2^{[1]} = p^2+p+4, \\
&V_3^{[1]} = p^3+p^2+4p+4, \quad V_4^{[1]} = p^4+p^3+5p^2+4p+6.
\end{aligned}$$

We can prove that $\sqrt[3]{V_3^{[1]}} > \sqrt[4]{V_4^{[1]}}$. Since the log-concavity of $\{V_n^{[1]}\}_{n\geq 3}$ is proved in [18], we show that $\{\sqrt[n]{V_n^{[1]}}\}_{n\geq 3}$ is decreasing by using Lemma 2.2. On the other hand, $V_1^{[1]} > \sqrt{V_2^{[1]}} > \sqrt[3]{V_3^{[1]}}$. Hence the sequence $\{\sqrt[n]{V_n^{[1]}}\}_{n\geq 1}$ is decreasing. ■

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