On Proper-Path Colorings in Graphs

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Abstract

Let G be an edge-colored connected graph. A path P is a proper path in G if no two adjacent edges of P are colored the same. If P is a proper $u-v$ path of length $d(u,v)$, then P is a proper $u-v$ geodesic. An edge coloring c is a proper-path coloring of a connected graph G if every pair u, v of distinct vertices of G are connected by a proper $u-v$ path in G and c is a strong proper coloring if every two vertices u and v are connected by a proper $u-v$ geodesic in G. The minimum number of colors used a proper-path coloring and strong proper coloring of G are called the proper connection number $\text{pc}(G)$ and strong proper connection number $\text{spc}(G)$ of G, respectively. These concepts are inspired by the concepts of rainbow coloring, rainbow connection number $\text{rc}(G)$, strong rainbow coloring and strong connection number $\text{src}(G)$ of a connected graph G. The numbers $\text{pc}(G)$ and $\text{spc}(G)$ are determined for several well-known classes of graphs G. We investigate the relationship among these four edge colorings as well as the well-studied proper edge colorings in graphs. Furthermore, several realization theorems are established for the five edge coloring parameters, namely $\text{pc}(G), \text{spc}(G), \text{rc}(G), \text{src}(G)$ and the chromatic index of a connected graph G.

Key Words: edge coloring, proper-path coloring, strong proper coloring.
AMS Subject Classification: 05C15, 05C38.

1 Introduction

The Department of Homeland Security was created in 2003 in response to weaknesses discovered in the transfer of classified information after the September 11, 2001 terrorist attacks. In [8] Ericksen made the following observation:

> *An unanticipated aftermath of those deadly attacks was the realization that law enforcement and intelligence agencies couldn't communicate with each other through their regular channels from radio systems to databases. The technologies utilized were separate entities and prohibited shared access, meaning there was no way for officers and agents to cross check information between various organizations.*

While the information needs to be protected since it relates to national security, there must also be procedures that permit access between appropriate parties. This two-fold issue can be addressed by assigning information transfer paths between agencies which may have other agencies as intermediaries while requiring a large enough number of passwords and firewalls that is prohibitive to intruders, yet small enough to manage (that is, enough so that one or more paths between every pair of agencies have no password repeated). An immediate question arises: What is the minimum number of passwords or firewalls needed that allows one or more secure paths between every two agencies so that the passwords along each path are distinct? This situation can be represented (modeled) by a graph and studied by means of what is called rainbow colorings introduced by Chartrand, Johns, McKeon and Zhang in [2]. A *rainbow coloring* of a connected graph G is an edge coloring of G with the property that for every two vertices u and v of G, there exists a $u - v$ *rainbow path* (no two edges of the path are colored the same). In this case, G is *rainbow-connected* (*with respect to* c) . The minimum number of colors in a rainbow coloring of G is referred to as the *rainbow connection number* of G and denoted by $\mathrm{rc}(G)$. In recently years, this topic has been studied by many (see [3, 5, 7, 10] for example).

An *edge coloring* of a graph G is an assignment c of colors to the edges of G, one color to each edge of G. Once the edges of G are assigned colors, an *edge-colored graph* results. Let G be a nontrivial connected graph of order n with an edge coloring $c: E(G) \to [k] = \{1, 2, \dots, k\}$. If adjacent edges of G are assigned different colors by c, then c is a *proper* (*edge*) *coloring*. The minimum number of colors needed in a proper coloring of G is referred to as the *chromatic index* of G and denoted by $\chi'(G)$. One property that a properly edge-colored graph G has is that for every two vertices u and v of G, every $u - v$ path of G is properly colored. However, if our main interest is that of having only at least one properly colored $u-v$ path in G for every two vertices u and v of G, then it is possible that this can be accomplished using fewer than $\chi'(G)$ colors. With regard to the national security discussion above, we are then interested in the answer to the following question: What is the minimum number of passwords or firewalls that allow one or more secure paths between every two agencies where as we progress from one agency to another along such a path, we are required to change passwords?

Inspired by rainbow colorings and proper colorings in graphs, we introduce the concept of proper-path colorings. Let G be an edge-colored connected graph, where adjacent edges may be colored the same. A path P in G is *properly colored* or, more simply, P is a *proper path* in G if no two adjacent edges of P are colored the same. An edge coloring c is a *proper-path coloring* of a connected graph G if every pair u, v of distinct vertices of G are connected by a proper $u-v$ path in G. If k colors are used, then c is referred to as a *proper-path k-coloring*. The minimum number of colors needed to produce a proper-path coloring of G is called the *proper connection number* $\mathrm{pc}(G)$ of G. A proper-path coloring using $\mathrm{pc}(G)$ colors is referred to as a *minimum proper-path coloring*.

Let G be a nontrivial connected graph of order n and size m. Since every rainbow coloring is a proper-path coloring, it follows that $\mathrm{pc}(G)$ exists and

$$1 \le \mathrm{pc}(G) \le \min\{\chi'(G), \mathrm{rc}(G)\} \le m. \qquad (1)$$

Furthermore, $\mathrm{pc}(G) = 1$ if and only if $G = K_n$ and $\mathrm{pc}(G) = m$ if and only if $G = K_{1,m}$ is a star of size m. To illustrate this concept, we consider the 3-regular graph G of Figure 1 and a proper-path 3-coloring of G using the colors 1, 2, 3 where the uncolored edges can be colored arbitrarily with these three colors. Since the three bridges in G must be assigned distinct colors, this coloring is minimum and so $\mathrm{pc}(G) = 3$. Note that this 3-regular graph G is not 1-factorable and so $\chi'(G) = 4$.

We refer to the books [4, 6] for graph theory notation and terminology not described in this paper. All graphs under consideration here are nontrivial connected graphs.

2 Preliminary Results

In this section, we present some preliminary observations and results on proper-path colorings of graphs. The example of Figure 1 illustrates the following two useful facts of the path connection numbers of graphs.

Proposition 2.1 *If G is a nontrivial connected graph and H is a connected spanning subgraph of G, then $\mathrm{pc}(G) \le \mathrm{pc}(H)$. In particular, $\mathrm{pc}(G) \le \mathrm{pc}(T)$ for every spanning tree T of G.*

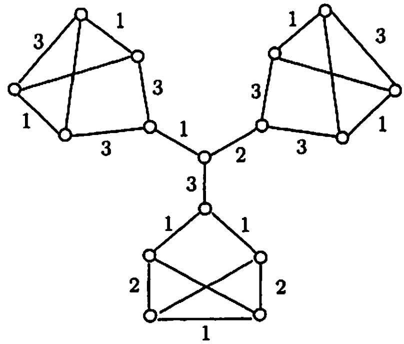

Figure 1: A proper-path 3-coloring

Proof. Let H be a spanning subgraph of G and c_H a minimum proper-path coloring of H. Define a coloring c of G by $c(e) = c_H(e)$ if $e \in E(H)$ and $c(e) = 1$ for the remaining edges of G. Then c is a proper-path coloring of G using $\mathrm{pc}(H)$ colors and so $\mathrm{pc}(G) \leq \mathrm{pc}(H)$.

Proposition 2.2 *Let G be a nontrivial connected graph that contains bridges. If b is the maximum number of bridges incident with a single vertex in G, then* $\mathrm{pc}(G) \geq b$.

Proof. Since $\mathrm{pc}(G) \geq 1$, the result is trivial when $b = 1$. Thus we may assume that $b \geq 2$. Let v be a vertex of G that is incident with b bridges and let vw_1 and vw_2 are two bridges incident with v. Since (w_1, v, w_2) is the only $w_1 - w_2$ path in G, it follows that every proper-path coloring of G assigns distinct colors to vw_1 and vw_2. Hence all b bridges incident with v are colored differently and so $\mathrm{pc}(G) \geq b$.

In a nontrivial tree T, every edge is a bridge and so $\mathrm{pc}(T) \geq \Delta(T)$ by Proposition 2.2. By König's theorem [9], $\chi'(G) = \Delta(G)$ for every nonempty bipartite graph G. Hence, the following is a consequence of (1), Proposition 2.2 and König's theorem.

Proposition 2.3 *If T is a nontrivial tree, then* $\mathrm{pc}(T) = \chi'(T) = \Delta(T)$.

Propositions 2.1 and 2.3 provide an upper bound for the proper connection number of a graph.

Proposition 2.4 *For a nontrivial connected graph G,*

$$\mathrm{pc}(G) \leq \min\{\Delta(T) : T \textit{ is a spanning tree of } G\}.$$

A *Hamiltonian path* in a graph G is a path containing every vertex of G and a graph having a Hamiltonian path is a *traceable graph*. The following is an immediate consequence of Proposition 2.4.

Corollary 2.5 *If G is a traceable graph that is not complete, then*

$$\text{pc}(G) = 2.$$

We saw in (1) that if G is a nontrivial connected graph that is not complete such that $\text{pc}(G) = a$ and $\text{rc}(G) = b$, then $2 \leq a \leq b$. In fact, this is the only restriction on these two parameters.

Proposition 2.6 *For every pair a, b of positive integers with $2 \leq a \leq b$, there is a connected graph G such that $\text{pc}(G) = a$ and $\text{rc}(G) = b$.*

Proof. By Proposition 2.3, if T is a tree of order at least 3, then $\text{pc}(T) = \Delta(T) \geq 2$. It is easy to see that $\text{rc}(T)$ is the size of T. Thus if a and b are integers with $2 \leq a \leq b$, then let T be a tree of size b and $\Delta(T) = a$. Hence $\text{pc}(T) = a$ and $\text{rc}(T) = b$. ■

By (1), if G is a nontrivial connected graph that is not complete such that $\text{pc}(G) = a$ and $\chi'(G) = b$, then $2 \leq a \leq b$. Again, this is the only restriction on these two parameters.

Proposition 2.7 *For every pair a, b of integers with $2 \leq a \leq b$, there exists a connected graph G such that $\text{pc}(G) = a$ and $\chi'(G) = b$.*

Proof. If $b = 2$, then $a = b = 2$ and any path of order at least 3 has the desired property by Corollary 2.5. Thus, we may assume that $b \geq 3$. Let G be the graph obtained from the path $(x_1, x_2, x_3, x_4, x_5)$ of order 5 by (i) adding the $b - 2$ new vertices $v_1, v_2, \ldots, v_{b-2}$ and joining each v_i $(1 \leq i \leq b - 2)$ to both x_2 and x_4 and (ii) adding the $a - 1$ new vertices $w_1, w_2, \ldots, w_{a-1}$ and joining each w_i $(1 \leq i \leq a - 1)$ to x_5. Since G is a bipartite graph and $\Delta(G) = b$, it follows that $\chi'(G) = b$. It remains to show that $\text{pc}(G) = a$. Define an edge coloring c by assigning (1) the color 1 to each of x_1x_2, x_2v_i $(1 \leq i \leq b-2)$, x_3x_4 and x_5w_1, (2) the color 2 to each of x_2x_3, x_4v_i $(1 \leq i \leq b - 2)$ and x_5w_2 (3) the color i to x_5w_i $(3 \leq i \leq a - 1$ if $a \geq 4)$ and (4) the color a to x_4x_5. Then every two vertices u and v are connected by a proper $u - v$ path. For example, v_1 and w_1 are connected by the proper path $(v_1, x_2, x_3, x_4, x_5, w_1)$. Hence c is a proper-path coloring of G using a colors and so $\text{pc}(G) \leq a$. Assume, to the contrary, that $\text{pc}(G) \leq a - 1$. Let c^* be a minimum proper-path coloring of G. Since $\deg x_5 = a$ and at most $a - 1$ colors are used by c^*, there are two edges e and f incident with x_5 that are colored the same, say $e = ux_5$ and $f = x_5v$. However then, there is no proper $u - v$ path in G, which is impossible. Thus $\text{pc}(G) \geq a$ and so $\text{pc}(G) = a$. ■

3 On Graphs with Proper Connection Number 2

We saw that a nontrivial connected graph G has proper connection number 1 if and only if $G = K_n$. Also, if G contains a Hamiltonian path and G is not complete, then $\mathrm{pc}(G) = 2$ by Corollary 2.5. However, there are connected graphs G without a Hamiltonian path for which $\mathrm{pc}(G) = 2$. The corona $\mathrm{cor}(K_n)$ of the complete graph K_n of order $n \geq 2$ is such an example. In fact, the coloring that assigns the color 1 to each edge that belongs to the subgraph K_n in $\mathrm{cor}(K_n)$ and the color 2 to each pendant edge in $\mathrm{cor}(K_n)$ is a proper-path coloring of $\mathrm{cor}(K_n)$. This example suggests that we can construct a new graph from each graph such that the resulting graph has proper connection number 2. For a given graph G of order n, let $\mathrm{com}(G)$ be the graph obtained from G by replacing the n vertices of G by n mutually disjoint complete graphs, where $v \in V(G)$ is replaced by $K(v)$ of order $\deg_G v$, such that (i) one vertex in $K(v)$ is adjacent to one vertex in $K(u)$ if and only if $uv \in E(G)$ and (ii) a vertex x in $\mathrm{com}(G)$ has degree k if and only if $x \in V(K(v))$ for some $v \in V(G)$ for which $\deg_G v = k$. This is illustrated in Figure 2. Thus if G contains three or more end-vertices, then $\mathrm{com}(G)$ contains no Hamiltonian path. In particular, if $G = K_{1,n}$ is a star, then $\mathrm{com}(G) = \mathrm{cor}(K_n)$. The coloring of $\mathrm{com}(G)$ that assigns the color 1 to each edge in the complete graph $K(v)$ of order $\deg_G v \geq 2$ for every vertex v of G and the color 2 to the remaining edges of $\mathrm{com}(G)$ is a proper-path coloring. This gives rise to the following observation.

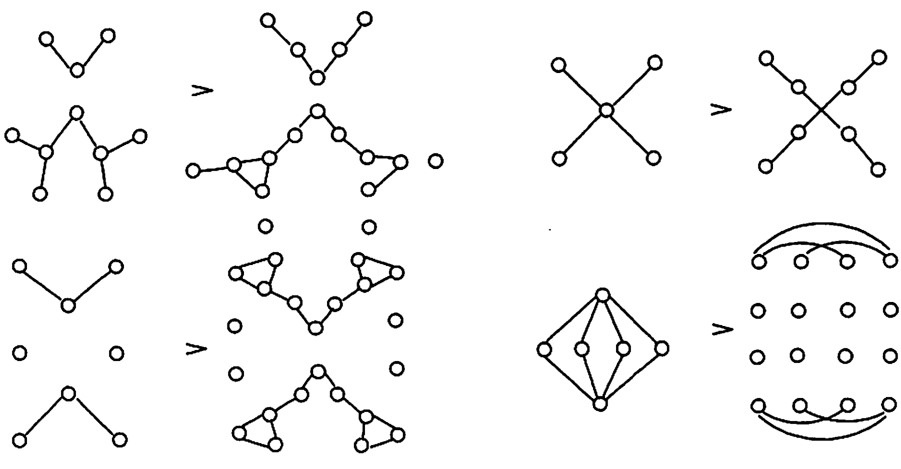

Figure 2: Constructing the graph $\mathrm{com}(G)$ from a given graph G

Observation 3.1 *If G is a nontrivial connected graph, then*

$$\mathrm{pc}(\mathrm{com}(G)) = 2.$$

In this section, we show that several classes of well-known graphs have proper connection number 2. We begin with complete multipartite graphs.

3.1 Complete Multipartite Graphs

In this subsection, we show that every complete multipartite graph that is neither a complete graph nor a tree has proper connection number 2. In order to do this, we first present two lemmas.

Lemma 3.2 *Let H be the graph obtained from the cycle $C_4 = (u_1, v_1, u_2, v_2, u_1)$ of order 4 and two empty graphs $\overline{K}_r$ and $\overline{K}_s$ of order r and s, respectively, by joining each of u_1 and u_2 to every vertex in $\overline{K}_s$ and joining each of v_1 and v_2 to every vertex in $\overline{K}_r$. Then* $\mathrm{pc}(H) = 2$.

Proof. Since H is not complete, it suffices to show that H has a proper-path 2-coloring. Define a coloring c by assigning the color 1 to (i) the edge u_iv_i for $i = 1, 2$, (ii) the edge u_1x for each vertex x of $\overline{K}_s$ and (iii) the edge v_1y for each vertex y of $\overline{K}_r$ and assigning the color 2 to the remaining edges of H. We show that c is a proper-path 2-coloring of H. Let $u, v \in V(H)$. Since every two vertices on C_4 are connected by a proper path, we may assume that at least one of u and v does not belong to C_4, say v is not in C_4.

First, suppose that u is a vertex of C_4. By symmetry, we may assume that $u = u_1$ or $u = u_2$. Assume first that $u = u_1$. If v is a vertex of $\overline{K}_r$, then (u, v_1, u_2, v_2, v) is a proper $u - v$ path, while if v is a vertex of $\overline{K}_s$, then (u, v) is a proper $u - v$ path. Next, assume that $u = u_2$. If v is a vertex of $\overline{K}_r$, then (u, v_2, v) is a proper $u - v$ path, while if v is a vertex of $\overline{K}_s$, then (u, v) is a proper $u - v$ path. Next, suppose that u is not a vertex of C_4. By symmetry, we may assume u is a vertex of $\overline{K}_r$. If v is a vertex of $\overline{K}_r$, then (u, v_1, u_2, v_2, v) is a proper $u - v$ path, while if v is a vertex of $\overline{K}_s$, then (u, v_2, u_2, v) is a proper $u - v$ path. Thus c is a proper-path 2-coloring of H and so $\mathrm{pc}(H) = 2$.

The proof of Lemma 3.2 provides the following lemma.

Lemma 3.3 *Let F be the graph obtained from the cycle $(v_1, v_2, v_3, v_4, v_1)$ of order 4 and an empty graphs $\overline{K}_r$ of order r by joining each of v_1 and v_3 to every vertex in $\overline{K}_r$. Then* $\mathrm{pc}(F) = 2$.

Theorem 3.4 *If G is a complete multipartite graph that is neither a complete graph nor a tree, then* $\mathrm{pc}(G) = 2$.

Proof. Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph that is not complete, where $k \geq 2$ and $n = n_1 + n_2 + \cdots + n_k$ is the order of G. Suppose that $V_1, V_2, \ldots, V_k$ be the partite sets of G where $|V_i| = n_i$ for $1 \leq i \leq k$ and $|V_1| \geq |V_2| \geq \cdots \geq |V_k|$ and $|V_1| \geq 2$.

First, suppose that $k = 2$. Since G is not a tree, $|V_2| \geq 2$. By Corollary 2.5, we may assume that $G \neq C_4$ and so $n \geq 5$. Thus either G contains the graph H in Lemma 3.2 as a spanning subgraph or contains the graph F in Lemma 3.3 as a spanning subgraph. It then follows by Proposition 2.1 and Lemmas 3.2 and 3.3 that $\text{pc}(G) \leq 2$. Hence $\text{pc}(G) = 2$.

Next, suppose that $k \geq 3$. If $n = 4$, then G contains a 4-cycle as a spanning subgraph and so $\text{pc}(G) = 2$ by Corollary 2.5 and Proposition 2.1. Thus, we may assume that $n \geq 5$. First, suppose that $n_1 \geq 2$ and $n_i = 1$ for $2 \leq i \leq k$. Define a coloring c of G by assigning the color 1 to each edge that is incident with a vertex in V_1 and the color 2 to the remaining edges of G. Let u and v be two nonadjacent vertices of G. Then $u, v \in V_1$. Let $x \in V_2$ and $y \in V_3$. Then (u, x, y, v) is a proper $u - v$ path and so c is a proper-path 2-coloring of G. Next, suppose that $n_1 \geq 2$ and $n_2 \geq 2$. Let C_4 be a cycle of order 4 in the subgraph K_{n_1,n_2} of G where the partite sets of K_{n_1,n_2} are V_1 and V_2. Then either G contains the graph H in Lemma 3.2 as a spanning subgraph or contains the graph F in Lemma 3.3 as a spanning subgraph. By Proposition 2.1 and Lemmas 3.2 and 3.3, $\text{pc}(G) = 2$ in either case.

3.2 Joins of Graphs

The *join* $G \vee H$ of two graphs G and H has vertex set $V(G) \cup V(H)$ and its edge set consists of $E(G) \cup E(H)$ and the set $\{uv : u \in V(G) \text{ and } v \in V(H)\}$. With the aid of Lemmas 3.2 and 3.3, we present the following result.

Theorem 3.5 *If G and H are connected graphs such that $G \vee H$ is not complete, then* $\text{pc}(G \vee H) = 2$.

Proof. If G and H are both nontrivial connected graphs such that $G \vee H$ is not complete, then $G \vee H$ contains either the graph in Lemma 3.2 as a spanning subgraph or the graph in Lemma 3.3 as a spanning subgraph. By Proposition 2.1 and Lemmas 3.2 and 3.3, it follows that $\text{pc}(G \vee H) = 2$ in either case. Thus, we may assume that G is a nontrivial connected graph of order at least 3 that is not complete and $H = K_1$ where $V(H) = \{w\}$. Since $G \vee K_1$ is not complete, it follows that $\text{pc}(G \vee K_1) \geq 2$ and so it remains to show that $\text{pc}(G \vee K_1) \leq 2$. Let T be a spanning tree of G. By Proposition 2.1, it suffices to show that $\text{pc}(T \vee K_1) \leq 2$. For a vertex v of T, let $e_T(v)$ denote the eccentricity of v in T (that is, the distance between v and a vertex farthest from v in T). For each integer i with $1 \leq i \leq e_T(v)$, let $V_i = \{u : d(v,u) = i\}$. Hence $V_0 = \{v\}$. Define a 2-coloring c of $T \vee K_1$ by

$$c(wx) = \begin{cases} 1 & \text{if } x \in V_i,\ i \text{ is odd and } 1 \le i \le e_T(v) \\ 2 & \text{if } x \in V_i,\ i \text{ is even and } 0 \le i \le e_T(v); \end{cases}$$
$$c(xy) = \begin{cases} 1 & \text{if } x \in V_i,\ y \in V_{i+1},\ i \text{ is even and } 0 \le i \le e_T(v) - 1 \\ 2 & \text{if } x \in V_i,\ y \in V_{i+1},\ i \text{ is odd and } 1 \le i \le e_T(v) - 1. \end{cases}$$

Let x and y be two vertices of $T \vee K_1$. Since w is adjacent to every vertex in T, we may assume $x \ne w$ and $y \ne w$ and so $x, y \in V(T)$. First, suppose that $x \in V_i$ and $y \in V_j$, where $0 \le i < j$. If i and j are of opposite parity, then (x, w, y) is a proper $x - y$ path in $T \vee K_1$. Thus, we may assume that i and j are of the same parity and so $j - i \ge 2$. Let $z \in V_{j-1}$ such that yz is an edge of T. Then (x, w, z, y) is a proper $x - y$ path in $T \vee K_1$. Next, suppose that $x, y \in V_i$ for some i with $1 \le i \le e_T(v)$. Let $z \in V_{i-1}$ such that xz is an edge of T. Then (x, z, w, y) is a proper $x - y$ path in $T \vee K_1$. Hence, c is a proper-path 2-coloring of $T \vee K_1$ and so $\text{pc}(T \vee K_1) = 2$. Therefore, $\text{pc}(G \vee K_1) = 2$. ∎

3.3 Cartesian Products of Graphs

The *Cartesian product* $G \,\square\, H$ of two graphs G and H has vertex set $V(G \,\square\, H) = V(G) \times V(H)$ and two distinct vertices (u, v) and (x, y) of $G \,\square\, H$ are adjacent if either (1) $ux \in E(G)$ and $v = y$ or (2) $vy \in E(H)$ and $u = x$. The Cartesian product $G \,\square\, K_2$ of a graph G and K_2 is a special case of a more general class of graphs. We partition the edge set $G \,\square\, H$ into two sets E_1 and E_2 such that E_1 is the set of edges $(u, v)(x, y)$ in $G \,\square\, H$ such that $ux \in E(G)$ and $v = y$ and E_2 is the set of edges $(u, v)(x, y)$ in $G \,\square\, H$ such that $u = x$ and $vy \in E(H)$. We show that $\text{pc}(G \,\square\, H) = 2$ for every two nontrivial connected graphs G and H. In order to do this, we first present a lemma.

Lemma 3.6 *For integers s and t with $s \ge t \ge 2$, let $P_s = (u_1, u_2, \ldots, u_s)$ be a path of order s and $P_t = (v_1, v_2, \ldots, v_t)$ be a path of order t. Define the coloring of $P_s \,\square\, P_t$ by assigning the color 1 to all edges in E_1 and the color 2 to all edges in E_2. Then there is a proper path from (u_1, v_1) to (u_s, v_t) in $P_s \,\square\, P_t$. Furthermore, if $s = t$, then there are two proper paths (u_1, v_1) to (u_s, v_s), one of which has its initial edge colored 1 and the other one has its initial edge colored 2.*

Proof. We consider three cases, according to whether $t = 2$, $s = t$ or $s > t \ge 3$.

Case 1. $t = 2$. If s is even, then $P = ((u_1, v_1), (u_2, v_1), (u_2, v_2), (u_3, v_2), (u_3, v_1), \ldots, (u_{s-1}, v_1), (u_s, v_1), (u_s, v_2))$ is a proper (u_1, v_1)-(u_s, v_2) path that the colors of edges alternate 1 and 2. If s is odd, then $P' = ((u_1, v_1), (u_2, v_1), (u_2, v_2), (u_3, v_2), (u_3, v_1), \ldots, (u_{s-1}, v_2), (u_s, v_2))$ is a proper (u_1, v_1)-(u_s, v_2) path that the colors of edges that alternates 1 and 2.

Case 2. $s = t$. Observe that $P' = ((u_1, v_1), (u_2, v_1), (u_2, v_2), (u_3, v_2), (u_3, v_3), \ldots, (u_{s-1}, v_{s-1}), (u_s, v_{s-1}), (u_s, v_s))$ is a proper (u_1, v_1)-(u_s, v_s) path whose initial edge is colored 1. Furthermore, $P'' = ((u_1, v_1), (u_1, v_2), (u_2, v_2), (u_2, v_3), (u_3, v_3), \ldots, (u_{s-1}, v_{s-1}), (u_{s-1}, v_s), (u_s, v_s))$ is a proper (u_1, v_1)-(u_s, v_s) path whose initial edge is colored 2.

Case 3. $s > t \geq 3$. Then $s = t + p$ for some positive integer p. By the same argument used in Case 1, we consider the subgraph $P_{p+1} \square P_2$ of $P_s \square P_t$, where $P_{p+1} = (u_1, u_2, \ldots, u_{p+1})$ and $P_2 = (v_1, v_2)$. If $p+1$ is even, then $((u_1, v_1), (u_2, v_1), (u_2, v_2), (u_3, v_2), (u_3, v_1), \ldots, (u_p, v_1), (u_{p+1}, v_1))$ is a proper (u_1, v_1)-(u_{p+1}, v_1) path that the colors of edges alternate 1 and 2. If $p + 1$ is odd, then $((u_1, v_1), (u_2, v_1), (u_2, v_2), (u_3, v_2), (u_3, v_1), \ldots, (u_p, v_2), (u_{p+1}, v_2), (u_{p+1}, v_1))$ is a proper (u_1, v_1)-(u_{p+1}, v_1) path the colors of whose edges alternate 1 and 2. By Case 2, there are two proper (u_{p+1}, v_1)-(u_{p+t}, v_t) paths P' and P'' such that the initial edge of P' is colored 1 and the initial edge of P'' is colored 2. If the terminal edge of P is colored 1, then P followed by P'' is a proper path from (u_1, v_1) to (u_s, v_t); while if the terminal edge of P is colored 2, then P followed by P' is a proper path from (u_1, v_1) to (u_s, v_t). ∎

Theorem 3.7 *If G and H are nontrivial connected graphs, then*

$$\mathrm{pc}(G \square H) = 2.$$

Proof. As we saw, it suffices to show that $G \square H$ has a proper-path 2-coloring. Let $V(G) = \{u_1, u_2, \ldots, u_s\}$ and $V(H) = \{v_1, v_2, \ldots, v_t\}$ for some integers $s, t \geq 2$. Define a coloring c of $G \square H$ by assigning the color 1 to all edges in E_1 and the color 2 to all edges in E_2. We show that c is a proper-path 2-coloring of $G \square H$. Let (u_i, v_p) and (u_j, v_q) be two vertices of $G \square H$, where $i, j \in \{1, 2, \ldots, s\}$ and $p, q \in \{1, 2, \ldots, t\}$.

First, suppose that either $u_i \neq u_j$ and $v_p = v_q$ or $u_i = u_j$ and $v_p \neq v_q$. We may assume, without loss of generality, $u_i \neq u_j$. Furthermore, we can assume that v_p is adjacent to v_{p+1} in H. Since G is connected, there is a $u_i - u_j$ path P in G. Note that $P \square (v_p, v_{p+1})$ is a subgraph of $G \square H$ and by the proof of Case 3 of Lemma 3.6, there exists a proper path from (u_i, v_p) to $(u_j, v_p) = (u_j, v_q)$. Next, suppose that $u_i \neq u_j$ and $v_p \neq v_q$. Since G is connected, there is a $u_i - u_j$ path P_G in G. Similarly, since H is connected, there is a $v_p - v_q$ path P_H in H. Thus $P_G \square P_H$ is a subgraph of $G \square H$ and by Lemma 3.6, there is a proper path from (u_i, v_p) to (u_j, v_q).

3.4 Permutation Graphs

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let α be a permutation of the set $S = \{1, 2, \ldots, n\}$. The *permutation graph* $P_\alpha(G)$ of a graph G is the graph of order $2n$ obtained from two copies of G, where the second copy of G is denoted by G' and the vertex v_i in G is denoted by u_i in G' and v_i is joined to the vertex $u_{\alpha(i)}$ in G'. The edges $v_i u_{\alpha(i)}$ are called the *permutation edges* of $P_\alpha(G)$. This concept was first introduced by Chartrand and Harary [1]. Therefore, if α is the identity permutation on S, then $P_\alpha(G) = G \,\square\, K_2$ is the Cartesian product of a graph G and K_2. We show that every permutation graph of a traceable graph has proper connection number 2. In order to do this, we first present a lemma. A connected graph of order 3 or more is *unicyclic* if it contains exactly one cycle.

Lemma 3.8 *If H is a bipartite unicyclic graph with maximum degree* 3 *such that H contains exactly two vertices of degree* 3 *each of which lies on the cycle in H, then* $\text{pc}(H) = 2$.

Proof. Let $C_p = (v_1, v_2, \ldots, v_p, v_{p+1} = v_1)$ be the unique cycle in H, where then p is even. We may assume that $\deg_H v_1 = \deg_H v_i = 3$ where $2 \le i \le p$. Suppose that $P = (v_1, u_1, \ldots, u_s)$ and $P' = (v_i, w_1, \ldots, w_t)$ are paths in H, where $s, t \ge 1$, such that $E(P) \cap E(C_p) = \emptyset$ and $E(P') \cap E(C_p) = \emptyset$. Define a 2-coloring c of G by first coloring C_p properly and then coloring P and P' properly such that $c(v_1 u_1) = c(v_p v_1)$ and $c(v_i w_1) = c(v_i v_{i+1})$. Each of the paths $(P, v_2, \ldots, v_{i-1}, P')$, $(P, v_2, \ldots, v_p)$ and $(P', v_{i-1}, \ldots, v_1, v_p, \ldots, v_{i+1})$ is a proper path. Since every two vertices of H lie on one of these three proper paths, it follows that c is a proper-path 2-coloring and so $\text{pc}(H) = 2$.

Theorem 3.9 *If G is a nontrivial traceable graph of order n, then*

$$\text{pc}(P_\alpha(G)) = 2$$

for each permutation α of the set $\{1, 2, \ldots, n\}$. In particular, $\text{pc}(P_\alpha(G)) =$ 2 *if G is Hamiltonian.*

Proof. For a nontrivial traceable graph G of order n, let $(v_1, v_2, \ldots, v_n)$ be a Hamiltonian path in G and let $(v'_1, v'_2, \ldots, v'_n)$ be the corresponding Hamiltonian path in the second copy G' of G. Since $P_\alpha(G)$ is not complete for each permutation α of $\{1, 2, \ldots, n\}$, it remains to show that $\text{pc}(P_\alpha(G)) \le 2$. We consider two cases.

Case 1. $\{\alpha(1), \alpha(n)\} \cap \{1, n\} \ne \emptyset$. We may assume that $\alpha(1) = 1$ or $\alpha(n) = 1$. If $\alpha(1) = 1$, then $(v_n, v_{n-1}, \ldots, v_1, v'_1, v'_2, \ldots, v'_n)$ is a Hamiltonian path of $P_\alpha(G)$; while if $\alpha(n) = 1$, then $(v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n)$

is a Hamiltonian path of $P_\alpha(G)$. It then follows by Corollary 2.5 that $\text{pc}(P_\alpha(G)) = 2$.

Case 2. $\{\alpha(1), \alpha(n)\} \cap \{1, n\} = \emptyset$. Suppose $\alpha(1) = i$ and $\alpha(n) = j$ where $2 \le i \ne j \le n-1$. We will only consider the case when $i < j$ (since the argument for the case when $i > j$ is similar and we use the path $(v'_n, v'_{n-1}, \ldots, v'_2, v'_1)$ in the proof). Furthermore, assume that $\alpha(k) = 1$ for some k with $2 \le k \le n-1$. We consider three cases, depending on the parities of two of the integers $k-1$, $i-1$ and $n-j$.

Subcase 2.1. *$k-1$ and $i-1$ are of the same parity.* Let H be the subgraph of $P_\alpha(G)$ consisting of the even cycle $(v_1, v_2, \ldots, v_k, v'_1, v'_2, \ldots, v'_i, v_1)$ and two paths $(v_k, v_{k+1}, \ldots, v_n)$ and $(v'_i, v'_{i+1}, \ldots, v'_n)$. Then H is a spanning subgraph of $P_\alpha(G)$. By Lemma 3.8, $\text{pc}(H) = 2$. It then follows by Proposition 2.1 that $\text{pc}(P_\alpha(G)) \le 2$ and so $\text{pc}(P_\alpha(G)) = 2$.

Subcase 2.2. *$k-1$ and $n-j$ are of the same parity.* Let H be the subgraph of $P_\alpha(G)$ consisting of the even cycle $(v_k, v_{k+1}, \ldots, v_n, v'_j, v'_{j-1}, \ldots, v'_1, v_k)$ and the two paths $(v_1, v_2, \ldots, v_k)$ and $(v'_j, v'_{j+1}, \ldots, v'_n)$. Then H is a spanning subgraph of $P_\alpha(G)$. By Lemma 3.8, $\text{pc}(H) = 2$. It then follows by Proposition 2.1 that $\text{pc}(P_\alpha(G)) \le 2$ and so $\text{pc}(P_\alpha(G)) = 2$.

Subcase 2.3. *$i-1$ and $n-j$ are of the same parity.* Let H be the subgraph of $P_\alpha(G)$ consisting of the even cycle $(v_1, v_2, \ldots, v_n, v'_j, v'_{j-1}, \ldots, v'_i, v_1)$ and the two paths $(v'_1, v'_2, \ldots, v'_i)$ and $(v'_j, v'_{j+1}, \ldots, v'_n)$. Then H is a spanning subgraph of $P_\alpha(G)$. By Lemma 3.8, $\text{pc}(H) = 2$. It then follows by Proposition 2.1 that $\text{pc}(P_\alpha(G)) \le 2$ and so $\text{pc}(P_\alpha(G)) = 2$.

By Theorem 3.9, every permutation graph of a traceable graph has proper connection number 2. However, traceable graphs are not only connected graphs with this property, as we show next.

Proposition 3.10 *Every permutation graph of a star of order at least* 4 *has proper connection number* 2.

Proof. For an integer $m \ge 3$, let $G = K_{1,m}$ be the star with vertex set $\{v_0, v_1, \ldots, v_m\}$, where v_0 is the central vertex. Then there are exactly two non-isomorphic permutation graphs, namely $P_{\alpha_1}(G) = G \,\square\, K_2$ where α_1 is the identity permutation on the set $\{0, 1, \ldots, m\}$ and $P_{\alpha_2}(G)$ where $\alpha_2 = (0, 1)$. By Theorem 3.7, $\text{pc}(P_{\alpha_1}(G)) = 2$. It remains to show that $P_{\alpha_2}(G) = 2$. Let $\{v'_0, v'_1, \ldots, v'_m\}$ be the corresponding vertex set in the second copy G' of G. Since $P_{\alpha_2}(G)$ is not complete, it remains to show that $\text{pc}(P_{\alpha_2}(G)) \le 2$.

Define an edge 2-coloring c by assigning the color 1 to (i) the edge $v_0v'_1$, $v_1v'_0$, $v_2v'_2$, (ii) the edge v_0v_i for each $i \ge 3$ and (iii) the edge $v'_0v'_i$ for each $i \ge 3$ and assigning the color 2 to the remaining edges of $P_{\alpha_2}(G)$. We show that c is a proper-path 2-coloring of $P_{\alpha_2}(G)$. Let $u, v \in V(P_{\alpha_2}(G))$. Since every two vertices on $C_4 = (v_0, v_1, v'_0, v'_1, v_0)$ are connected by a proper path, we may assume that at least one of u and v does not belong to C_4, say v is not in C_4.

First, suppose that u is a vertex of C_4. By symmetry, we may assume that $u = v_0$ or $u = v_1$. Assume first that $u = v_0$. If $v = v_i$ where $i \geq 2$, then (u, v) is a proper $u - v$ path, while if $v = v'_i$ where $i \geq 2$, then (u, v_i, v) is a proper $u - v$ path. Next, assume that $u = v_1$. If $v = v_2$, then (u, v'_0, v'_2, v) is a proper $u - v$ path, while if $v = v'_2$, then (u, v'_0, v) is a proper $u - v$ path. If $v = v_i$ where $i \geq 3$, then (u, v_0, v) is a proper $u - v$ path, while if $v = v'_i$ where $i \geq 3$, then (u, v_0, v_i, v) is a proper $u - v$ path. Next, suppose that u is not a vertex of C_4. By symmetry, we may assume $u = v_2$ or $u = v_3$. We first assume that $u = v_2$. If $v = v'_2$, then u and v are adjacent and so there is a proper $u - v$ path. If $v = v_i$ where $i \geq 3$, then (u, v_0, v) is a proper $u - v$ path, while if $v = v'_i$ where $i \geq 3$, then (u, v_0, v_i, v) is a proper $u - v$ path. Now, we assume that $u = v_3$. If $v = v'_2$, then (u, v_0, v_2, v) is a proper $u - v$ path, while if $v = v'_3$, then (u, v) is a proper $u - v$ path. If $v = v_i$ where $i \geq 4$, then $(u, v_0, v_2, v'_2, v'_0, v'_i, v)$ is a proper $u - v$ path, while if $v = v'_i$ where $i \geq 4$, then $(u, v_0, v_2, v'_2, v'_0, v)$ is a proper $u - v$ path. Thus c is a proper-path 2-coloring of $P_{\alpha_2}(G)$ and so $\text{pc}(P_{\alpha_2}(G)) = 2$. ∎

We conclude this subsection with the following question: Is there a class of nontrivial connected graphs G such that $\text{pc}(P_\alpha(G)) \geq 3$ for some permutation graph $P_\alpha(G)$ of G?

4 The Strong Proper Connection Numbers of Graphs

A related concept concerning rainbow colorings was introduced in [2]. Let c be a rainbow coloring of a connected graph G. For two vertices u and v of G, a *rainbow* $u - v$ *geodesic* in G is a rainbow $u - v$ path of length $d(u, v)$, where $d(u, v)$ is the distance between u and v (the length of a shortest $u - v$ path in G). The graph G is *strongly rainbow-connected* if G contains a rainbow $u - v$ geodesic for every two vertices u and v of G. In this case, the coloring c is called a *strong rainbow coloring* of G. The minimum k for which there exists a coloring $c : E(G) \to \{1, 2, \ldots, k\}$ of the edges of G such that G is strongly rainbow-connected is the *strong rainbow connection number* $\text{src}(G)$ of G. A strong rainbow coloring of G using $\text{src}(G)$ colors is called a *minimum strong rainbow coloring* of G. Thus $\text{rc}(G) \leq \text{src}(G)$ for every connected graph G.

Inspired by this concept, we consider an analogous concept in proper-path colorings. Let c be a proper-path coloring of a nontrivial connected graph G. For two vertices u and v of G, a *proper* $u - v$ *geodesic* in G is a proper $u-v$ path of length $d(u,v)$. If there is a proper $u-v$ geodesic for every two vertices u and v of G, then c is called a *strong proper coloring* of G or a *strong proper k-coloring* if k colors are used. The minimum number of colors needed to produce a strong proper coloring of G is called the *strong proper connection number* $\text{spc}(G)$ of G. A strong proper coloring using $\text{spc}(G)$ colors is referred to as a *minimum strong proper coloring.* In general, if G is a nontrivial connected graph, then $1 \leq \text{pc}(G) \leq \text{spc}(G) \leq \chi'(G)$. Since every strong rainbow coloring of G is a strong proper coloring of G, it follows that $\text{spc}(G) \leq \text{src}(G)$. Therefore, if G is nontrivial connected graph, then

$$1 \leq \text{pc}(G) \leq \text{spc}(G) \leq \min\{\chi'(G), \text{src}(G)\}. \tag{2}$$

We present several useful observations on the strong connection numbers of graphs. The *diameter* $\text{diam}(G)$ of a connected graph G is the largest distance between two vertices of G.

Observation 4.1 *Let G be a nontrivial connected graph of order n and size m. Then*

(1) $\text{spc}(G) = \text{pc}(G) = 1$ *if and only if* $G = K_n$*;*

(2) $\text{spc}(G) = \text{pc}(G) = m$ *if and only if* $G = K_{1,m}$*;*

(3) *if G is a tree, then* $\text{spc}(G) = \text{pc}(G) = \Delta(G)$*;*

(4) *if G is a connected graph with* $\text{diam}(G) = 2$*, then* $\text{spc}(G) = \text{src}(G)$*;*

(5) *if b is the maximum number of bridges incident with a vertex in G, then* $\text{spc}(G) \geq b$.

By Observation 4.1 and (2), if G is a connected graph of order n that is not complete such that $\text{pc}(G) = a$ and $\text{spc}(G) = b$, then $2 \leq a \leq b$. In [2], it is shown that $\text{src}(K_{s,t}) = \lceil \sqrt[s]{t} \rceil$ for all integers s and t with $2 \leq s \leq t$. Since $\text{diam}(K_{s,t}) = 2$, it follows by Observation 4.1 that $\text{spc}(K_{s,t}) = \text{src}(K_{s,t}) = \lceil \sqrt[s]{t} \rceil$ for $2 \leq s \leq t$. Moreover, we saw in Theorem 3.4 that $\text{pc}(K_{s,t}) = 2$ for $2 \leq s \leq t$. Thus, if $s = 2$ and $t = b^2$ where $b \geq 2$, then $\text{spc}(K_{s,t}) - \text{pc}(K_{s,t}) = b - 2$ which can be arbitrarily large. In fact, more can be said. Next, we show that every pair a, b of integers where $2 \leq a \leq b$ is realizable as the strong proper connection number and proper connection number, respectively, of some connected graph. In order to do this, we first present a lemma.

Lemma 4.2 *For each integer $t \geq 2$, let $G = K_{2,t^2}$ be the complete bipartite graph of order $2 + t^2$ with partite sets U and W, where $|U| = 2$ and $|W| = t^2$. If c is a strong proper t-coloring of G using the colors $1, 2, \ldots, t$, then $\{c(uw) : w \in W\} = \{1, 2, \ldots, t\}$ for each vertex $u \in U$.*

Proof. Let $U = \{u_1, u_2\}$. Since $\text{spc}(G) = t$, every strong proper coloring of G uses at least t colors. Assume, to the contrary, that there is a strong proper t-coloring c of G using the colors $1, 2, \ldots, t$ such that $\{c(uw) : w \in W\} \neq \{1, 2, \ldots, t\}$ for some $u \in U$, say $t \notin \{c(u_1w) : w \in W\}$. For each vertex $w \in W$, we can associate an ordered pair $\text{code}(w) = (a_1(w), a_2(w))$ called the color code of w, where $a_i(w) = c(u_iw)$ for $i = 1, 2$. Since $1 \leq a_1(w) \leq t - 1$ for each $w \in W$, the number of distinct color codes of the vertices of W is at most $(t-1)t$. However, because $t^2 > (t-1)t$, there exists at least two distinct vertices w' and w'' of W such that $\text{code}(w') = \text{code}(w'')$. Since $c(u_iw') = c(u_iw'')$ for $i = 1, 2$, it follows that G contains no proper $w' - w''$ geodesic in G, contradicting our assumption that c is a strong proper t-coloring of G. ■

Theorem 4.3 *For every pair a, b of integers where $2 \leq a \leq b$, there exists a connected graph G such that* $\text{pc}(G) = a$ *and* $\text{spc}(G) = b$.

Proof. If $a = b \geq 2$, then let G be a tree with maximum degree a. Thus $\text{spc}(G) = \text{pc}(G) = a$. Thus, we may assume that $2 \leq a < b$, where then $b \geq 3$. Let $H = K_{2,(b-1)^2}$ be the complete bipartite graph with partite sets $U = \{u_1, u_2\}$ and $W = \{w_1, w_2, \ldots, w_{(b-1)^2}\}$ and let $F = K_{1,a-1}$ with $V(F) = \{v, v_1, v_2, \ldots, v_{a-1}\}$ where v is the central vertex of F. Now let G be the graph obtained from H and F by adding the edge u_2v.

First, we show that $\text{pc}(G) = a$. Since the vertex v is incident with a bridges in G, it follows by Observation 4.1 that $\text{pc}(G) \geq a$. Next, define an edge coloring c of G by assigning (1) the color 1 to each of the edges u_1w_i $(2 \leq i \leq (b-1)^2)$, w_1u_2 and vv_1, (2) the color 2 to each of u_1w_1, u_2w_i $(2 \leq i \leq (b-1)^2)$ and vv_2 (3) the color i to vv_i $(3 \leq i \leq a-1$ if $a \geq 4)$ and (4) the color a to u_2v. Then every two vertices x and y are connected by a proper $x - y$ path. For example, $(v_1, v, u_2, w_1, u_1, w_2)$ is a proper $v_1 - w_2$ path in G. Hence c is a proper-path coloring of G using a colors and so $\text{pc}(G) \leq a$. Thus, $\text{pc}(G) = a$.

Next, we show that $\text{spc}(G) = b$. First, we show that $\text{spc}(G) \leq b$. Since $\text{spc}(K_{2,(b-1)^2}) = b - 1$, there is a strong proper $(b-1)$-coloring c_0 of the subgraph H of G using colors $1, 2, \ldots, b-1$. Define an edge coloring c_1 of G by assigning (1) the color $c_0(e)$ to each edge e of H, (2) the color b to the edge u_2v and (3) the color i to vv_i for $1 \leq i \leq a-1$. It is easy to see that c_1 is a strong proper b-coloring of G and so $\text{spc}(G) \leq b$. Next, we show that $\text{spc}(G) \leq b$. Let c be a strong proper k-coloring of G. For every two vertices x and y in the subgraph H of G, each $x - y$ geodesic lies completely in H. Hence the restriction c_H of c to H is a strong proper coloring of H and so $k \geq b - 1$. Assume, to the contrary, that $k = b - 1$. By Lemma 4.2, $\{c(u_iw) : w \in W\} = \{1, 2, \ldots, b-1\}$ for $i = 1, 2$. Since $c(u_2v) \in \{1, 2, \ldots, b-1\}$, there exists $w \in W$ such that $c(u_2w) = c(u_2v)$.

However then, G contains no proper $w-v$ geodesic in G, contradicting our assumption that c is a strong proper coloring of G. ■

By Observation 4.1 and (2), if G is a connected graph of order n that is not complete such that $\text{spc}(G) = a$ and $\chi'(G) = b$, then $2 \le a \le b < n$. In fact, this is the only restriction on these three parameters.

Theorem 4.4 *For every triple a, b, n of integers where $2 \le a \le b < n$, there exists a connected graph G of order n such that $\text{spc}(G) = a$ and $\chi'(G) = b$.*

Proof. First suppose that $2 \le a = b < n$. Let G be the graph obtained from the star $K_{1,a}$ of order $a+1$ and the path P_{n-a-1} of order $n-a-1$ by adding an edge joining an end-vertex of $K_{1,a}$ and an end-vertex of P_{n-a-1}. Then G is a tree of order n with $\Delta(G) = a$ and so $\text{spc}(G) = \chi'(G) = a$ by Observation 4.1.

Next, suppose that $2 \le a < b < n$. We begin with a graph H constructing from the complete graph K_{b-a+2} by adding $a-1$ pendant edges at a vertex v of K_{b-a+2}. Then the graph G is obtained from H and P_{n-b-1} by adding an edge joining an end-vertex u of H and an end-vertex w of P_{n-b-1}. Then G has order n and $\Delta(G) = b$.

We show that $\chi'(G) = \Delta(G) = b$. It suffices to provide an edge b-coloring of G. First, suppose that $b-a+2 \ge 4$ is even. Since $\chi'(K_{b-a+2}) = b-a+1$, there is a proper edge coloring c_1 of K_{b-a+2} using the colors 1, 2, ..., $b-a+1$. Since $T = G - E(K_{b-a+2})$ is a tree of maximum degree $a-1$, there is a proper edge coloring c_2 of T using the $a-1$ colors $b-a+2, b-a+3, \ldots, b$. Then the coloring c of G defined by $c(e) = c_1(e)$ if $e \in E(K_{b-a+2})$ and $c(e) = c_2(e)$ if $e \in E(T)$ is a proper b-coloring of G. Next, suppose that $b-a+2 \ge 3$ is odd. Since $\chi'(K_{b-a+2}) = b-a+2$, there is a proper edge coloring c_1 of K_{b-a+2} using the colors 1, 2, ..., $b-a+2$. Since v is incident with $b-a+1$ edges in K_{b-a+2}, there is a color $i \in \{1, 2, \ldots, b-a+2\}$ that is not used to color any edge incident with v in K_{b-a+2}, say $i = b-a+2$. Now, the subgraph $T = G - E(K_{b-a+2})$ is a tree of maximum degree $a-1$. Hence there is a proper edge coloring c_2 of T using the $a-1$ colors $b-a+2, b-a+3, b-a+4, \ldots, b$. Then the coloring c of G defined by $c(e) = c_1(e)$ if $e \in E(K_{b-a+2})$ and $c(e) = c_2(e)$ if $e \in E(T)$ is a proper b-coloring of G. In either case, $\chi'(G) = \Delta(G) = b$.

Next, we show that $\text{spc}(G) = a$. First, define the coloring c by (i) assigning the color 1 to each edge of K_{b-a+2} and distinct colors from the set $\{2, 3, \ldots, a\}$ of colors to the $a-1$ pendant edges of H and (ii) assigning the color 1 to the edge uw and the colors 1 and 2 properly to the edges of P_{n-b-1} such that the initial edge of P_{n-b-1} (that is adjacent to uw) is colored 2. Since c is a strong proper a-coloring of G, it follows that $\text{spc}(G) \le a$. Next, we show that $\text{spc}(G) \ge a$. Since there are $a-1$ pendant edges incident with v, it follows by Observation 4.1 that $\text{spc}(G) \geq a-1$. Suppose that $\text{spc}(G) = a-1$. Let c' be a strong proper $(a-1)$-coloring of G. Thus, c' must assign $a-1$ distinct colors to the $a-1$ edges incident with v that do not long to K_{b-a+2}. This implies that there are edges e and f incident with v in G such that e is an edge K_{b-a+2} and f is not an edge K_{b-a+2} for which $c'(e) = c'(f)$. Let $e = uv$ and $f = vw$. However then, there is no proper $u-w$ geodesic in G, such that which is not possible. Thus $\text{spc}(G) = a$.

By Observation 4.1, if G is a tree, then $\text{spc}(G) = \chi'(G) = \Delta(G)$. However, there are many connected graphs G that are not trees and $\text{spc}(G) = \chi'(G)$. For example, for the cycle C_n of order $n \geq 4$, it can be shown that

$$\text{spc}(C_n) = \chi'(G) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

The following result provides a sufficient condition for a non-tree connected graph G such that $\text{spc}(G) = \chi'(G)$. The *girth* $g(G)$ of a graph G having cycles is the length of the smallest cycle in G. In particular, $g(C_n) = n$ for each $n \geq 3$.

Proposition 4.5 *If G is a connected graph having $g(G) \geq 5$, then*

$$\text{spc}(G) = \chi'(G).$$

Proof. Since $\text{spc}(G) \leq \chi'(G)$, it suffices to show that $\chi'(G) \leq \text{spc}(G)$. Let c be a minimum strong proper coloring of G using $\text{spc}(G)$ colors. We show that c is proper; for otherwise, there are adjacent edges e and f such that $c(e) = c(f)$. Let $e = uv$ and $f = vw$. Since $g(G) \geq 5$, it follows that (u, v, w) is the only $u-w$ geodesic in G. However then, there is no proper $u-w$ geodesic in G, which is a contradiction. Thus, $\chi'(G) \leq \text{spc}(G)$. ■

Since the girth of the Petersen graph P is 5, it then follows by Proposition 4.5 that $\text{spc}(P) = \chi'(P) = 4$. The lower bound 5 for the girth of a graph is best possible. For example, the complete bipartite graph $G = K_{2,t^2}$ (where $t \geq 2$) has $g(G) = 4$, $\text{spc}(G) = t$ and $\chi'(G) = t^2$. Also, notice that the converse of Proposition 4.5 is not true. For example, the 4-cycle C_4 has girth 4 and $\text{spc}(C_4) = \chi'(C_4) = 2$.

We saw that $\text{spc}(G) \leq \text{src}(G)$ for every connected graph G and if G is a connected graph with $\text{diam}(G) = 2$, then $\text{spc}(G) = \text{src}(G)$. Also, as one may expected, the value $\text{src}(G) - \text{spc}(G)$ can be arbitrarily large. In fact, more can be said. First, we present a result on the rainbow connection number and strong rainbow connection number of a graph.

Proposition 4.6 [2] *Let G be a nontrivial connected graph of size m. Then*

(1) $s\,\mathrm{rc}(G) = 1$ *if and only if G is a complete graph;*

(2) $\mathrm{rc}(G) = m$ *if and only if G is a tree.*

Proposition 4.6 implies that the only connected graphs G for which $\mathrm{rc}(G) = 1$ are the complete graphs and that the only connected graphs G of size m for which $\mathrm{src}(G) = m$ are trees. Thus, if T is a tree of size $b \geq 2$ and maximum degree a, then $\mathrm{spc}(T) = a$ and $\mathrm{src}(T) = b$ by Observation 4.1 and Proposition 4.6. If we replace an end-vertex of such a tree T by a complete graph, the resulting graph G has $\mathrm{spc}(G) = a$ and $\mathrm{src}(G) = b$ as well. This observation gives rise to the following result.

Theorem 4.7 *For every triple a, b, n of integers $2 \leq a \leq b < n$, there exists a connected graph G of order n such that $\mathrm{spc}(G) = a$ and $\mathrm{src}(G) = b$.*

Acknowledgments We are grateful to Professor Gary Chartrand for suggesting the concepts of proper-path coloring and strong proper coloring to us and kindly providing useful information on this topic.

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