On path-supermagic labelings of cycles

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Abstract

A graph G admits an H-covering if every edge in $E(G)$ belongs to a subgraph of G isomorphic to H. The graph G is said to be H-magic if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that for every subgraph H' of G isomorphic to H, $\sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$ is constant. When $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$, then G is said to be H-supermagic. In this paper, we investigate path-supermagic cycles. We prove that for two positive integers m and t with $m > t \geq 2$, if C_m is P_t-supermagic, then C_{3m} is also P_t-supermagic. Moreover, we show that for $t \in \{3, 4, 9\}$, C_n is P_t-supermagic if and only if n is odd with $n > t$.

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1. Introduction

We consider finite undirected graphs without loops or multiple edges. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively. We denote the path and the cycle on n vertices by P_n and C_n, respectively.

An *edge-covering* of a graph G is a family of subgraphs $H_1, H_2, \ldots, H_k$ of G such that each edge of G belongs to at least one of the subgraphs H_i, $1 \leq i \leq k$. Then it is said that G admits an $(H_1, H_2, \ldots, H_k)$-*edge-covering*. If every H_i is isomorphic to a given graph H, then we say that G admits

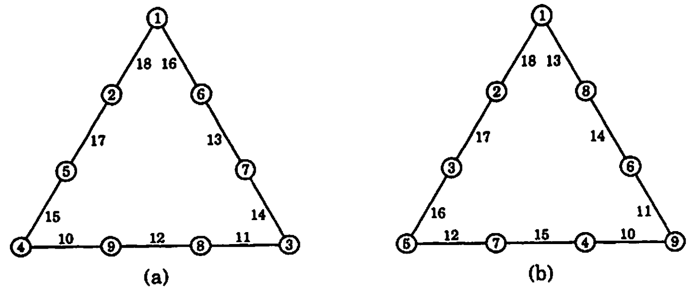

Fig. 1 (a) A P_3-supermagic labeling of C_9. (b) A P_4-supermagic labeling of C_9.

an *H-covering.* Suppose that G admits an H-covering. A bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ is called an *H-magic labeling* of G if there exists a constant $m(f)$, called the *magic sum*, such that for every subgraph H' of G isomorphic to H, $\sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) = m(f)$. An H-magic labeling f of G is called an *H-supermagic labeling* of G if $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$ and $f(E(G)) = \{|V(G)| + 1, |V(G)| + 2, \ldots, |V(G)| + |E(G)|\}$. The magic sum of an H-supermagic labeling f of G is called the *supermagic sum* and we denote it by $s(f)$. A graph G which admits an H-covering is called *H-magic* (resp. *H-supermagic*) if there exists an H-magic (resp. H-supermagic) labeling of G. In Fig. 1(a) and (b), we show P_3-supermagic and P_4-supermagic labelings of C_9, respectively. When $H = P_2$, an H-magic graph and an H-supermagic graph are called an *edge-magic graph* and a *super edge-magic graph*, respectively. Surveys of H-magic and related topics are included in Gallian [2].

In this paper, we investigate path-supermagic labelings of cycles. The following results are known.

Proposition 1 (Enomoto et al. [1]). *The cycle C_n is super edge-magic, i.e. P_2-supermagic, if and only if n is odd with $n \geq 3$.*

Proposition 2 (Gutiérrez and Lladó [3]).

(i) *Let G be a P_t-magic graph, $t > 2$. Then G is C_t-free.*

(ii) *The cycle C_n is P_t-supermagic for any $2 \leq t < n$ such that* $\gcd(n, t(t-1)) = 1$.

Proposition 3 (Ngurah et al. [4]). *If C_n is P_t-supermagic, $2 \leq t \leq n-1$, then n is odd.*

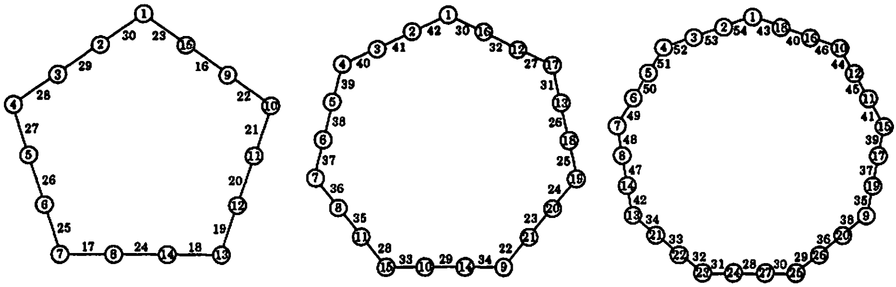

Fig. 2. P_9-supermagic labelings of C_{15}, C_{21}, and C_{27}.

Moreover, Ngurah et al. [4] proposed an open problem that for $3 \le t \le n-1$, determine whether there is a P_t-supermagic labeling of C_n such that $\gcd(n, t(t-1)) \ne 1$.

We get the following theorem.

Theorem 1. *Let m and t be two positive integers with $m > t \ge 2$. If C_m is P_t-supermagic, then C_{3m} is also P_t-supermagic.*

In Fig. 1, we show that C_9 is P_3-supermagic and P_4-supermagic. Furthermore, in Fig. 2, we show that C_{15}, C_{21}, and C_{27} are P_9-supermagic. From Propositions 2, 3, and Theorem 1 with the facts, we obtain the following theorem.

Theorem 2. *For $t \in \{3,4,9\}$, C_n is P_t-supermagic if and only if n is odd with $n > t$.*

In the next section, we prove Theorem 1.

2. Proof of Theorem 1

We use the following notations. For two integers n and m with $n < m$, let $[n,m]$ denote the set of all consecutive integers from n to m. When some pattern of integers $(x_1, x_2, \ldots, x_k)$ is repeated r times, we write $(x_1, x_2, \ldots, x_k)^r$. For instance, the sequence of integers $(0,1,2,0,1,2,3,4)$ is denoted by $(0,1,2)^2(3,4)$. For a graph G and a mapping f from $V(G) \cup E(G)$ to $\mathbb{Z}$, we define $\sum f(V(G)) = \sum_{v \in V(G)} f(v)$, $\sum f(E(G)) = \sum_{e \in E(G)} f(e)$, and $\sum f(G) = \sum f(V(G)) + \sum f(E(G))$.

Let $V(C_n) = \{v_i : 0 \le i \le n-1\}$ and $E(C_n) = \{e_i = v_iv_{i+1} : 0 \le i \le n-1\}$, where the subscripts are taken modulo n.

Lemma 1. *Let n and t be two positive integers with $n > t \ge 2$. If there is a mapping f from $V(C_n) \cup E(C_n)$ to $\mathbb{Z}$ such that $f(v_k) + f(e_k) = f(v_{k+t}) + f(e_{k+t-1})$ for $0 \le k \le n-1$, then for every subgraph H of C_n isomorphic to P_t, $\sum f(H) = \frac{1}{n}(t \sum f(V(C_n)) + (t-1)\sum f(E(C_n)))$.*

Proof. Let $P_t^{(i)}$ be the subpath of C_n with $V(P_t^{(i)}) = \{v_i, v_{i+1}, \ldots, v_{i+t-1}\}$ and $E(P_t^{(i)}) = \{e_i, e_{i+1}, \ldots, e_{i+t-2}\}$ for $0 \le i \le n-1$. By $f(v_k) + f(e_k) = f(v_{k+t}) + f(e_{k+t-1})$ for $0 \le k \le n-1$, we have $\sum f(P_t^{(k)}) - \sum f(P_t^{(k+1)}) = f(v_k) + f(e_k) - f(v_{k+t}) - f(e_{k+t-1}) = 0$ for $0 \le k \le n-1$. Therefore, we get $\sum f(P_t^{(i)})$ is constant for $0 \le i \le n-1$. We can verify that each vertex of C_n is contained t different subpaths $P_t^{(i)}$ and each edge of C_n is contained $t-1$ different subpaths $P_t^{(i)}$. Hence, $\sum f(P_t^{(i)}) = \frac{1}{n}(t\sum f(V(C_n)) + (t-1)\sum f(E(C_n)))$ for $0 \le i \le n-1$. □

Lemma 2. *Let m and t be two positive integers with $m > t \ge 2$. There is a mapping f from $V(C_{3m}) \cup E(C_{3m})$ to $\{0,1,2\} \cup \{2,3,4\}$ such that $f(\{v_j, v_{m+j}, v_{2m+j}\}) = \{0,1,2\}$ and $f(\{e_j, e_{m+j}, e_{2m+j}\}) = \{2,3,4\}$ for $0 \le j \le m-1$, and $f(v_k) + f(e_k) = f(v_{k+t}) + f(e_{k+t-1})$ for $0 \le k \le 3m-1$.*

Proof. Let a be the integer such that $m \equiv a \pmod t$ with $0 \le a \le t-1$. We denote $\boldsymbol{V_i} = (f(v_{im}), f(v_{im+1}), \ldots, f(v_{im+m-1}))$ and $\boldsymbol{E_i} = (f(e_{im}), f(e_{im+1}), \ldots, f(e_{im+m-1}))$ for $i = 0, 1, 2$. We divide our proof into four cases depending upon the values of a and m.

Case 1: $a = 0$. We define a mapping f from $V(C_{3m}) \cup E(C_{3m})$ to $\{0,1,2\} \cup \{2,3,4\}$ as follows:

$$\boldsymbol{V_0} = (0, \overbrace{0, \ldots, 0}^{t-2}, 1)^{\frac{m}{t}}, \quad \boldsymbol{E_0} = (\overbrace{4, 4, \ldots, 4}^{m-1}, 2),$$
$$\boldsymbol{V_1} = (2, \overbrace{1, \ldots, 1}^{t-2}, 0)^{\frac{m}{t}}, \quad \boldsymbol{E_1} = (\overbrace{3, 3, \ldots, 3}^{m-1}, 4),$$
$$\boldsymbol{V_2} = (1, \overbrace{2, \ldots, 2}^{t-2}, 2)^{\frac{m}{t}}, \quad \boldsymbol{E_2} = (\overbrace{2, 2, \ldots, 2}^{m-1}, 3).$$

We remark that $f(\{v_j, v_{m+j}, v_{2m+j}\}) = \{0,1,2\}$ and $f(\{e_j, e_{m+j}, e_{2m+j}\}) =$

$\{2,3,4\}$ for $0 \le j \le m-1$. Moreover, we can verify that for $0 \le k \le 3m-1$,

$$f(v_k)+f(e_k) = \begin{cases} 5 & \text{if } k \equiv t-1 \pmod t \\ & \quad \text{with } t-1 \le k \le m-t-1 \text{ or } k = 3m-1, \\ & \quad k \equiv 0 \pmod t \text{ with } m \le k \le 2m-t, \\ 3 & \text{if } k \equiv t-1 \pmod t \text{ with } m-1 \le k \le 2m-t-1, \\ & \quad k \equiv 0 \pmod t \text{ with } 2m \le k \le 3m-t, \\ 4 & \text{otherwise} \end{cases}$$

and

$$f(v_k)+f(e_{k-1}) = \begin{cases} 5 & \text{if } k \equiv t-1 \pmod t \text{ with } t-1 \le k \le m-1, \\ & \quad k \equiv 0 \pmod t \text{ with } m+t \le k \le 2m, \\ 3 & \text{if } k \equiv t-1 \pmod t \\ & \quad \text{with } m+t-1 \le k \le 2m-1, \\ & \quad k \equiv 0 \pmod t \\ & \quad \text{with } k = 0 \text{ or } 2m+t \le k \le 3m-t, \\ 4 & \text{otherwise.} \end{cases}$$

Therefore, we get $f(v_k)+f(e_k) = f(v_{k+t})+f(e_{k+t-1})$ for $0 \le k \le 3m-1$.

Case 2: $1 \le a \le t-2$ *and* $m = t+a$. We define a mapping f from $V(C_{3m}) \cup E(C_{3m})$ to $\{0,1,2\} \cup \{2,3,4\}$ as follows:

$$\begin{aligned}
V_0 &= (\overbrace{0,0,\ldots,0,0,\ldots,0}^{t-1=m-a-1},1)(\overbrace{0,\ldots,0,0}^{a}), \\
V_1 &= (2,\overbrace{1,\ldots,1}^{t-a-2},\overbrace{2,\ldots,2}^{a},0)(\overbrace{2,\ldots,2,2}^{a}), \\
V_2 &= (1,\overbrace{2,\ldots,2}^{t-a-2},\overbrace{1,\ldots,1}^{a},2)(\overbrace{1,\ldots,1,1}^{a}), \\
E_0 &= (\overbrace{4,4,\ldots,4,4,\ldots,4,4}^{t=m-a})(\overbrace{4,\ldots,4}^{a-1},2), \\
E_1 &= (\overbrace{3,3,\ldots,3}^{t-a-1},\overbrace{2,\ldots,2}^{a},3)(\overbrace{2,\ldots,2}^{a-1},3), \\
E_2 &= (\overbrace{2,2,\ldots,2}^{t-a-1},\overbrace{3,\ldots,3}^{a},2)(\overbrace{3,\ldots,3}^{a-1},4).
\end{aligned}$$

Note that $f(\{v_j, v_{m+j}, v_{2m+j}\}) = \{0,1,2\}$ and $f(\{e_j, e_{m+j}, e_{2m+j}\}) = \{2,3,4\}$ for $0 \le j \le m-1$. Furthermore, we can check that for $0 \le k \le 3m-1$,

$$f(v_k)+f(e_k) = \begin{cases} 5 & \text{if } k = t-1,\ m,\ 2m-1,\ 3m-1, \\ 3 & \text{if } k = m+t-1,\ 2m, \\ 2 & \text{if } k = m-1, \\ 4 & \text{otherwise} \end{cases}$$

and

$$f(v_k)+f(e_{k-1})=\begin{cases}5 & \text{if } k=t-1,\ m+t-a-1(=2t-1),\ m+t,\\ & \qquad 2m+t-1,\\ 3 & \text{if } k=2m+t-a-1(=m+2t-1),\ 2m+t,\\ 2 & \text{if } k=m+t-1,\\ 4 & \text{otherwise.}\end{cases}$$

Hence, we have $f(v_k)+f(e_k)=f(v_{k+t})+f(e_{k+t-1})$ for $0\le k\le 3m-1$.

Case 3: $1\le a\le t-2$ *and* $m>t+a$. Let $r=\frac{m-a}{t}-1$. We define a mapping f from $V(C_{3m})\cup E(C_{3m})$ to $\{0,1,2\}\cup\{2,3,4\}$ as follows:

$$\begin{aligned}
V_0 &= (\overbrace{\underbrace{0,\dots,0}_{t-a-1},0,\underbrace{0,\dots,0}_{a-1}}^{t-1},1)^r(\overbrace{0,\underbrace{0,\dots,0}_{t-a-2},\underbrace{0,\dots,0}_{a}}^{t-1},1)(\overbrace{0,\dots,0,0}^{a}),\\
V_1 &= (\overbrace{1,\dots,1}^{t-a-1},2,\overbrace{1,\dots,1}^{a-1},0)^r(2,\overbrace{1,\dots,1}^{t-a-2},\overbrace{2,\dots,2}^{a},0)(\overbrace{2,\dots,2,2}^{a}),\\
V_2 &= (\overbrace{2,\dots,2}^{t-a-1},1,\overbrace{2,\dots,2}^{a-1},2)^r(1,\overbrace{2,\dots,2}^{t-a-2},\overbrace{1,\dots,1}^{a},2)(\overbrace{1,\dots,1,1}^{a}),\\
E_0 &= (\overbrace{4,\dots,4,4,4,\dots,4,4}^{m-t-a})\ (\overbrace{4,4,\dots,4,4,\dots,4,4}^{t})(\overbrace{4,\dots,4}^{a-1},3),\\
E_1 &= (\overbrace{3,\dots,3,3,3,\dots,3}^{m-t-a-1},2)\ (\overbrace{3,3,\dots,3}^{t-a-1},\overbrace{2,\dots,2}^{a},3)(\overbrace{2,\dots,2}^{a-1},2),\\
E_2 &= (\overbrace{2,\dots,2,2,2,\dots,2}^{m-t-a-1},3)\ (\overbrace{2,2,\dots,2}^{t-a-1},\overbrace{3,\dots,3}^{a},2)(\overbrace{3,\dots,3}^{a-1},4).
\end{aligned}$$

We remark that $f(\{v_j,v_{m+j},v_{2m+j}\})=\{0,1,2\}$ and $f(\{e_j,e_{m+j},e_{2m+j}\})=\{2,3,4\}$ for $0\le j\le m-1$. Moreover, we can show that for $0\le k\le 3m-1$,

$$f(v_k)+f(e_k)=\begin{cases}5 & \text{if } k\equiv t-1 \pmod t \text{ with } t-1\le k\le m-a-1,\\ & \quad k-m\equiv t-a-1 \pmod t\\ & \qquad \text{with } m+t-a-1\le k\le 2m-t-2a-1,\\ & \quad k=2m-t-a,\ 3m-t-a-1,\ 3m-1,\\ 3 & \text{if } k-m\equiv t-1 \pmod t\\ & \qquad \text{with } m-1\le k\le 2m-2t-a-1,\\ & \quad k-2m\equiv t-a-1 \pmod t\\ & \qquad \text{with } 2m-a-1\le k\le 3m-t-2a-1,\\ & \quad k=3m-t-a,\\ 2 & \text{if } k=2m-t-a-1,\\ 4 & \text{otherwise}\end{cases}$$

and

$$f(v_k)+f(e_{k-1}) = \begin{cases} 5 & \text{if } k \equiv t-1 \pmod t \text{ with } t-1 \le k \le m-a-1, \\ & \quad k-m \equiv t-a-1 \pmod t \\ & \quad\ \text{with } m+t-a-1 \le k \le 2m-2a-1, \\ & \quad k = 2m-a,\ 3m-a-1, \\ 3 & \text{if } k-m \equiv t-1 \pmod t \\ & \quad\ \text{with } m+t-1 \le k \le 2m-t-a-1, \\ & \quad k-2m \equiv t-a-1 \pmod t \\ & \quad\ \text{with } 2m+t-a-1 \le k \le 3m-2a-1, \\ & \quad k = 3m-a, \\ 2 & \text{if } k = 2m-a-1, \\ 4 & \text{otherwise.} \end{cases}$$

Therefore, we obtain $f(v_k)+f(e_k) = f(v_{k+t})+f(e_{k+t-1})$ for $0 \le k \le 3m-1$.

Case 4: $a = t-1$. Let $r = \frac{m+1}{t} - 1$. We define a mapping f from $V(C_{3m}) \cup E(C_{3m})$ to $\{0,1,2\} \cup \{2,3,4\}$ as follows:

$$V_0 = (0, \overbrace{0,\dots,0}^{t-2}, 1)^r(\overbrace{0,\dots,0,0}^{t-1}), \quad E_0 = (\overbrace{4,4,\dots,4,4}^{m-t+1})(\overbrace{4,\dots,4}^{t-2}, 3),$$
$$V_1 = (2, \overbrace{1,\dots,1}^{t-2}, 0)^r(\overbrace{2,\dots,2,2}^{t-1}), \quad E_1 = (\overbrace{3,3,\dots,3,3}^{m-t+1})(\overbrace{2,\dots,2}^{t-2}, 2),$$
$$V_2 = (1, \overbrace{2,\dots,2}^{t-2}, 2)^r(\overbrace{1,\dots,1,1}^{t-1}), \quad E_2 = (\overbrace{2,2,\dots,2,2}^{m-t+1})(\overbrace{3,\dots,3}^{t-2}, 4).$$

Note that $f(\{v_j, v_{m+j}, v_{2m+j}\}) = \{0,1,2\}$ and $f(\{e_j, e_{m+j}, e_{2m+j}\}) = \{2,3,4\}$ for $0 \le j \le m-1$. Furthermore, we can verify that for $0 \le k \le 3m-1$,

$$f(v_k)+f(e_k) = \begin{cases} 5 & \text{if } k \equiv t-1 \pmod t \text{ with } t-1 \le k \le m-t, \\ & \quad k-m \equiv 0 \pmod t \text{ with } m \le k \le 2m-2t+1, \\ & \quad k = 3m-1, \\ 3 & \text{if } k-m \equiv t-1 \pmod t \text{ with } m-1 \le k \le 2m-t, \\ & \quad k-2m \equiv 0 \pmod t \text{ with } 2m \le k \le 3m-2t+1, \\ 4 & \text{otherwise} \end{cases}$$

and

$$f(v_k)+f(e_{k-1}) = \begin{cases} 5 & \text{if } k \equiv t-1 \pmod t \text{ with } t-1 \le k \le m-t, \\ & \quad k-m \equiv 0 \pmod t \text{ with } m \le k \le 2m-t+1, \\ 3 & \text{if } k-m \equiv t-1 \pmod t \\ & \quad\ \text{with } m+t-1 \le k \le 2m-t, \\ & \quad k-2m \equiv 0 \pmod t \\ & \quad\ \text{with } 2m \le k \le 3m-t+1, \\ 4 & \text{otherwise.} \end{cases}$$

Hence, we get $f(v_k)+f(e_k)=f(v_{k+t})+f(e_{k+t-1})$ for $0 \le k \le 3m-1$. $\square$

We are now ready to prove Theorem 1. Let $V(C_m)=\{x_i : 0 \le i \le m-1\}$ and $E(C_m)=\{x_i x_{i+1} : 0 \le i \le m-1\}$, where the subscripts are taken modulo m. Let $V(C_{3m})=\{v_i : 0 \le i \le 3m-1\}$ and $E(C_{3m})=\{e_i = v_i v_{i+1} : 0 \le i \le 3m-1\}$, where the subscripts are taken modulo $3m$. Let f be a P_t-supermagic labeling of C_m with supermagic sum $s(f)$. By Lemmas 1 and 2, there is a mapping g from $V(C_{3m}) \cup E(C_{3m})$ to $\{0,1,2\} \cup \{2,3,4\}$ such that $g(\{v_j, v_{m+j}, v_{2m+j}\}) = \{0,1,2\}$ and $g(\{e_j, e_{m+j}, e_{2m+j}\}) = \{2,3,4\}$ for $0 \le j \le m-1$, and for every subgraph H of C_{3m} isomorphic to P_t, $\sum g(H) = \frac{1}{3m}(t \cdot 3m + (t-1) \cdot 9m) = 4t-3$. For $0 \le i \le 3m-1$, let i_m be the integer such that $i \equiv i_m \pmod{m}$ with $0 \le i_m \le m-1$. We define a bijection h from $V(C_{3m}) \cup E(C_{3m})$ to $[1, 6m]$ as follows: For $0 \le i \le 3m-1$,

$$h(v_i) = f(x_{i_m}) + g(v_i) \cdot m,$$
$$h(e_i) = f(x_{i_m} x_{i_m+1}) + g(e_i) \cdot m.$$

Then, we can check that $h(V(C_{3m})) = [1, 3m]$ and $h(E(C_{3m})) = [3m+1, 6m]$, and for every subgraph H of C_{3m} isomorphic to P_t, $\sum h(H) = s(f) + m \sum g(H) = s(f) + (4t-3)m$. Therefore, h is a P_t-supermagic labeling of C_{3m}.

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