Some upper bounds for 3-rainbow index of graphs

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Abstract

A tree T, in an edge-colored graph G, is called a *rainbow tree* if no two edges of T are assigned the same color. A *k-rainbow coloring* of G is an edge coloring of G having the property that for every set S of k vertices of G, there exists a rainbow tree T in G such that $S \subseteq V(T)$. The minimum number of colors needed in a k-rainbow coloring of G is the *k-rainbow index of G*, denoted by $rx_k(G)$. In this paper, we investigate the 3-rainbow index $rx_3(G)$ of a connected graph G. For a connected graph G, it is shown that a sharp upper bound of $rx_3(G)$ is $rx_3(G[D]) + 4$, where D is a connected 3-way dominating set and a connected 2-dominating set of G. Moreover, we determine a sharp upper bound for $K_{s,t}$ $(3 \leq s \leq t)$ and a better bound for (P_5, C_5)-free graphs, respectively. Finally, a sharp bound for 3-rainbow index of general graphs is obtained.

Keywords: connectivity, edge-coloring, Steiner tree, connected dominating set, 3-rainbow index.

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1 Introduction

All graphs considered in this paper are simple, connected and undirected. We follow the terminology and notation of Bondy and Murty [1]. An edge-colored graph G is *rainbow connected* if any two vertices are connected by a path whose edges have distinct colors. The *rainbow connection number* $rc(G)$ of G, introduced by Chartrand et al. [6], is the minimum number of colors that results in a rainbow connected graph G.

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The concept of rainbow index, introduced by Chartrand et al. [7] in 2009, is a natural generalization of rainbow connection number. A tree T is a *rainbow tree* if no two edges of T are colored the same. For a fixed integer k with $2 \leq k \leq n$, an edge coloring of G with order n is called a *k-rainbow coloring* if for every set S of k vertices of G, there exists a rainbow tree in G containing the vertices of S. The *k-rainbow index* $rx_k(G)$ of G is the minimum number of colors needed in a k-rainbow coloring of G. Thus $rc(G) = rx_2(G)$. There is a rather simple and trivial bound for $rx_k(G)$ in terms of the order n of G, regardless of the value of k, i.e., $rx_k(G) \leq n-1$.

Let k be a positive integer. A dominating set D of G is called a *k-way dominating set* if $d(v) \geq k$ for every vertex $v \in V(G) \setminus D$. In addition, if $G[D]$ is connected, we call D a *connected k-way dominating set.* A subset $D \subseteq V(G)$ is a *k-dominating set* of the graph G if $|N_G(v) \cap D| \geq k$ for every $v \in V(G) \setminus D$. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality among the k-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ is the usual domination number $\gamma(G)$. Furthermore, a subset S is a *connected k-dominating set* if it is a k-dominating set and the graph induced by S is connected. The *connected k-domination number* $\gamma_k^c(G)$ denotes the cardinality of a minimum connected k-dominating set. For $k = 1$, we write γ_c instead of $\gamma_1^c(G)$.

Chakraborty et al. [4] showed that computing the rainbow connection number of a connected graph is NP-hard. So it is NP-hard to compute the k-rainbow index ($k \geq 3$) of an arbitrary connected graph as well. For this reason, one of the most important goals for studying rainbow connection number and rainbow index is to obtain good upper and lower bounds.

As with many other interesting graph coloring problems [2], rainbow connection has received lots of attention. Many results have been obtained, see [12] for a recent survey. Chartrand et al. [6] determined that for integers s and t with $2 \leq s \leq t$, $rc(K_{s,t}) = min\{\sqrt[s]{t}, 4\}$. It follows that $rx_2(K_{s,t}) = rc(K_{s,t}) \leq 4$. Caro et al. [3] observed that $rc(G)$ can be bounded by a function of $\delta(G)$, the minimum degree of G, and proved that if $\delta(G) \geq 3$ then $rc(G) \leq \alpha n$ where $\alpha \leq 1$ is a constant and $n = V(G)$. For a graph G without any cut vertex, i.e., a 2-connected graph, of order n, it was shown that $rc(G) \leq \frac{n}{2}$ and the bound is sharp in [9]. Li et al. [11] showed that for a connected graph G of order n with cut vertices, $rc(G) \leq \frac{n+r-1}{2}$, where r is the number of blocks of G with even orders, and the upper bound is sharp. Chandran et al. [5] used a strengthened connected dominating set (connected 2-way dominating set D) to prove $rc(G) \leq rc(G[D]) + 3$. This led us to the investigation of what strengthening of a connected dominating set can apply to the 3-rainbow index of a connected graph.

Compared with rainbow connection number, rainbow index is a new research subject. For the 3-rainbow index of G, Chen et al. [8] determined some basic results and obtained the following theorem.

Theorem 1.1. *[8] Let G be a 2-connected graph of order n ($n \geq 4$). Then $rx_3(G) \leq n-2$, with equality if and only if $G = C_n$ or G is a spanning subgraph of 3-sun or G is a spanning subgraph of $K_5 \setminus e$ or G is a spanning subgraph of K_4, where a 3-sun is a graph G which is defined from $C_6 = v_1v_2 \cdots v_6v_1$ by adding three edges v_2v_4, v_2v_6 and v_4v_6.*

In the same paper, Chen et al. also considered the regular complete bipartite graphs $K_{r,r}$. They showed $rx_3(K_{r,r}) = 3$ for any integer r with $r \geq 3$.

In this paper, we focus our attention on the 3-rainbow index of a connected graph G. In Section 2, we adopt connected dominating sets to investigate the 3-rainbow index of G. A coloring strategy is obtained which uses only a constant number of extra colors outside the dominating set. We prove that $rx_3(G) \leq rx_3(G[D]) + 4$, where D is a connected 3-way dominating set and a connected 2-dominating set of G, as well as obtain an upper bound for graphs with $\delta(G) \geq 3$ by the above result. Sharp upper bounds of $K_{s,t}$ ($3 \leq s \leq t$) and (P_5, C_5)-free graphs are given in Section 3. In Section 4, we show a sharp upper for $rx_3(G)$ of general connected graphs by block decomposition.

2 A sharp upper bound for 3-rainbow index of graphs in terms of connected dominating set

Chartrand et al. [6] obtained that for every nontrivial connected graph G of order n, $rx_2(G) \leq rx_3(G) \leq \cdots \leq rx_n(G)$. Since $rc(K_2) = 1$, logically, we can define $rx_3(K_2) = 1$ in our paper as well.

Theorem 2.1. *Let G be a connected graph. If D is a connected 3-way dominating set and a connected 2-dominating set of G, then $rx_3(G) \leq rx_3(G[D]) + 4$. Moreover, the bound is sharp.*

Proof. We prove the theorem by demonstrating that G has a 3-rainbow coloring with $rx_3(G[D]) + 4$ colors. For $x \in V(G) \setminus D$, its neighbors in D will be called *foots* of x, and the corresponding edges will be called *legs* of x.

Color the edges in $G[D]$ with k distinct colors from $1, 2, \ldots, k$ ($k = rx_3(G[D])$) such that for every triple of vertices in D, there exists a rainbow tree in $G[D]$ connecting them. Let $H := G[V(G) \setminus D]$. Partition $V(H)$ into sets X, Y, Z as follows. Let Z be the set of all isolated vertices of H. In every nonsingleton connected component of H, choose a spanning tree. So we construct a forest on $W := V(H) \setminus Z$ and choose X and Y as any one of the bipartitions defined by this forest. Color every $X - D$ edge with $k+1$ or $k+2$ where each of $k+1, k+2$ appears at least once at each vertex, every $Y - D$ edge with $k+1$ or $k+3$ where each of $k+1$, $k+3$ appears at least once at each vertex, and every edge between X and Y with $k+4$. Since D is a connected 3-way dominating set, every vertex in Z will have at least three legs. Color two of them with $k+1$ and $k+3$ and all the others with $k+4$. Next, we will show that under such an edge coloring, for any three vertices in G, there exists a rainbow tree containing them.

For three vertices $(x, y, z) \in D \times D \times D$, there is already a rainbow tree containing them in $G[D]$. For other cases, we first suppose that vertices in $V(H)$ do not have common foots. For three vertices $(x, y, z) \in D \times D \times V(H)$ (or $D \times V(H) \times V(H)$), join any one leg of z (or $k+1$, $k+3$ $(k+2)$ legs of y and z) with a rainbow tree containing the corresponding foot (or two foots), x and y (or x) in $G[D]$. Now we consider the case of three vertices $(x, y, z) \in V(H) \times V(H) \times V(H)$. For three vertices $(x, y, z) \in Z \times Z \times Z$, join three edges colored $k+1, k+4$ and $k+3$ with a rainbow tree containing the corresponding foots (x', y', z') in D. For two vertices $(x, y) \in Z \times Z$, $z \in W$, join a $k+1$ leg of z and $k+3$, $k+4$ legs of x, y with a rainbow tree containing the corresponding foots in $G[D]$. Consider one vertex $x \in Z$, two vertices $(y, z) \in W \times W$. If $(y, z) \in X \times X$, join a $k+4$ leg of x and $k+1, k+2$ legs of y and z with a rainbow tree containing the corresponding foots in $G[D]$. If $(y, z) \in X \times Y$ or $(y, z) \in Y \times Y$, join a $k+4$ leg of x and $k+1, k+3$ legs of y and z with a rainbow tree containing the corresponding foots in $G[D]$. Then consider three vertices $(x, y, z) \in W \times W \times W$. If $(x, y, z) \in X \times X \times X$, we know, for $x \in X$, x has a neighbor $y(x) \in Y$. Join an x-$y(x)$ edge (colored $k+4$), a $k+3$ leg of $y(x)$, a $k+1$ leg of y and a $k+2$ leg of z with a rainbow tree containing the corresponding foots in $G[D]$. Similarly, in remaining cases, we still can find a rainbow tree containing S. Note that for two vertices in $V(H) \cap S$, we can find their corresponding legs, which hold different colors. If the corresponding legs have a common foot v in D, then these two vertices can be connected by the vertex v to construct the rainbow tree containing S.

Note that $|V(G[D])| = 2$ is trivial since the edge e in D can be given a color different from colors used in $E(H) \cup E[D, V(H)]$. Hence, G has a 3-rainbow coloring with $rx_3(G[D]) + 4$ colors.

The sharpness of this theorem will be given in the next section. □

Obviously, for a connected graph G with minimum degree $\delta(G) \geq 3$, a connected 2-dominating set of G is also a connected 3-way dominating set of G. We finish this section with general graphs with minimum degree at least 3. Here, we denote as $q_{max}(G)$ the maximum number of components of $G\backslash u$ among all vertices $u \in V$. The following result is needed in the sequel.

Theorem 2.2. *[10] Let G be a connected graph on n vertices with minimum degree $\delta \geq 2$ and let k be an integer with $1 \leq k \leq \delta$. Then $\gamma_k^c \leq n - q_{max}(G)(\delta - k + 1)$*

For general graphs with $\delta \geq 3$, we obtain an upper bound for 3-rainbow index of a connected graph from Theorem 2.1 and Theorem 2.2.

Theorem 2.3. *Let G be a connected graph with minimum degree $\delta \geq 3$. Then $rx_3(G) \leq n - q_{max}(\delta - 1) + 3$.*

Note that the bound of 3-rainbow index of a connected graph is better for the graphs with cut vertices and larger minimum degree.

3 Upper bounds for 3-rainbow index of some classes of graphs

In this section, we consider two classes of graphs: complete bipartite graphs $K_{s,t}$ and (P_5, C_5)-free graphs.

Theorem 3.1. *For any complete bipartite graph $K_{s,t}$ with $3 \leq s \leq t$, $rx_3(K_{s,t}) \leq min\{6, s + t - 3\}$. Moreover, the bound is sharp.*

Proof. As $K_{s,t}$ with $3 \leq s \leq t$ is a 2-connected graph, then by Theorem 1.1, $rx_3(K_{s,t}) \leq s + t - 3$. The equality clearly holds for $s = t = 3$ since $rx_3(K_{3,3}) = 3$. Thus, to complete the proof, it suffices to show $rx_3(K_{s,t}) \leq 6$, $3 \leq s \leq t$. Let U and W be the two partite sets of $K_{s,t}$, where $|U| = s$ and $|W| = t$. Set $U = \{u_1, u_2, \ldots, u_s\}$, $W = \{w_1, w_2, \ldots, w_t\}$.

Clearly, $D = \{u_1, u_2, w_1, w_2\}$ is both a connected 2-dominating set and connected 3-way dominating set of $K_{s,t}$. Hence, by Theorem 2.1, $rx_3(K_{s,t}) \leq rx_3(G[D]) + 4 = 6$.

To show the sharpness of the above upper bound, we prove the following claim.

Claim. For any $s \geq 3$, $t > 2 \times 6^s$, $rx_3(K_{s,t}) = 6$.

Firstly, we consider the graph $K_{3,t}$. We may assume that there exists a 3-rainbow coloring c of $K_{3,t}$ with k colors. Corresponding to this 3-rainbow coloring, for every vertex w in W, there is a color code, *code*(w)=$(a_1$, a_2, $a_3)$:=$(c(u_1w)$, $c(u_2w)$, $c(u_3w))$. Observe that any three vertices have at least three distinct colors in their color codes. Thus, we know that at most two vertices have a common code except possibly when $a_1 \neq a_2 \neq a_3$. Otherwise, there is no rainbow tree containing these three vertices which have the same code and at most two colors in their color code. Therefore, when $t > 2k^3$, there must exist three vertices w', w'', w''' such that $code(w')$=$code(w'')$=$code(w''')$=(a_1, a_2, a_3) and $a_1 \neq a_2 \neq a_3$. If a rainbow tree contains $S = \{w', w'', w'''\}$, it must contain u_1, u_2, u_3 and w_i to guarantee its connectivity, where w_i belongs to W and $code(w_i)$=(b_1, b_2, b_3), where a_i, b_j are different from each other, $i = 1,2,3$; $j = 1,2,3$. Thus $k \geq 6$. So $rx_3(K_{3,t}) = 6$ for $t > 2 \times 6^3$. Similarly, we can prove $rx_3(K_{s,t}) = 6$ for $s \geq 4$, $t > 2 \times 6^s$. Thus, this claim also provides a sharp example for Theorem 2.1. □

Here, we can simply check that the upper bound cannot be generalized to the graphs $K_{2,t}$. By the same method used in the above claim, we may assume that there exists a 3-rainbow coloring c of $K_{2,t}$ with k colors. Corresponding to this 3-rainbow coloring, there is a color code, $code(w)$=(a_1, a_2) :=$(c(u_1w), c(u_2w))$. Observe that at most two vertices have the common code. It follows that $t \leq 2k^2$. Thus, k is not less than a certain constant when t is enough large.

To state next theorem, we need to introduce the following concepts. A graph G is called a *perfect connected dominant graph* if $\gamma(X) = \gamma_c(X)$, for each connected induced subgraph X of G. If G and H are two graphs, we say that G is *H-free* if H does not appear as an induced subgraph of G. Furthermore, if G is H_1-free and H_2-free, we say that G is *(H_1, H_2)-free*. We proceed with an upper bound for the 3-rainbow index of (P_5, C_5)-free graphs. Zverovich [16] obtained the following result.

Theorem 3.2. *[16] A graph G is a perfect connected-dominant graph if and only if G contains no induced path P_5 and induced cycle C_5.*

As shown in Theorem 2.1, in order to obtain a better bound of 3-rainbow index of G with $\delta(G) \geq 3$, we may turn to a smallest possible connected 2-dominating set. For a graph with minimum degree $\delta \geq 3$ and $\delta \geq 1$, Reed and Ore proved the following conclusions, respectively.

Theorem 3.3. *[15] If G is connected graph with $\delta \geq 3$, then $\gamma(G) \leq \frac{3n}{8}$.*

Theorem 3.4. *[14] If G is a graph on n vertices and without isolated vertices, then $\gamma(G) \leq \frac{n}{2}$.*

For (P_5, C_5)-free graphs with $\delta \geq 3$, it follows that $\gamma_c(G) \leq \frac{3n}{8}$ from Theorem 3.3. Inspired by this result, the extension of the idea of a connected dominating set to a connected 2-dominating set is what gives the following lemma.

Lemma 3.1. *Let G be a connected graph of order n with minimum degree $\delta \geq 2$. If D is a connected dominating set in a graph G, then there is a set of vertices $D' \supseteq D$ such that D' is a connected 2-dominating set and $|D'| \leq \frac{1}{2}n + \frac{1}{2}|D|$.*

Proof. Let Q be the set of nonisolated vertices of $V(G) \setminus D$ and let P be a minimum dominating set of $G[Q]$. Then $D' = D \cup P$ is a connected 2-dominating set of G. By Theorem 3.4, we have $|P| \leq \frac{1}{2}|V(G[Q])| \leq \frac{1}{2}|V(G)\setminus D| = (n-|D|)/2$, which implies $|D'| = |D \cup P| \leq |D| + \frac{1}{2}(n-|D|) = \frac{1}{2}n + \frac{1}{2}|D|$. □

For a connected (P_5, C_5)-free graph with $\delta \geq 3$, we can derive the following result by Theorem 2.1, Theorem 3.2, Theorem 3.3 and Lemma 3.1.

Theorem 3.5. *For every connected (P_5, C_5)-free graph G with $\delta(G) \geq 3$, $rx_3(G) \leq \frac{11}{16}n + 3$.*

Proof. For every connected (P_5, C_5)-free graph G with $\delta(G) \geq 3$, $\gamma(G) = \gamma_c(G)$ by Theorem 3.2. Also, by Theorem 3.3, we have $\gamma(G) \leq \frac{3n}{8}$. Thus, $\gamma_c(G) \leq \frac{3n}{8}$. Combining this with Lemma 3.1, G has a connected 2-dominating set D with order less than $\frac{11}{16}n$. Observe that the connected 2-dominating set D has a 3-rainbow coloring using $|D| - 1$ colors by ensuring that every edge of some spanning tree gets a distinct color. So the upper bound follows immediately from Theorem 2.1. □

4 An upper bound for 3-rainbow index of general graphs

In this section, we derive a sharp bound for 3-rainbow index of general graphs by block decomposition. Let $\mathcal{A}$ be the set of blocks of G isomorphic to K_2; let $\mathcal{B}$ be the set of blocks of G isomorphic to K_3; let $\mathcal{C}$ be the set of blocks of G isomorphic to X, which is a cycle or a block of order $4 \leq |V(X)| \leq 6$; and let $\mathcal{D}$ be the set of blocks of G isomorphic to X, which is not a cycle and $|V(X)| \geq 7$.

Theorem 4.1. *Let G be a connected graph of order n $(n \geq 3)$. If G has a block decomposition $B_1, B_2, \cdots, B_q$, then $rx_3(G) \leq n - |\mathcal{C}| - 2|\mathcal{D}| - 1$, and the upper bound is sharp.*

Proof. Let G be a connected graph of order n with q blocks in its block decomposition. If $q = 1$, then it was done by Theorem 1.1 and $rx_3(K_3) = 2$, which satisfies the above bound. Thus, we can suppose $q \geq 2$.

Note that $|\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}| = q$. By Theorem 1.1, we get $rx_3(X) \leq |X| - 2$ for $X \in \mathcal{C}$ and $rx_3(X) \leq |X| - 3$ for $X \in \mathcal{D}$. Hence, it follows that

$$\begin{aligned} rx_3(G) &\leq \sum_{X \in \mathcal{A}} 1 + \sum_{X \in \mathcal{B}} 2 + \sum_{X \in \mathcal{C}} (|X| - 2) + \sum_{X \in \mathcal{D}} (|X| - 3) \\ &= n - |\mathcal{C}| - 2|\mathcal{D}| - 1. \end{aligned}$$

In order to prove that the upper bound is sharp, we construct the graph G of order n, as shown in Figure 1, consisting of a path of length $n - 4r - 7$, r cycles of order 4, one 7-length-cycle with a chord and an edge connecting every previous cycle and the subsequent one. As we can see, $|\mathcal{C}| = r$, $|\mathcal{D}| = 1$. We consider the size of a rainbow tree T contain the vertices $\{u, v, w\}$. Since $|E(T)| \geq n - 4r - 7 + 3r + 4 = n - r - 3$ and $rx_3(G) \leq n - |\mathcal{C}| - 2|\mathcal{D}| - 1 = n - r - 3$ by Theorem 4.1, it follows that $rx_3(G) = n - |\mathcal{C}| - 2|\mathcal{D}| - 1$. □

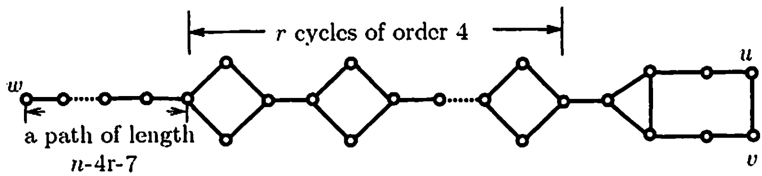

Figure 1: Graph for Theorem 4.1

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