Exact Values for the ε-Ascent Chromatic Index of Complete Graphs

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Abstract

Following a problem introduced by Schurch [M. Schurch, *On the Depression of Graphs*, Doctoral Dissertation, University of Victoria, 2013], we find exact values of the minimum number of colours required to properly edge colour K_n, $n \geq 6$, using natural numbers, such that the length of a shortest maximal path of increasing edge labels is equal to three. This result improves the result of Breytenbach and Mynhardt [A. Breytenbach and C. M. Mynhardt, On the ε-Ascent Chromatic Index of Complete Graphs, *Involve*, to appear].

1 Introduction

An edge ordering of a graph $G = (V, E)$ is an injection $f : E \to \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of positive integers. A path $v_1, e_1, \ldots, e_{k-1}, v_k$ (with $v_1 \neq v_k$) in G for which the edge ordering $f(e_1) < \cdots < f(e_{k-1})$ increases along its edge sequence is called an *f-ascent*; an f-ascent is maximal if it is not contained in a longer f-ascent. The *flatness* of f, denoted $h(f)$, is the length of a shortest maximal ascent. The *depression* $\varepsilon(G)$ of G is the smallest integer k such that any edge ordering f has a maximal f-ascent of length at most k.

An edge ordering for a graph G is also a proper edge colouring: no two adjacent edges have the same label. The minimum number of labels, or colours, in a proper edge colouring is called the *edge chromatic number* or the *chromatic index* $\chi'(G)$. The minimum number of colours in a proper

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edge colouring c such that $h(c) = \varepsilon(G)$ is called the *ε-ascent chromatic index* of G and is denoted by $\chi_\varepsilon(G)$.

As shown in [3], $\varepsilon(K_n) = 3$ for all $n \geq 4$. This fact prompted Schurch [4, 5] to introduce the following problem, where $r(n)$ is the same as $\chi_\varepsilon(K_n)$:

Question 1. For $n \geq 4$, what is the smallest integer $r(n)$ for which there exists a proper edge colouring of K_n in colours $1, \ldots, r(n)$ such that a shortest maximal path of increasing edge labels has length three?

Schurch [4, 5] showed that $r(n) \leq 2n - 3$ for all $n \geq 4$, which allowed him to determine $r(n)$ for $n \in \{4, 5\}$ as well as bound the value of $r(6)$. In [1], Breytenbach and Mynhardt provided a lower bound for $r(n) = \chi_\varepsilon(K_n)$:

Theorem 2. *If $n \geq 4$, then*

$$\chi_\varepsilon(K_n) \geq \begin{cases} n & \text{if } n \equiv 0 \pmod 4 \\ n+1 & \text{if } n \equiv 1,2 \pmod 4 \\ n+2 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Further, they improved the general upper bound to $r(n) \leq \lceil \frac{3n-3}{2} \rceil$. For even n, they provided better bounds: in the case $n \equiv 2 \pmod 4$ they show that $r(n) = n + 1$, and in the case $n \equiv 0 \pmod 4$ they show that $n \leq r(n) \leq n + 1$. Using these bounds, they also achieve $r(7) = 9$. Breytenbach and Mynhardt conclude with the following conjecture:

Conjecture 3. For all $n \geq 4$, $\chi_\varepsilon(K_n) = \chi'(K_n) + 2$.

Since it is well known (see, e.g. [2], Section 10.2) that $\chi'(K_{2n}) = 2n - 1$ and $\chi'(K_{2n+1}) = 2n + 1$, a colouring of K_8 that illustrates $\chi_\varepsilon(K_8) \leq 8$ is a counter-example to the conjecture, and similarly a colouring of K_9 that illustrates $\chi_\varepsilon(K_9) \leq 10$ is another. For K_8 such a colouring is contained in the proof for the case $n \equiv 0 \pmod 4$ in Section 2.1, for K_9, such a colouring is provided in Appendix B as Figure 5; it was verified by computer that this colouring has flatness equal to three. Motivated by these counter-examples, we determine the exact values of χ_ε for all $n \geq 4$.

2 Improved Upper Bounds

For $n \geq 6$, we show the existence of proper edge colourings with flatness equal to three and with the number of colours equal to the lower bound on $\chi_\varepsilon(K_n)$ established by Breytenbach and Mynhardt [1], and thus complete the computation of the exact value of $\chi_\varepsilon(K_n)$ for all n. The value was determined exactly for $n \leq 5$ in [4, 5] and for $n = 7$ and $n \equiv 2 \pmod 4$

in [1]. We consider $n \equiv 0 \pmod 4$ in Section 2.1, $n \equiv 1 \pmod 4$ in Section 2.2, and $n \equiv 3 \pmod 4$ in Section 2.3. In each case, we make use of the following fact [1].

Fact 4. *To prove that $h(c) = 3$, where c is a proper edge colouring of K_n, it is sufficient to prove the following statement:*

S: *For any $y \in V(K_n)$ and edges $e = xy$ and $f = yz$ such that $c(e) < c(f)$, there exists*

(a) *an edge $tx, t \notin \{x, y, z\}$, such that $c(tx) < c(e)$, or*

(b) *an edge $zt, t \notin \{x, y, z\}$, such that $c(f) < c(zt)$.*

2.1 The case $n \equiv 0 \pmod 4$

Say $n = 4m$, $m \geq 2$, and $V(K_n) = \{v_0, \ldots, v_{4m-1}\}$. Let G and H be the subgraphs of K_n induced by $\{v_0, \ldots, v_{2m-1}\}$ and $\{v_{2m}, \ldots, v_{4m-1}\}$, respectively. Then $G \cong H \cong K_{2m}$ and each of them is $(2m-1)$-edge colourable. We describe a colouring c of K_n in the colours $1, \ldots, 4m$ as follows.

- In G, let c be any proper edge colouring of K_{2m} in the $2m-1$ colours $\{1\} \cup \{m+2, \ldots, 3m-1\}$.
- In H, let c be any proper edge colouring of K_{2m} in the $2m-1$ colours $\{4m\} \cup \{m+2, \ldots, 3m-1\}$.
- We still need to colour the edges of the complete bipartite graph $F \cong K_{2m,2m}$ induced by the edges $v_i v_j, i \in \{0, \ldots, 2m-1\}, j \in \{2m, \ldots 4m-1\}$. But $\chi'(K_{2m,2m}) = 2m$ and there are $2m$ unused colours $2, \ldots, m+1$ and $3m, \ldots, 4m-1$. Colour the edges of F with these colours in such a way that the graph induced by the edges assigned the colours $1, 2, 4m-1, 4m$ is triangle free. This can be achieved by partitioning the vertices of K_n into sets X and Y such that each edge labelled with 1, 2, $4m-1$, and $4m$ has an end in X and an end in Y, which is possible as $m \geq 2$. Then the graph induced by the edges assigned the colours $1, 2, 4m-1, 4m$ is bipartite, and hence triangle free.

As an example, a colouring of K_8 is given in Figure 1. It is clear that c is a proper edge colouring of K_{4m} in $4m$ colours.

Theorem 5. *For all $m \geq 2$, the colouring c of K_{4m} has flatness equal to three.*

Proof. Let F, G, and H be the subgraphs of K_{4m} defined above and let $e, f \in E(K_{4m})$ be adjacent edges such that $c(e) < c(f)$. By Fact 4, it is

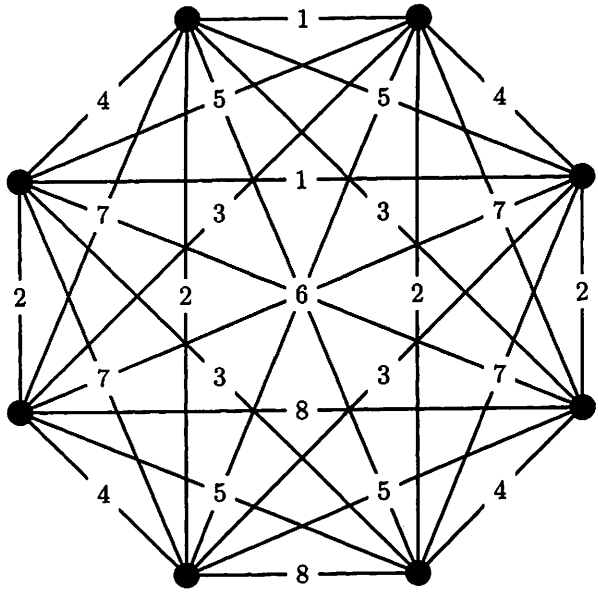


Figure 1: Edge colouring c of K_8

sufficent to show $\mathbf{S(a)}$ or $\mathbf{S(b)}$ holds. Let $e = v_j v_i$ and $f = v_i v_k$. Observe at each vertex, every colour is used at an incident edge, except exactly one of 1 or $4m$ is used. Thus if $c(e) \geq 4$, at least one of 2, 3 is not used to colour edge $v_j v_k$, and thus v_j is adjacent to some vertex v_l, $l \neq k$, such that $c(v_l v_j) < c(e)$, so $\mathbf{S(a)}$ holds. Similarly, if $c(f) \leq 4m-3$, at least one of $4m-2, 4m-1$ is not used to colour edge $v_k v_j$, and thus v_k is adjacent to some vertex v_l, $l \neq j$, such that $c(f) < c(v_k v_l)$, so $\mathbf{S(b)}$ holds. Thus we consider $c(e) \in \{1,2,3\}$ and $c(f) \in \{4m-2, 4m-1, 4m\}$.

Suppose $c(e) = 1$. By construction, $c(f) \neq 4m$. Thus $c(f) \in \{4m-2, 4m-1\}$, so $e \in E(G)$ and $f \in E(F)$. Thus there exists an edge $v_k v_l$ such that $c(v_k v_l) = 4m$. As $j \neq l$, $\mathbf{S(b)}$ holds. Similarly, if $c(f) = 4m$, $c(e) \in \{2,3\}$, so $f \in E(H)$ and $e \in E(F)$. Thus there exists an edge $v_l v_j$ such that $c(v_l v_j) = 1$. As $j \neq l$, $\mathbf{S(a)}$ holds.

Now we consider $c(e) \in \{2,3\}$ and $c(f) \in \{4m-2, 4m-1\}$, and thus $e, f \in E(F)$. If $c(e) = 3$, then there exists an edge $v_l v_j \in E(F)$ such that $c(v_l v_j) = 2$ and $l \neq k$ as F is bipartite. Thus $\mathbf{S(a)}$ holds. Similarly, if $c(f) = 4m-2$, then there exists an edge $v_k v_l \in E(F)$ such that $c(v_k v_l) = 4m-1$ and $l \neq k$ as F is bipartite. Thus $\mathbf{S(b)}$ holds.

Finally, we consider $c(e) = 2$ and $c(f) = 4m-1$. If $v_i \in V(G)$, then there exists an edge $v_k v_l \in E(H)$ such that $c(v_k v_l) = 4m$ and $l \neq j$ by construction, so **S(b)** holds. If $v_i \in V(H)$, then there exists an edge $v_l v_j \in E(G)$ such that $c(v_l v_j) = 1$ and $l \neq k$ by construction, so **S(a)** holds. □

Thus we conclude the following.

Corollary 6. *For all $n \geq 8$ and $n \equiv 0 \pmod 4$, $\chi_\epsilon(K_n) = n$.*

2.2 The case $n \equiv 1 \pmod 4$

Say $n = 4m + 1$, $m \geq 3$, and $V(K_n) = \{u_0, \dots, u_{2m-1}, v_0, \dots, v_{2m-1}, w\}$. Let G and H be the subgraphs of K_n induced by $\{u_0, \dots, u_{2m-1}\}$ and $\{v_0, \dots, v_{2m-1}\}$, respectively. Then $G \cong H \cong K_{2m}$ and each of them is $(2m-1)$-edge colourable. Let F be the subgraph of K_n induced by the edges $u_i v_j, 0 \leq i, j \leq 2m-1$. Then $F \cong K_{2m,2m}$ and is $2m$-edge colourable. Let c be a colouring of $K_n - w$ with the following colour classes:

F: For $0 \leq k \leq 2m-1$, let $E_k^F = \{u_i v_{i+k} : 0 \leq i \leq 2m-1\}$, indices taken mod $2m$.

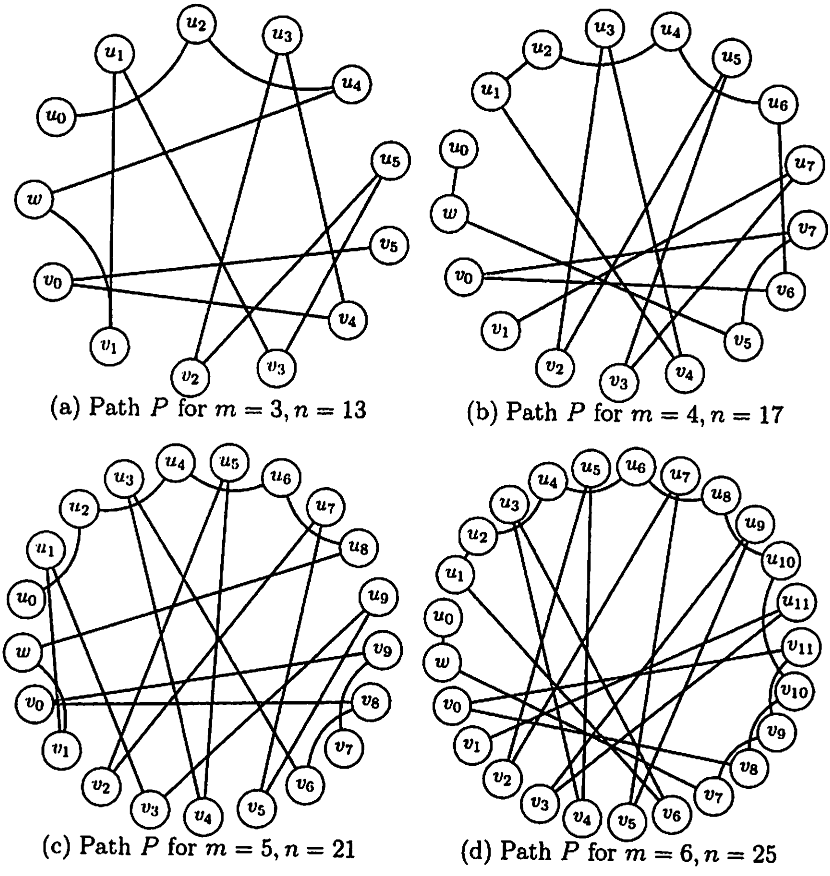
G and H: Let $\{a_j\}, 0 \leq j \leq 2m-2$, be the sequence $2m-2, 2m-4, \dots, 2, 1, 3, \dots, 2m-1$. For $0 \leq k \leq 2m-2$, let $E_k^G = \{u_0 u_{a_k}\} \cup \{u_{a_{k-i}} u_{a_{k+i}} : 1 \leq i \leq m-1\}$ and $E_k^H = \{v_0 v_{a_k}\} \cup \{v_{a_{k-i}} v_{a_{k+i}} : 1 \leq i \leq m-1\}$, indices taken mod $2m-1$.

Later we will pair the colour classes of G and H to get exactly $4m-1$ colours. We form the colouring c^* of K_n as follows. Assume c uses the colours $1, \dots, 2m, 2m+3, \dots, 4m+2$. Define the path P as follows: if $m \equiv 1 \pmod 2$, $P = u_0, u_2, \dots, u_{2m-2}, w, v_1, u_1, v_3, u_{2m-1}, v_5, u_{2m-3}, \dots, v_m, u_{m+2}, v_2, u_m, v_4, \dots, u_3, v_{m+1}, v_{m+3}, \dots, v_{2m-2}, v_0, v_{2m-1}, v_{2m-3}, \dots, v_{m+2}$, and if $m \equiv 0 \pmod 2$, $P = v_1, u_{2m-1}, v_3, u_{2m-3}, \dots, v_{m-1}, u_{m+1}, v_2, u_{m-1}, v_4, u_{m-3}, \dots, v_m, u_1, u_2, u_4, \dots, u_{2m-2}, v_{2m-2}, v_{2m-4}, \dots, v_{m+2}, v_0, v_{2m-1}, v_{2m-3}, \dots, v_{m+1}, w, u_0$. For small values of m, the path P is shown in Figure 2. For each edge xy of P, if xy occurs before w in the path, let $c^*(xw) = c(xy)$, otherwise, if xy occurs after w in the path, let $c^*(yw) = c(xy)$. Then if the edges on the path are enumerated, let $c^*(xy) = 2m+1$ if xy is an odd edge, otherwise let $c^*(xy) = 2m+2$ if xy is an even edge. Finally, if $e \in E(K_n - w)$ is not in P, let $c^*(e) = c(e)$.

For c^* to be a proper colouring of K_n, each edge of P must belong to a different colour class in c. We prove this statement in the following claim.

Claim 7. *Each edge in P that does not have w as an endpoint belongs to a different colour class.*

Proof. We consider each of the two paths separately.



(a) Path P for $m = 3, n = 13$

(b) Path P for $m = 4, n = 17$

(c) Path P for $m = 5, n = 21$

(d) Path P for $m = 6, n = 25$

Figure 2: The path P for small values of m

$m \equiv 1 \pmod 2$: The first $m-1$ edges are contained in G. The edge u_0u_2 is in colour class E^G_{m-1} and each edge of the form $u_{2j}u_{2(j+1)}, 1 \le j \le m-2$ is in colour class E^G_{2m-j-2}. The next two edges are incident with w. The following $2m$ edges are contained in F. Each edge of the form $v_{2j-1}u_{2m-2j+3}, 1 \le j \le \frac{m+1}{2}$ is in the colour class E^F_{4j-4}, each edge of the form $u_{2m-2j+3}v_{2j+1}, 1 \le j \le \frac{m-1}{2}$ is in the colour class E^F_{4j-2}, each edge of the form $u_{m-2j+4}v_{2j}, 1 \le j \le \frac{m+1}{2}$ is in the colour class E^F_{4j+m-4}, and each edge of the form $v_{2j}u_{m-2j+2}, 1 \le j \le \frac{m-1}{2}$ is in the colour class E^F_{4j+m-2}. The final $m-1$ edges are contained in H. Each edge of the form $v_{m+2j-1}v_{m+2j+1}, 1 \le j \le \frac{m-3}{2}$ is in the colour class $E^H_{\frac{3m-2j-3}{2}}$, the edge $v_{2m-2}v_0$ is in the colour class E^H_0, the edge v_0v_{2m-1} is in the colour class E^H_{2m-2}, and each edge of the form $v_{2m-2j+1}v_{2m-2j-1}, 1 \le j \le \frac{m-3}{2}$ is in the colour class E^H_{m-1-j}.

No two edges taken from F belong to the same colour class as the first half are consecutive even numbered colour classes, and the second half are consecutive odd numbered colour classes, mod $2m$, starting with m. No two edges taken from G belong to the same colour class as $m-1 = 2m-j-2 \implies j = m-1$. Finally, no two edges taken from H belong to the same colour class as $0 < \frac{m+1}{2}$, $m-2 < m$, and $\frac{3m-5}{2} < 2m-2$ for positive m.

$m \equiv 0 \pmod 2$: The first $2m-1$ edges are contained in F. Each edge of the form $v_{2j-1}u_{2m-2j+1}, 1 \le j \le \frac{m}{2}$ is in the colour class E^F_{4j-2}, each edge of the form $u_{2m-2j+1}v_{2j+1}, 1 \le j \le \frac{m-2}{2}$ is in the colour class E^F_{4j}, each edge of the form $u_{m-2j+3}v_{2j}, 1 \le j \le \frac{m}{2}$ is in the colour class E^F_{4j-m-3} and each edge of the form $v_{2j}u_{m-2j+1}, 1 \le j \le \frac{m}{2}$ is in the colour class E^F_{4j-m-1}. The next $m-1$ edges are contained in G. The edge u_1u_2 is in the colour class E^G_{2m-2} and each edge of the form $u_{2j}u_{2(j+1)}, 1 \le j \le m-2$ is in colour class E^G_{2m-j-2}. The following edge, $u_{2m-2}v_{2m-2}$ is contained in F and is in colour class E^F_0. The next $m-1$ edges are contained in H. Each edge of the form $v_{2m-2j}v_{2m-2j-2}, 1 \le j \le \frac{m-4}{2}$ is in the colour class E^H_{m+j-2}, the edge $v_{m+2}v_0$ is in the colour class $E^H_{\frac{m-4}{2}}$, the edge v_0v_{2m-1} is in the colour class E^H_{2m-2}, and each edge of the form $v_{2m-2j+1}v_{2m-2j-1}, 1 \le j \le \frac{m-2}{2}$ is in the colour class E^H_{m-1-j}. The final two edges are incident with w.

No two edges taken from F belong to the same colour class as the first half are consecutive even numbered colour classes starting from 2, the second half are consecutive odd numbered colour classes, mod $2m$, starting with $1-m \equiv m+1$, and the final edge is in the colour class 0. No two edges taken from G belong to the same colour class as

$2m-2 = 2m-j-2 \implies j = 0$. Finally, no two edges taken from H belong to the same colour class as $\frac{m-4}{2} < \frac{m}{2}$, $m-2 < m-1$, and $\frac{3m-8}{2} < 2m-2$ for positive m.

□

Therefore, c^* is a proper colouring of K_n. It remains to show that there is a colouring c which, when extended to c^*, allows us to avoid maximal 2-ascents. We assign the colours to the colour classes in the following manner.

- Let E^G_{m-1} be assigned colour 1 and E^H_{m-1} be assigned colour $4m+2$.
- If $m \equiv 0, 1 \pmod 4$, let E^F_0 be assigned colour 2, E^F_1 be assigned colour 3, E^F_2 be assigned colour $4m$, and E^F_3 be assigned colour $4m+1$. If $m \equiv 2, 3 \pmod 4$, let E^F_{2m-1} be assigned colour 2, E^F_0 be assigned colour 3, E^F_1 be assigned colour $4m$, and E^F_2 be assigned colour $4m+1$. As a result, when c is extended to c^*, the edges incident with w assigned 2 and 3 have their other endpoint in $V(G)$, and the edges incident with w assigned $4m$ and $4m+1$ have their other endpoint in $V(H)$. Assign the remaining colour classes of F from the colours $\{4, \ldots, m+1, 3m+2, \ldots, 4m-1\}$.
- For the remaining colour classes of G and H, assign from the colours $\{m+2, \ldots, 2m, 2m+3, \ldots, 3m+1\}$ such that each colour class with an edge in P is assigned a different colour.

As an example, a colouring of K_{13} is given in Figure 4 in Appendix A. Note that this proof cannot be applied to K_9. In place of a proof of this small case, we used a computer to search[1] for a 10-colouring of K_9 with flatness three, and a result is shown in Figure 5 in Appendix B.

Theorem 8. *For all $m \geq 3$, the colouring c^* of K_{4m+1} has flatness equal to three.*

Proof. Let F, G, and H be the subgraphs of K_{4m+1} defined above, let W be the subgraph induced by the edges incident with w, and let $e, f \in E(K_{4m+1})$ be adjacent edges such that $c^*(e) < c^*(f)$. By Fact 4, it is sufficent to show **S(a)** or **S(b)** holds. Let $e = xy$ and $f = yz$. Observe that at each vertex, exactly two colours are not incident with it, at least one of which is either 1 or $4m+2$. Thus if $c^*(e) \geq 5$, at least two of $2, 3, 4$ are incident with x, and thus x is adjacent to some $t \neq z$ such that $c^*(tx) < c^*(e)$, so **S(a)** holds. Similarly, if $c^*(f) \leq 4m-2$, at least two of $4m-1, 4m, 4m+1$ are incident with z, and thus z is adjacent to some $t \neq x$ such that $c^*(f) < c^*(zt)$, so **S(b)** holds. Thus we consider $c^*(e) \in \{1, 2, 3, 4\}$ and $c^*(f) \in \{4m-1, 4m, 4m+1, 4m+2\}$.

[1]The code is available at: `http://www.math.uvic.ca/~jgorzny/ascent/`

Suppose $c^*(e) = 1$. By construction, $c^*(f) \neq 4m+2$. Thus $c^*(f) \in \{4m-1, 4m, 4m+1\}$, so $e \in E(G)$ and $f \in E(F) \cup E(W)$. If $f \in E(W)$, then $c^*(f) = 4m-1$ and there exists an edge wt such that $c^*(wt) = 4m+1$. Otherwise, $f \in E(F)$, and there exists an edge zt such that $c^*(zt) = 4m+2$. As $t \neq x$ in either case, $\mathbf{S(b)}$ holds. Similarly, if $c^*(f) = 4m+2$, $c^*(e) \in \{2,3,4\}$, so $f \in E(H)$ and $e \in E(F) \cup E(W)$. If $e \in E(W)$, then $c^*(e) = 4$, and there exists an edge wt such that $c^*(wt) = 2$, otherwise, $e \in E(F)$, and there exists an edge tx such that $c^*(tx) = 1$. As $t \neq z$ in either case, $\mathbf{S(a)}$ holds.

Now we consider $c^*(e) \in \{2,3,4\}$ and $c^*(f) \in \{4m-1, 4m, 4m+1\}$, and thus $e, f \in E(F) \cup E(W)$. If $c^*(e) = 4$ and x is incident with both colours 2 and 3, it is clear that $\mathbf{S(a)}$ holds. If x is incident with only one of the colours 2 and 3, then $x \in V(H)$. If $y = w$, then $c^*(yz) = 4m-1$, and z is incident with at least one of the colours $4m, 4m+1$. Therefore, at least one of these four colours is incident with x or z but not assigned to xz, so there exists some t such that either $t \neq z$ and $c^*(tx) < c^*(e)$ or $t \neq x$ and $c^*(f) < c^*(zt)$, so either $\mathbf{S(a)}$ or $\mathbf{S(b)}$ holds. If $y \neq w$ then $y \in V(G)$ and $z \in V(H) \cup \{w\}$. Thus there exists an edge $tx \in E(F)$ such that $c^*(tx) \in \{2,3\}$ and $t \neq z$ as $t \in V(G)$; hence $\mathbf{S(a)}$ holds. Similarly, if $c^*(f) = 4m-1$, then clealy $\mathbf{S(b)}$ holds unless z is incident with only one of the colours $4m, 4m+1$, in which case $z \in V(G)$. We have shown $\mathbf{S(a)}$ or $\mathbf{S(b)}$ holds if $c^*(e) = 4$, thus $y \neq w$. Hence, $y \in V(H)$ and $x \in V(G)$, and there exists an edge $zt \in E(F)$ such that $c^*(zt) \in \{4m, 4m+1\}$ and $t \neq x$ as $t \in V(H)$ so $\mathbf{S(b)}$ holds.

We now consider $c^*(e) \in \{2,3\}$ and $c^*(f) \in \{4m, 4m+1\}$. If $y = w$, then $x \in V(G)$, $z \in V(H)$, and there is an edge $tx \in E(G)$ such that $c^*(tx) = 1$ and as $t \neq z$, $\mathbf{S(a)}$ holds. If $x = w$, then $y \in V(G)$, $z \in V(H)$, and there is an edge $zt \in E(H)$ such that $c^*(zt) = 4m+2$ and as $t \neq x$, $\mathbf{S(b)}$ holds. Similarly, if $z = w$, then $y \in V(H)$, $x \in V(G)$, and there is an edge $tx \in E(G)$ such that $c^*(tx) = 1$ and as $t \neq z$, $\mathbf{S(a)}$ holds.

Otherwise, $e, f \in E(F)$. Suppose $c^*(e) = 2$. If $x, z \in V(G)$, let $x = u_i$. Either $z = u_{i-2}$ or $z = u_{i-3}$, and as either $c^*(u_{i-1}u_i) = 1$ or $c^*(u_{i+1}u_i) = 1$, then $\mathbf{S(a)}$ holds. Otherwise, if $x, z \in V(H)$, let $z = v_i$. Either $x = v_{i-2}$ or $x = v_{i-3}$, and as either $c^*(v_{i-1}v_i) = 4m+2$ or $c^*(v_{i+1}v_i) = 4m+2$, then $\mathbf{S(b)}$ holds. Similarly, if $c^*(f) = 4m+1$, either $x, z \in V(G)$ and $\mathbf{S(a)}$ holds or $x, z \in V(H)$ and $\mathbf{S(b)}$ holds.

Finally, we consider $c^*(e) = 3$ and $c^*(f) = 4m$. If $x, z \in V(G)$, then there is an edge tx such that $c^*(tx) = 2$ and $t \neq z$, so $\mathbf{S(a)}$ holds. Otherwise, $x, z \in V(H)$, and there is an edge zt such that $c^*(zt) = 4m+1$ and $t \neq x$, so $\mathbf{S(b)}$ holds. □

Thus we conclude the following.

Corollary 9. *For all* $n \geq 13$ *and* $n \equiv 1 \pmod 4$, $\chi_\varepsilon(K_n) = n+1$.

2.3 The case $n \equiv 3 \pmod 4$

Say $n = 4m + 3$, $m \geq 1$, and $V(K_n) = \{v_0, \ldots, v_{4m+2}\}$. Let G and H be the subgraphs of K_n induced by $\{v_0, \ldots, v_{2m}\}$ and $\{v_{2m+1}, \ldots, v_{4m+2}\}$, respectively. Then $G \cong K_{2m+1}$, $H \cong K_{2m+2}$, and each of them is $(2m+1)$-edge colourable. We describe a colouring c of K_n in the colours $1, \ldots, 4m+5$ as follows.

- In G, let c be any proper edge colouring of K_{2m+1} in the $2m + 1$ colours $\{1, 2\} \cup \{m + 4, \ldots, 3m + 2\}$.
- In H, let c be any proper edge colouring of K_{2m+2} in the $2m + 1$ colours $\{4m + 4, 4m + 5\} \cup \{m + 4, \ldots, 3m + 2\}$.
- We still need to colour the edges of the complete bipartite graph $F \cong K_{2m+1,2m+2}$ induced by the edges v_iv_j, $i \in \{0, \ldots, 2m\}$, $j \in \{2m + 1, \ldots, 4m + 2\}$. But $\chi'(K_{2m+1,2m+2}) = 2m + 2$ and there are $2m + 2$ unused colours $3, \ldots, m + 3$ and $3m + 3, \ldots, 4m + 3$. Colour the edges of F with these colours such that the following conditions are satisfied:
    - Let $v_i \in V(G)$ be the vertex incident with no edge labelled 2. If $v_j \in V(G)$ such that $c(v_iv_j) = 1$ and $v_k \in V(H)$ such that $c(v_iv_k) = 3$, then $c(v_jv_k) \neq 4m + 3$.
    - Let $v_p \in V(G)$ be the vertex incident with no edge labelled 1. If $v_q \in V(G)$ such that $c(v_pv_q) = 2$ and $v_r \in V(H)$ such that $c(v_pv_r) = 3$, then $c(v_qv_r) \neq 4m + 3$.

  Such a colouring is easily found by arbitrarily assigning a proper colouring to F, and switching two colour classes if one of the two conditions is violated (there are at least four colour classes in F as $m \geq 1$).

As an example, a colouring of K_7 is given in Figure 3. It is clear that c is a proper edge colouring of K_{4m+3} in $4m + 5$ colours.

Theorem 10. *For all $m \geq 1$, the colouring c of K_{4m+3} has flatness equal to three.*

Proof. Let F, G, and H be the subgraphs of K_{4m+3} defined above and let $e, f \in E(K_{4m+3})$ be adjacent edges such that $c(e) < c(f)$. By Fact 4, it is sufficent to show **S(a)** or **S(b)** holds. Let $e = v_jv_i$ and $f = v_iv_k$. Observe that at each vertex, exactly two colours do not appear as colours of edges incident with it, at least one of which is either 1 or $4m + 5$. Thus if $c(e) \geq 5$, at least two of $2, 3, 4$ are incident with v_j, and thus v_j is adjacent to some vertex $v_l \neq v_k$ such that $c(v_lv_j) < c(e)$, so **S(a)** holds.

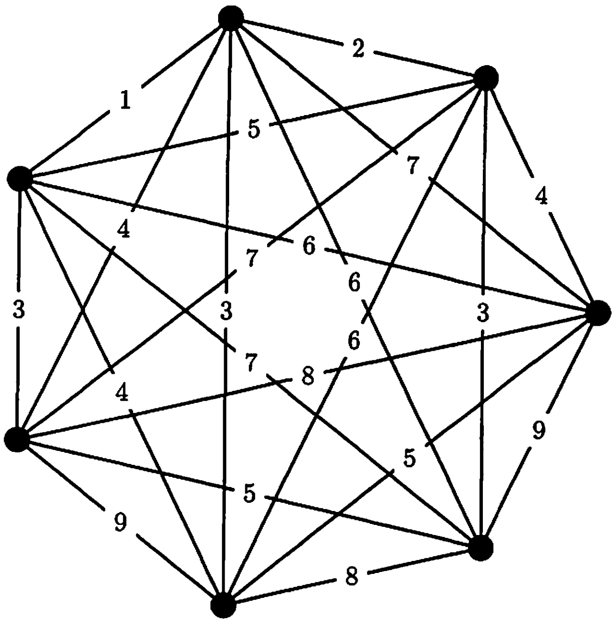


Figure 3: Edge colouring c of K_7

Similarly, if $c(f) \leq 4m+1$, at least two of $4m+2, 4m+3, 4m+4$ are incident with v_k, and thus v_k is adjacent to some vertex $v_l \neq v_j$ such that $c(f) < c(v_k v_l)$, so **S(b)** holds. Thus we consider $c(e) \in \{1,2,3,4\}$ and $c(f) \in \{4m+2, 4m+3, 4m+4, 4m+5\}$.

Suppose $c(e) \in \{1,2\}$. By construction, $c(f) \notin \{4m+4, 4m+5\}$. Thus $c(f) \in \{4m+2, 4m+3\}$, so $e \in E(G)$ and $f \in E(F)$. Thus there exists an edge $v_k v_l$ such that $c(v_k v_l) \in \{4m+4, 4m+5\}$. As $j \neq l$, **S(b)** holds. Similarly, if $c(f) \in \{4m+4, 4m+5\}$, $c(e) \in \{3,4\}$, so $f \in E(H)$ and $e \in E(F)$. Thus there exists an edge $v_l v_j$ such that $c(v_l v_j) \in \{1,2\}$. As $j \neq l$, **S(a)** holds.

Now we consider $c(e) \in \{3,4\}$ and $c(f) \in \{4m+2, 4m+3\}$, and thus $e, f \in E(F)$. If $c(e) = 4$, then either there exists an edge $v_l v_j \in E(F)$ such that $c(v_l v_j) = 3$ and $l \neq k$ as F is bipartite, so **S(a)** holds; otherwise v_j and v_k are both in $V(H)$, and there exists a vertex v_p, $p \neq j$, such that $c(v_k v_p) \in \{4m+4, 4m+5\}$ and **S(b)** holds. Similarly, if $c(f) = 4m+2$, then either there exists an edge $v_k v_l \in E(F)$ such that $c(v_k v_l) = 4m+3$ and $l \neq k$ as F is bipartite, so **S(b)** holds; otherwise v_j and v_k are both in $V(H)$, and there exists a vertex v_p, $p \neq j$, such that $c(v_k v_p) \in \{4m+4, 4m+5\}$ and **S(b)** holds.

Finally, we consider $c(e) = 3$ and $c(f) = 4m+3$. If $v_i \in V(G)$, then there exists an edge $v_k v_l \in E(H)$ such that $c(v_k v_l) \in \{4m+4, 4m+5\}$ and $l \neq j$, so **S(b)** holds. If $v_i \in V(H)$, then there exists an edge $v_l v_j \in E(G)$ such that $c(v_l v_j) \in \{1, 2\}$ and $l \neq k$ by construction, so **S(a)** holds. ☐

Thus we conclude the following.

Corollary 11. *For all* $n \geq 7$ *and* $n \equiv 3 \pmod 4$, $\chi_\varepsilon(K_n) = n+2$.

3 Conclusion

From Corollaries 6, 9, and 11, together with previous results, we obtain exact values of χ_ε for all $n \geq 6$:

Theorem 12. *If* $n \geq 6$, *then*

$$\chi_\varepsilon(K_n) = \begin{cases} n & \text{if } n \equiv 0 \pmod 4 \\ n+1 & \text{if } n \equiv 1,2 \pmod 4 \\ n+2 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

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A K_{13}

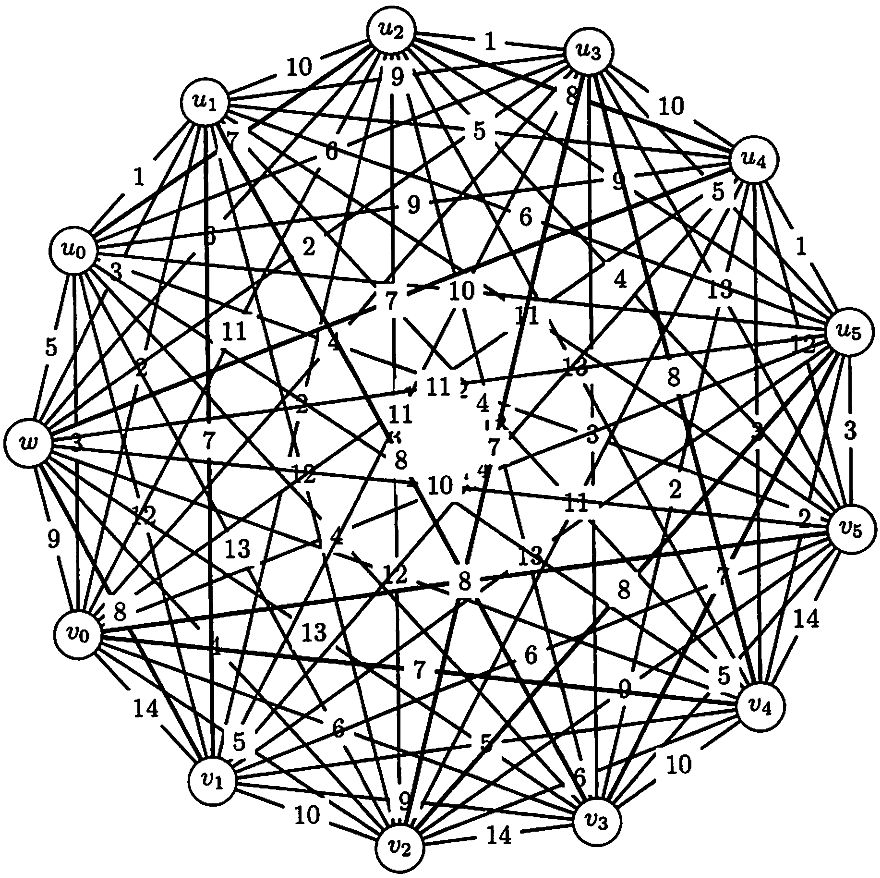

Figure 4: Edge colouring c^* of K_{13} with flatness three.

B K_9

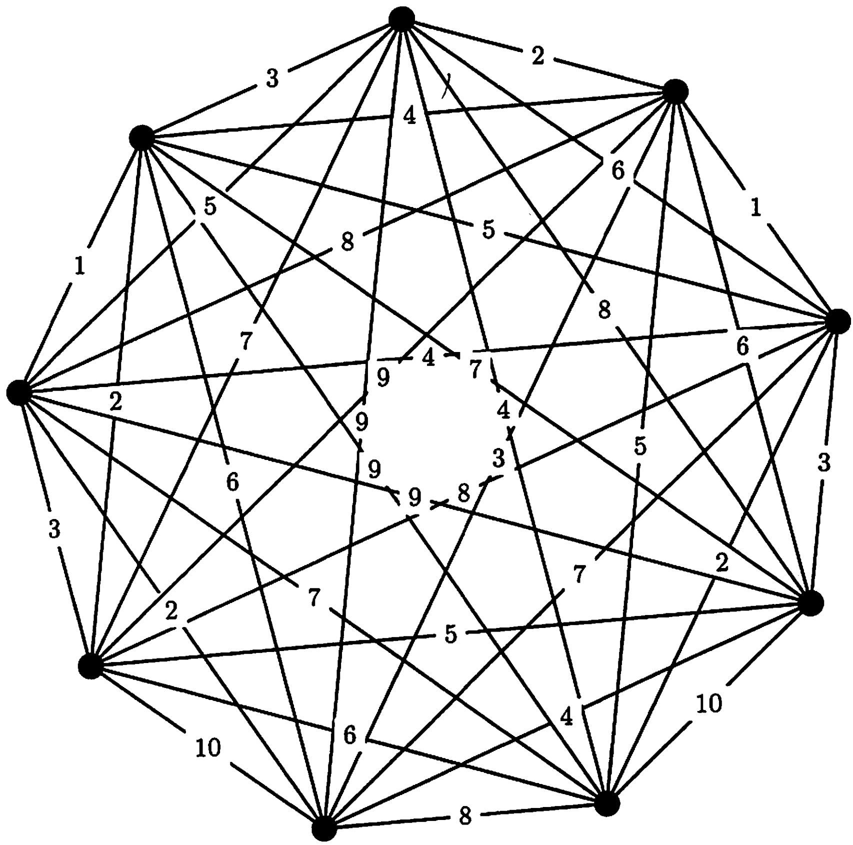

Figure 5: Edge colouring of K_9 with flatness three.