Turán number for pS_r *

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Abstract. The *Turán number* $ex(m,G)$ of the graph G is the maximum number of edges of an m-vertex simple graph having no G as a subgraph. A *star* S_r is the complete bipartite graph $K_{1,r}$ (or a tree with one internal vertex and r leaves) and pS_r denotes the disjoint union of p copies of S_r. A result of Lidický et al. (Electron. J. Combin. 20(2)(2013) P62) implies that $ex(m,pS_r) = \lfloor\frac{(m-p+1)(r-1)}{2}\rfloor + (p-1)m - \binom{p}{2}$ for m sufficiently large. In this paper, we give another proof and show that $ex(m,pS_r) = \lfloor\frac{(m-p+1)(r-1)}{2}\rfloor + (p-1)m - \binom{p}{2}$ for all $r \geq 1$, $p \geq 1$ and $m \geq \frac{1}{2}r^2p(p-1) + p - 2 + \max\{rp, r^2+2r\}$.
Keywords. Turán number, Disjoint copies, pS_r.

1. Introduction

Graphs in this paper are finite and simple. Terms and notation not defined here are from [1]. Let S_r be the star on $r+1$ vertices, that is, the complete bipartite graph $K_{1,r}$ (or a tree with one internal vertex and r leaves). For graphs G and H, $G \cup H$ denotes the disjoint union of G and H, $H \subseteq G$ denotes that H is a subgraph of G, pG denotes the disjoint union of p copies of G, and $G+H$ denotes the *join* of G and H, that is, the graph obtained from $G \cup H$ by joining each vertex of G with each vertex of H. For $v \in V(G)$ and $H \subseteq G$, the neighbor set of v in H is denoted by $N_H(v)$ and the degree of v in H is denoted by $d_H(v)$. Clearly, $d_H(v) = |N_H(v)|$. For $V' \subseteq V(G)$, the subgraph of G induced by V' is denoted by $G[V']$. For a set S by $|S|$ we denote the cardinality of S. For subgraphs H_1 and H_2 of G, $e_G(H_1, H_2)$ denotes the number of edges in G with one end in H_1 and the other end in H_2.

The *Turán number* $ex(m,G)$ of the graph G is the maximum number of edges of an m-vertex simple graph having no G as a subgraph. Let

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$H_{ex}(m,G)$ denote a graph on m vertices with $ex(m,G)$ edges not containing G. We call this graph *an extremal graph for* G. Let $T_r(m)$ denote the complete r-partite graph on m vertices in which all parts are as equal in size as possible. Turán [5] determined the value $ex(m,K_{r+1})$ and showed that $T_r(m)$ is the unique extremal graph for K_{r+1}, where K_{r+1} is the complete graph on $r+1$ vertices. Turán's theorem is regarded as the basis of a significant branch of graph theory known as *extremal graph theory*. It was shown by Simonovits [4] that if m is sufficiently large then $K_{p-1}+T_r(m-p+1)$ is the unique extremal graph for pK_{r+1}. Gorgol [2] further considered the Turán number for p disjoint copies of any connected graph G and gave a lower bound for $ex(m,pG)$ by simply counting the number of edges of the graphs $H_{ex}(m-pn+1,G)\cup K_{pn-1}$ and $H_{ex}(m-p+1,G)+K_{p-1}$ which do not contain pG.

Theorem 1.1 [2] *Let G be an arbitrary connected graph on n vertices, p be an arbitrary positive integer and m be an integer such that $m \geq pn$. Then $ex(m,pG) \geq \max\{ex(m-pn+1,G)+\binom{pn-1}{2}, ex(m-p+1,G)+(p-1)m-\binom{p}{2}\}$.*

Lidický et al. [3] investigated the Turán number for a star forest. Their result implies that $ex(m,pS_r) = \lfloor\frac{(m-p+1)(r-1)}{2}\rfloor + (p-1)m - \binom{p}{2}$ for m sufficiently large. Lidický et al. [3] also pointed out that they make no attempt to minimize the bound on m in their proof. In this paper, we present another proof of $ex(m,pS_r) = \lfloor\frac{(m-p+1)(r-1)}{2}\rfloor + (p-1)m - \binom{p}{2}$ and give a lower bound on m. That is the following Theorem 1.2.

Theorem 1.2 *If $m \geq \frac{1}{2}r^2p(p-1)+p-2+\max\{rp, r^2+2r\}$, then $ex(m,pS_r) = \lfloor\frac{(m-p+1)(r-1)}{2}\rfloor + (p-1)m - \binom{p}{2}$.*

2. Proof of Theorem 1.2

The lower bound follows from $ex(m,S_r) = \lfloor\frac{m(r-1)}{2}\rfloor$ and Theorem 1.1. To show the upper bound, we consider an arbitrary graph G on m vertices with $\lfloor\frac{(m-p+1)(r-1)}{2}\rfloor + (p-1)m - \binom{p}{2} + 1$ edges, where $m \geq \frac{1}{2}r^2p(p-1)+p-2+\max\{rp, r^2+2r\}$. We use induction on p to show that $pS_r \subseteq G$. If $p=1$, then $|E(G)| = \lfloor\frac{m(r-1)}{2}\rfloor + 1$, and so $S_r \subseteq G$. Assume that $p \geq 2$. Since $m \geq \frac{1}{2}r^2p(p-1)+p-2+\max\{rp, r^2+2r\} \geq \frac{1}{2}r^2(p-1)(p-2)+(p-1)-2+\max\{r(p-1), r^2+2r\}$ and $\lfloor\frac{(m-p+1)(r-1)}{2}\rfloor + (p-1)m - \binom{p}{2} + 1 \geq \lfloor\frac{(m-(p-1)+1)(r-1)}{2}\rfloor + ((p-1)-1)m - \binom{p-1}{2} + 1$, by the induction hypothesis, we have that $(p-1)S_r \subseteq G$. Let $V((p-1)S_r) = \{v_{10}, v_{11}, \ldots, v_{1r}, v_{20}, v_{21}, \ldots, v_{2r}, \ldots, v_{(p-1)0}, v_{(p-1)1}, \ldots, v_{(p-1)r}\}$ and $E((p-1)S_r) = \{v_{10}v_{11}, \ldots, v_{10}v_{1r}, v_{20}v_{21}, \ldots, v_{20}v_{2r}, \ldots, v_{(p-1)0}v_{(p-1)1}, \ldots, v_{(p-1)0}v_{(p-1)r}\}$. Denote $H = G[V((p-1)S_r)]$ and $H' = G - H$. To the contrary, we assume that G contains no pS_r as a subgraph.

Claim 1. $|E(H')| \leq \lfloor\frac{(m-(p-1)(r+1))(r-1)}{2}\rfloor$.

Proof of Claim 1. If $|E(H')| > \lfloor\frac{(m-(p-1)(r+1))(r-1)}{2}\rfloor$, then H' contains an S_r, implying that $pS_r \subseteq G$, a contradiction to our assumption. □

Claim 2. For each $i \in \{1, 2, \ldots, p-1\}$, there exists one vertex $x \in \{v_{i0}, v_{i1}, \ldots, v_{ir}\}$ with $|N_{H'}(x)| \geq pr$ and $|N_{H'}(y)| \leq r-1$ for $y \in \{v_{i0}, v_{i1}, \ldots, v_{ir}\} \setminus \{x\}$.

Proof of Claim 2. If there exists an $i \in \{1, 2, \ldots, p-1\}$ such that there exist $x, y \in \{v_{i0}, v_{i1}, \ldots, v_{ir}\}$ with $|N_{H'}(x)| \geq 2r$ and $|N_{H'}(y)| \geq r$, then $pS_r \subseteq G$, a contradiction. Hence for each $i \in \{1, 2, \ldots, p-1\}$, we have that (i) there exists one vertex $x \in \{v_{i0}, v_{i1}, \ldots, v_{ir}\}$ with $|N_{H'}(x)| \geq 2r$ and $|N_{H'}(y)| \leq r-1$ for $y \in \{v_{i0}, v_{i1}, \ldots, v_{ir}\} \setminus \{x\}$, or (ii) $|N_{H'}(z)| \leq 2r-1$ for $z \in \{v_{i0}, v_{i1}, \ldots, v_{ir}\}$. It follows from $m \geq \frac{1}{2}r^2p(p-1)+p-2+\max\{rp, r^2+2r\} \geq (r+1)p + r^2 + r - 2$ that

$$\begin{aligned}
\textstyle\sum_{j=0}^{r} |N_{H'}(v_{ij})| &\leq \max\{m-(p-1)(r+1)+r(r-1), (r+1)(2r-1)\} \\
&= m-(p-1)(r+1)+r(r-1) \\
&= m-p(r+1)+r^2+1
\end{aligned}$$

for each $i \in \{1, 2, \ldots, p-1\}$. If there exists an $k \in \{1, 2, \ldots, p-1\}$ such that $|N_{H'}(z)| \leq 2r-1$ for $z \in \{v_{k0}, v_{k1}, \ldots, v_{kr}\}$, by Claim 1 and $m \geq \frac{1}{2}r^2p(p-1)+p-2+\max\{rp, r^2+2r\}$, then

$$\begin{aligned}
|E(G)| &= |E(H)| + \textstyle\sum_{i=1}^{p-1}\sum_{j=0}^{r} |N_{H'}(v_{ij})| + |E(H')| \\
&\leq \frac{(p(r+1)-r-1)(p(r+1)-r-2)}{2} + (p-2)(m-p(r+1)+r^2+1) \\
&\quad +(r+1)(2r-1) + \lfloor\frac{(m-(p-1)(r+1))(r-1)}{2}\rfloor \\
&= \lfloor\frac{(m-p+1)(r-1)}{2}\rfloor + (p-1)m - \binom{p}{2} - m \\
&\quad + \frac{(p(r+1)-r-1)(p(r+1)-r-2)}{2} - (p-2)(p(r+1)) \\
&\quad +(p-2)(r^2+1) + (r+1)(2r-1) - \frac{(p-1)r(r-1)}{2} + \binom{p}{2} \\
&= \lfloor\frac{(m-p+1)(r-1)}{2}\rfloor + (p-1)m - \binom{p}{2} - m + \frac{1}{2}r^2p(p-1) \\
&\quad +p-2+r^2+2r \\
&\leq \lfloor\frac{(m-p+1)(r-1)}{2}\rfloor + (p-1)m - \binom{p}{2},
\end{aligned}$$

a contradiction to $|E(G)| = \lfloor\frac{(m-p+1)(r-1)}{2}\rfloor + (p-1)m - \binom{p}{2} + 1$. Thus for each $i \in \{1, 2, \ldots, p-1\}$, there exists one vertex $x \in \{v_{i0}, v_{i1}, \ldots, v_{ir}\}$ with $|N_{H'}(x)| \geq 2r$ and $|N_{H'}(y)| \leq r-1$ for $y \in \{v_{i0}, v_{i1}, \ldots, v_{ir}\} \setminus \{x\}$. If there exists an $k \in \{1, 2, \ldots, p-1\}$ such that there exists one vertex $x \in \{v_{k0}, v_{k1}, \ldots, v_{kr}\}$ with $2r \leq |N_{H'}(x)| \leq pr-1$ and $|N_{H'}(y)| \leq r-1$ for $y \in \{v_{k0}, v_{k1}, \ldots, v_{kr}\} \setminus \{x\}$, by Claim 1 and $m \geq \frac{1}{2}r^2p(p-1)+p-2+$

$\max\{rp, r^2+2r\}$, then

$$
\begin{aligned}
|E(G)| &= |E(H)| + \sum_{i=1}^{p-1}\sum_{j=0}^{r}|N_{H'}(v_{ij})| + |E(H')| \\
&\leq \tfrac{(p(r+1)-r-1)(p(r+1)-r-2)}{2} + (p-2)(m-p(r+1)+r^2+1) \\
&\quad +(pr-1+r(r-1)) + \lfloor \tfrac{(m-(p-1)(r+1))(r-1)}{2} \rfloor \\
&= \lfloor \tfrac{(m-p+1)(r-1)}{2} \rfloor + (p-1)m - \tbinom{p}{2} - m \\
&\quad + \tfrac{(p(r+1)-r-1)(p(r+1)-r-2)}{2} - (p-2)(p(r+1)) \\
&\quad +(p-2)(r^2+1) + (pr-1+r(r-1)) - \tfrac{(p-1)r(r-1)}{2} + \tbinom{p}{2} \\
&= \lfloor \tfrac{(m-p+1)(r-1)}{2} \rfloor + (p-1)m - \tbinom{p}{2} - m + \tfrac{1}{2}r^2p(p-1) \\
&\quad +p-2+rp \\
&\leq \lfloor \tfrac{(m-p+1)(r-1)}{2} \rfloor + (p-1)m - \tbinom{p}{2},
\end{aligned}
$$

a contradiction. This completes the proof of Claim 2. □

Denote $U = \{x | x \in V((p-1)S_r)$ and $|N_{H'}(x)| \geq pr\}$ and $W = V((p-1)S_r) \setminus U$. By Claim 2, $|U| = p-1$, $|W| = (p-1)r$ and $|N_{H'}(y)| \leq r-1$ for $y \in W$. If there exists one vertex $z \in V(H')$ such that $|N_{G[W]}(z)| \geq r$, then by $|N_{H'}(u) \setminus \{z\}| \geq pr-1 \geq (p-1)r$ for $u \in U$, we can find p disjoint copies of S_r in G, a contradiction. Hence $|N_{G[W]}(y)| \leq r-1$ for $y \in V(H')$. For $0 \leq i \leq r-1$, we denote $X_i = \{y | y \in W$ and $|N_{H'}(y)| = i\}$, $X_i' = \{y | y \in V(H')$ and $|N_{G[W]}(y)| = i\}$, $\ell_i = |X_i|$ and $\ell_i' = |X_i'|$. Then $\sum_{i=0}^{r-1}\ell_i = (p-1)r$, $\sum_{i=0}^{r-1}\ell_i' = m-(p-1)(r+1)$ and $\sum_{i=0}^{r-1} i\ell_i = \sum_{i=0}^{r-1} i\ell_i'$.

Claim 3. For each $i \in \{0, \ldots, r-1\}$, we have that $d_{G[W]}(x) \leq r-i-1$ for $x \in X_i$ and $d_{H'}(y) \leq r-i-1$ for $y \in X_i'$.

Proof of Claim 3. If there exists one vertex $x \in X_i$ with $d_{G[W]}(x) \geq r-i$, let $N_{H'}(x) = \{x_1', \ldots, x_i'\}$, then by $|N_{H'}(u) \setminus \{x_1', \ldots, x_i'\}| \geq pr-i \geq (p-1)r$ for $u \in U$, we can find p disjoint copies of S_r in G, a contradiction. Assume that there exists one vertex $y \in X_i'$ with $d_{H'}(y) \geq r-i$. If $i=0$, then H' contains an S_r, implying that $pS_r \subseteq G$, a contradiction. If $i \geq 1$, let $y_1', \ldots, y_{r-i}' \in N_{H'}(y)$, then by $|N_{H'}(u) \setminus \{y, y_1', \ldots, y_{r-i}'\}| \geq pr-(r-i+1) \geq (p-1)r$ for $u \in U$, we can find p disjoint copies of S_r in G, a contradiction. This completes the proof of Claim 3. □

By Claim 3, $\sum_{i=0}^{r-1}\ell_i = (p-1)r$ and $\sum_{i=0}^{r-1}\ell_i' = m-(p-1)(r+1)$, we have that

$$
\begin{aligned}
|E(H)| &= |E(G[U])| + e_G(G[U], G[W]) + |E(G[W])| \\
&\leq \tbinom{p-1}{2} + (p-1)(p-1)r + \lfloor \tfrac{\sum_{i=0}^{r-1}(r-i-1)\ell_i}{2} \rfloor \\
&= \tbinom{p-1}{2} + r(p-1)^2 + \tfrac{r(r-1)(p-1)}{2} + \lfloor \tfrac{-\sum_{i=0}^{r-1} i\ell_i}{2} \rfloor,
\end{aligned}
$$

$$
\begin{aligned}
e_G(H, H') &= e_G(G[U], H') + e_G(G[W], H') \\
&\leq (p-1)(m-(p-1)(r+1)) + \sum_{i=0}^{r-1} i\ell_i \\
&= (p-1)m - (r+1)(p-1)^2 + \sum_{i=0}^{r-1} i\ell_i
\end{aligned}
$$

and $|E(H')| \leq \lfloor \frac{\sum_{i=0}^{r-1}(r-i-1)\ell'_i}{2} \rfloor = \lfloor \frac{(m-p+1)(r-1)-\sum_{i=0}^{r-1} i\ell'_i}{2} \rfloor - \frac{r(r-1)(p-1)}{2}$. Denote $\sum_{i=0}^{r-1} i\ell_i = \sum_{i=0}^{r-1} i\ell'_i = \ell$. Thus,

$$\begin{aligned}
|E(G)| &= |E(H)| + e_G(H,H') + |E(H')| \\
&\leq \lfloor \tfrac{(m-p+1)(r-1)-\sum_{i=0}^{r-1} i\ell'_i}{2} \rfloor + (p-1)m - \tbinom{p}{2} \\
&\quad + (\tbinom{p}{2} + \tbinom{p-1}{2} + \lfloor \tfrac{-\sum_{i=0}^{r-1} i\ell_i}{2} \rfloor - (p-1)^2 + \textstyle\sum_{i=0}^{r-1} i\ell_i) \\
&= \lfloor \tfrac{(m-p+1)(r-1)-\sum_{i=0}^{r-1} i\ell'_i}{2} \rfloor + (p-1)m - \tbinom{p}{2} \\
&\quad + (\lfloor \tfrac{-\sum_{i=0}^{r-1} i\ell_i}{2} \rfloor + \textstyle\sum_{i=0}^{r-1} i\ell_i) \\
&= \lfloor \tfrac{(m-p+1)(r-1)-\ell}{2} \rfloor + (p-1)m - \tbinom{p}{2} + (\lfloor \tfrac{-\ell}{2} \rfloor + \ell).
\end{aligned}$$

If ℓ is even, then

$$\begin{aligned}
|E(G)| &\leq \lfloor \tfrac{(m-p+1)(r-1)-\ell}{2} \rfloor + (p-1)m - \tbinom{p}{2} + (\lfloor \tfrac{-\ell}{2} \rfloor + \ell) \\
&= \lfloor \tfrac{(m-p+1)(r-1)}{2} \rfloor - \tfrac{\ell}{2} + (p-1)m - \tbinom{p}{2} + (-\tfrac{\ell}{2} + \ell) \\
&= \lfloor \tfrac{(m-p+1)(r-1)}{2} \rfloor + (p-1)m - \tbinom{p}{2}, \text{ a contradiction.}
\end{aligned}$$

Assume that ℓ is odd. If $(m-p+1)(r-1)$ is even, then

$$\begin{aligned}
|E(G)| &\leq \lfloor \tfrac{(m-p+1)(r-1)-\ell}{2} \rfloor + (p-1)m - \tbinom{p}{2} + (\lfloor \tfrac{-\ell}{2} \rfloor + \ell) \\
&= \lfloor \tfrac{(m-p+1)(r-1)}{2} \rfloor - \tfrac{\ell+1}{2} + (p-1)m - \tbinom{p}{2} + (-\tfrac{\ell+1}{2} + \ell) \\
&= \lfloor \tfrac{(m-p+1)(r-1)}{2} \rfloor + (p-1)m - \tbinom{p}{2} - 1, \text{ a contradiction.}
\end{aligned}$$

If $(m-p+1)(r-1)$ is odd, then

$$\begin{aligned}
|E(G)| &\leq \lfloor \tfrac{(m-p+1)(r-1)-\ell}{2} \rfloor + (p-1)m - \tbinom{p}{2} + (\lfloor \tfrac{-\ell}{2} \rfloor + \ell) \\
&= \lfloor \tfrac{(m-p+1)(r-1)}{2} \rfloor - \tfrac{\ell-1}{2} + (p-1)m - \tbinom{p}{2} + (-\tfrac{\ell+1}{2} + \ell) \\
&= \lfloor \tfrac{(m-p+1)(r-1)}{2} \rfloor + (p-1)m - \tbinom{p}{2}, \text{ a contradiction again.}
\end{aligned}$$

This contradiction completes the proof of Theorem 1.2. □

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