Some New Results on Fault-tolerant Cycles Embedding in Folded Hypercubes

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Abstract: Under the conditions looser than previous works, this paper shows that the n-dimensional folded hypercube networks have a cycle with length at least $2^n - 2|F_v|$ when the number of faulty vertices and non-critical edges is at most $2n - 4$, where $|F_v|$ is the number of faulty vertices. Meanwhile, this paper proves that FQ_n contains a fault-free cycle with length at least $2^n - 2|F_v|$, under the constraints that (1) The number of both faulty nodes and faulty edges is no more than $2n - 3$ and there is at least one faulty edge; (2) every node in FQ_n is incident to at least two fault-free links whose other end nodes are fault-free. These results have improved the present results with further theoretical evidence of the fact that FQ_n has excellent node-fault-tolerance and edge-fault-tolerance when used as a topology of large scale computer networks.

key words: Node-fault-tolerance; Edge-fault-tolerance; Cycle; Folded hypercube networks

1 Introduction

Fault tolerance is the most important factor to be considered in computer network design and management. Good quality of hardware system helps to prevent failures, but network topology is also of great significance in preventing network crash when fault happens. So it is vital to choose the topologies having high fault tolerance for computer networks, particularly for those complex large scale computer network systems. Since faults may happen on both vertices and edges in a network, apart from faulty node, faulty edges also need to be considered. An edge uv is said faulty if the communication link between the end nodes u and v is broken. Note that u and v may neither be faulty. Usually, two models are used for fault-tolerance analysis. One is standard fault model in which the distribution of faulty edges and faulty vertices is not restricted. The other is the conditional fault

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model in which each fault-free vertex must be incident to at least two fault-free edges([1], [2], [3],[4], [5], [6]).

The n-dimensional hypercube (or n-cube, denoted by Q_n) consists of 2^n nodes that are labeled with 2^n binary numbers from 0 to $2^n - 1$, which is one of the most well-known interconnection network architectures yet developed for multiprocessor system and large computation in industrial ([7]). To improve the performance of Q_n, many variants of Q_n have been proposed. Folded hypercube, denoted by FQ_n, is one of the most popular variants. FQ_n can be constructed by adding one edge to every pair of the farthest nodes of Q_n, i.e., two nodes with all bits different (or, complementary addresses). It has been shown that FQ_n has improved performance over a regular hypercube in many measurements ([8],[9],[10],[11], [12]).

Previous researches on the fault-tolerant embedding problems for the Q_n have been fruitful. Let F_v be the set of fault vertices. Fu ([13]) considers the case of faulty vertices, and shows that there exists a cycle of length at least $2^n - 2|F_v|$ if $|F_v| \leq 2n - 4$ in Q_n. Taking into account the existence of both faulty vertices and faulty edges, Sengupta ([14]) shows that Q_n contains a cycle of length at least $2^n - 2|F_v|$ when the number of faulty edges is at most $n - 2$ and the number of faulty nodes is at least one. Hsieh ([15]) extends the results by proving that there is a cycle of length at least $2^n - 2|F_v|$ in Q_n if the number of faulty edges is no more than $n - 2$ and the number of both faulty vertices and faulty edges is no more than $2n - 4$. Hsieh ([16]) discusses the path embedding and shows that there exists a path of length at least $2^n - 2|F_v| - 1$ joining any fault-free nodes $u = u_1u_2 \cdots u_n$, $v = v_1v_2 \cdots v_n$ only if $\sum_{i=1}^{n} u_i$ and $\sum_{i=1}^{n} v_i$ have different parity in Q_n when the number of faulty vertices and faulty edges is most $n - 2$.

Many efforts have been made on exploring the features of folded hypercube FQ_n. [17] investigates the 1-vertex-fault-tolerant cycles embedding into FQ_n, and shows that FQ_n contains a fault-free cycle of every even length from 4 to $2^n - 2$, and also contains a fault-free cycle of every odd length from $n + 1$ to $2^n - 1$ if n is even. By restricting the forbidden faulty set (resp. forbidden faulty edge set) to the sets of neighbors (resp. neighboring edges)of any spanning subgraph with no more than g-vertices in the faulty networks, g−extra connectivity (resp. g−extra edge connectivity) of FQ_n obtained attention [12], and showed that the 3-extra connectivity (resp. 3-extra edge connectivity) of FQ_n is $4n - 5$ for $n \geq 6$ (resp. $4n - 1$ for $n \geq 5$) . [18] has proved that there exists a cycle with length at least $2^n - 2|F_v|$ when the faulty edges is most $n - 1$ and the number of faulty vertices and faulty edges is most $2n - 4$. In the conditional fault model, [19] further shows that FQ_n has a cycle with length at least $2^n - 2|F_v|$ under the constraints that (1) the number of faulty vertices and faulty edges is most $2n-4$ and (2) every node in FQ_n is incident to at least two fault-free links, that is, the conditional fault-tolerance.

This paper aims to further explore the node-fault-tolerance and edge-fault-tolerance of FQ_n by cycle embedding. In this paper, the fault-tolerant model is generalized so that the two types of faulty edges, denoted by F_e and f_e, can be distinguished, where F_e is the set of faulty edges with at least one faulty end, and f_e is the set of faulty edges whose both ends are fault-free. An edge in FQ_n is said to be critical if it is either fault-free or in F_e, otherwise, it is called non-critical. Note that all the edges in f_e are non-critical. Some preliminary discussion for Q_n can be found in [20], which has proved that Q_n contains a cycle of length at least $2^n - 2|F_v|$ if $|f_e| \leq 2n-5$ and $|f_e| + |F_v| \leq 2n-4$ in which any vertex is incident to at least two critical edges.

It is shown in this paper that when $|f_e| + |F_v| \leq 2n-4$, FQ_n contains a fault-free cycle of length at least $2^n - 2|F_v|$ in which any node is incident to at least three critical edges. This paper also shows that FQ_n contains a fault-free cycles with length at least $2^n - 2|F_v|$, under the constraints that (1) $|F_v| + |FF_e| \leq 2n-3$, $|FF_e| \geq 1$, where $FF_e = f_e \cup F_e$ and (2) every node in FQ_n is incident to at least two fault-free links, which improves the results in [19]. The results obtained in this paper provide further theoretical evidence for the fact that FQ_n has excellent node-fault-tolerance and edge-fault-tolerance when used as a topology of large scale computer networks. Compared to previous works, the contributions of this work are: It considers larger set FQ_n rather than Q_n; It uses the generalized fault model, which distinguishes the two types of faulty edges, rather than using the conditional fault model in which each node is incident to at least two fault-free links; It has more relaxed condition since it does not require $|f_e| \leq 2n-5$ in FQ_n (this inequality is required in Q_n).

The rest of this paper is organized as follows. In the next section, necessary definitions and notations used in our discussion are introduced. In section 3, Cycle construction are presented. Finally, some concluding remarks are given in section 4.

2 Preliminaries

A network is usually modeled by a connected graph $G = (V, E)$, where V denotes the set of processors and E denotes the set of communication links between the processors. In this paper the terms of networks and graphs, nodes and vertices, links and edges are used interchangeably. Two vertices x and y are adjacent if $xy \in E$. A path $P[x, y] = xw_1w_2 \cdots w_ky$ is a sequence of distinct vertices in which any two consecutive vertices are adjacent. The length of a path, denoted by $|P|$, is the number of edges on the path. A path $xw_1w_2 \cdots w_ky$ forms a cycle if $x = y$ and $k \geq 2$. A path (respectively, a cycle) is called a Hamilton path (respectively, a Hamilton cycle) if it passes each vertex of G exactly once. A graph $G = (X \cup Y, E)$ is bipartite if $X \cap Y = \emptyset$ and $E \subseteq \{uv : u \in X, v \in Y\}$. $X$and Y is called its bipartition.

A network is faulty if it contains faulty elements, otherwise, it is fault-free. A path (cycle) is said to be faulty if it contains faulty elements, otherwise, it is fault-free.

An n-cube Q_n is a graph with vertex set $V(Q_n) = \{x_1x_2\cdots x_n : x_i \in \{0,1\}$, for $i = 1,2,\ldots,n\}$, with two vertices $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_n$ being adjacent if and only if they differ in exactly one bit, that is, $\sum_{i=1}^{n}|x_i - y_i| = 1$. Each node can be labeled with a unique n-bit binary string $x_1x_2\cdots x_n$ to indicate its address. Let $x = x_1x_2\cdots x_i\cdots x_n$, if $xy \in E$ and $y = x_1x_2\cdots \bar{x}_i\cdots x_n$, where $\bar{x}_i = 1 - x_i$, then the edge xy is called an i-dimensional edge, and denoted by $y = x^i$. An i-partition on Q_n splits the n-cube along a dimension i for some $i \in \{1,2,\cdots,n\}$ into two $(n-1)$-cubes Q^{i0}_{n-1} and Q^{i1}_{n-1}, where $V(Q^{i0}_{n-1}) = \{x_1x_2\cdots x_n : x_i = 0,\ x_j = 0,1,\ \text{for } 1 \leq j(\neq i) \leq n\}$ and $V(Q^{i1}_{n-1}) = \{x_1x_2\cdots x_n : x_i = 1,\ x_j = 0,1,\ \text{for } 1 \leq j(\neq i) \leq n\}$.

Suppose $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_n$ are two bit strings of length n. The Hamming distance between x and y is defined as $H(x,y) = \sum_{i=1}^{n}|x_i - y_i|$ which is the number of the different bits in the corresponding strings of x and y. Let x and y be two vertices of graph G, the distance between x and y is denoted by $d_G(x,y)$ ($d(x,y)$ if no confusion). It is the length of the shortest (x,y)-path in G. By the definition of Hamming distance, it is obviously that $H(x,y) = d_{Q_n}(x,y)$.

Let $x = x_1x_2\cdots x_n$ be a node in Q_n. x is called to be even (or odd) if $\sum_{i=1}^{n} x_i$ is even (or odd).

Folded hypercube $FQ_n = (V,E)$ has the same vertex set as Q_n, that is, $V(FQ_n) = \{x_1x_2\cdots x_n : x_i \in \{0,1\}$, for $1 \leq i \leq n\}$. Two vertices $x(= x_1x_2\cdots x_n)$ and y are connected by an edge of E if and only if y satisfies one of the following two conditions:

(i) $y = x^i = x_1x_2\cdots x_{i-1}\bar{x}_ix_{i+1}\ldots x_n,\ 1 \leq i \leq n$; or

(ii) $y = \bar{x} = \bar{x}_1\bar{x}_2\cdots\ \bar{x}_n$.

Hypercube Q_n is a subgraph of folded hypercube FQ_n, obtained by removing all edges of $x\bar{x}$. These removed edges are called compliment edges of FQ_n. The first kind of edges are called the edges of Q_n. For the sake of convenience, denote $E_i = \{xx^i\}(i = 1,2,\ldots,n)$, and $E_c = \{x\bar{x}\}$.

Obviously, folded hypercube FQ_n has 2^n vertices and each vertex is incident to $(n+1)$ edges and $FQ_n - E_i$ is isomorphic to Q_n.

To analyze FQ_n, it is essential to decompose FQ_n into lower dimensional graphs. Similar to i-partition of Q_n, FQ_n can be split along some dimension i for $i \in \{1,2,\cdots,n\}$ into two $(n-1)$-cubes Q^{i0}_{n-1} and Q^{i1}_{n-1}. For convenience, denote $FQ_n = Q^{i0}_{n-1} \uplus Q^{i1}_{n-1}$. In fact, $E_i \cup E_c$ consists of all edges between Q^{i0}_{n-1} and Q^{i1}_{n-1}.

Define the set of faulty elements F in FQ_n as $F_v \cup f_e$, that is, F consists of all the faulty nodes and non-critical edges in FQ_n. Note that F does not include F_e (the set of faulty edges incident to the faulty nodes). Suppose that Q^{i0}_{n-1} and Q^{i1}_{n-1} are $(n-1)$-cube derived after executing an i-partition on FQ_n, that is, $FQ_n = Q^{i0}_{n-1} \uplus Q^{i1}_{n-1}$. Denote $F^0_v = V(Q^{i0}_{n-1}) \cap F_v$, $F^1_v = V(Q^{i1}_{n-1}) \cap F_v$. Similarly, $f_e{}^0 = E(Q^{i0}_{n-1}) \cap f_e$ and $f_e{}^1 = E(Q^{i1}_{n-1}) \cap f_e$. Let $F^0 = F^0_v \cup f_e{}^0$ and $F^1 = F^1_v \cup f_e{}^1$. For a node x, let $F_v * [x]$ be the set of faulty nodes that are adjacent to x and in the same $(n-1)$-cube as x, and let $f_e * [x]$ be the set of non-critical edges that are incident to x and belong to the same $(n-1)$-cube where vertex x belongs to.

From the definition, it is easy to prove the following lemma.

Lemma 1. If $|F_v| = 0$, then $|F_e| = 0$.

Proof: If $|F_v| = 0$, then the ends of a faulty edge will be fault-free. Therefore $|F_e| = 0$.

A vertex in FQ_n is said to be 2-critical if it is incident to exactly two critical edges. Note that each 2-critical vertex in FQ_n is fault-free. A vertex in FQ_n is said to be 3-critical if it is incident to exactly three critical edges (i.e., it is incident to exactly $n-2$ non-critical edges).

As the basis of the analysis, we need to establish the following lemma.

Lemma 2. Let $FQ_n(n \geq 5)$ be an n-dimensional folded hypercube with $|F_v| \geq 0$ and $1 \leq |f_e| \leq 2n-4$, and each vertex in $FQ_n - F_v - f_e$ is incident to at least three critical edges. Then $FQ_n(n \geq 5)$ can be partitioned into Q_{n-1}^{i0} and Q_{n-1}^{i1} for some $1 \leq i \leq n$ such that

1) $|(E_i \cup E_c) \cap f_e| \geq 1$, and

2) each vertex in Q_{n-1}^{i0} (respectively, Q_{n-1}^{i1}) is incident to at least two critical edges in Q_{n-1}^{i0} (respectively, Q_{n-1}^{i1}).

Proof: If each vertex is incident to at least four critical edges, then the result is true. Assume that FQ_n contains at least one 3-critical vertex. Because $|f_e| \leq 2n-4$, then there are at most two 3-critical vertices in FQ_n. Two cases need to be considered.

Case 1. There is exactly one 3-critical vertex, say u. Then $uu^i \in E_i \cap f_e$ for some $i(1 \leq i \leq n)$. Let $FQ_n = Q_{n-1}^{i0} \uplus Q_{n-1}^{i1}$ after executing i–partition on FQ_n. Without loss of generality, let $u \in Q_{n-1}^{i0}$. If $u\bar{u} \in f_e$, then u is still 3-critical vertex in Q_{n-1}^{i0}. If $u\bar{u}$ is fault-free, then u is 2-critical vertex in Q_{n-1}^{i0}, since no matter for what i, after i–partition, $u\bar{u}$ is always between Q_{n-1}^{i0} and Q_{n-1}^{i1}.

Case 2. There are exactly two 3-critical vertices, say u and v. Consider the following scenarios.

Case 2.1. $d(u, v) \geq 2$. Then all edges incident to u and v are distinct.

When $u\bar{u}$, $v\bar{v} \notin f_e$, suppose $uu^{i_1}, uu^{i_2}, \cdots, uu^{i_{n-2}}$ and $vv^{j_1}, vv^{j_2}, \cdots, vv^{j_{n-2}}$ be non-critical edges. If $\{i_1, i_2, \cdots, i_{n-2}\} \cap \{j_1, j_2, \cdots, j_{n-2}\} = \emptyset$, then $(n-2)+(n-2) \leq n$, that is, $n \leq 4$, which contradicts to $n \geq 5$. This implies that there exists some dimension i such that uu^i and vv^i are non-critical edges. We execute i-partition $FQ_n = Q_{n-1}^{i0} \uplus Q_{n-1}^{i1}$, then uu^i and vv^i are edges between Q_{n-1}^{i0} and Q_{n-1}^{i1}, which means that u and v are incident to at least two critical edges in their $(n-1)$-cube.

When $u\bar{u}$, $v\bar{v} \in E_i \cap f_e$, select an edge such that $uu^i \in f_e$. We partition $FQ_n = Q_{n-1}^{i0} \uplus Q_{n-1}^{i1}$, then u is a 3-critical vertex in $(n-1)$-cube, and v is incident to at least two critical edges in $(n-1)$-cube.

When $u\bar{u} \in f_e$ and $v\bar{v} \notin f_e$ (similarly for $u\bar{u} \notin f_e$, $v\bar{v} \in f_e$), select $vv^i \in E_i \cap f_e$ (or $uu^j \in E_j \cap f_e$). Partition $FQ_n = Q_{n-1}^{i0} \uplus Q_{n-1}^{i1}$ (or $FQ_n = Q_{n-1}^{j0} \uplus Q_{n-1}^{j1}$). Then u and v are incident to at least two critical vertices.

Case 2.2. $d(u, v) = 1$.

Case 2.2.1. $uv \notin f_e$.

If $uv \in E_i$ for some $i(1 \leq i \leq n)$, since $n \geq 5$, then $(n-2)+(n-2) \geq n-1$, which means that there exists some j such that $uu^j, vv^j \in f_e$. We partition $FQ_n = Q_{n-1}^{j0} \uplus Q_{n-1}^{j1}$, then u and v are 2-critical vertices in their $(n-1)$-cube.

If $uv \in E_c$, since $n \geq 5$, we have $(n-2)+(n-2) > n$, then there exists some j such that $uu^j, vv^j \in f_e$. We partition $FQ_n = Q_{n-1}^{j0} \uplus Q_{n-1}^{j1}$, then u and v are 2-critical vertices in their respective $(n-1)$-cubes.

Case 2.2.2. $uv \in f_e$.

If $uv \in E_i \cap f_e$, after i-partition $FQ_n = Q_{n-1}^{i0} \uplus Q_{n-1}^{i1}$, then u and v are incident to at least two critical edges in their respective $(n-1)$-cubes.

If $uv \in E_c \cap f_e$, suppose $uu^i \in E_i \cap f_e$. Execute i-partition on $FQ_n = Q_{n-1}^{j0} \uplus Q_{n-1}^{j1}$, then u and v are incident to at least two critical edges in their respective $(n-1)$-cubes.

The proof of lemma 2 has been completed.

Some previous results obtained by related works will also be used. They are listed below.

Lemma 3 [21]. Let u and v be two nodes in $Q_n(n \geq 1)$ such that $d(u,v) = l$. Then there exists paths of length $l, l+2, l+4, \cdots, L$, where $L = 2^n - 1$ if l is odd, and $L = 2^n - 2$ if l is even.

Lemma 4 [20]. Let $Q_n(n \geq 3)$ be an n–cube with $|f_e| + |F_v| \leq 2n-4$ and $|f_e| \leq 2n-5$ in which any node is incident to at least two critical edges. Then Q_n contains a fault-free cycle of length at least $2^n - 2|F_v|$.

Lemma 5 [22]. Let x, x', y, y' be four distinct vertices of Q_n. If both $H(x,x')$ and $H(y,y')$ are odd, then there exist two vertex-disjoint paths $P[x,x']$ and $P[y,y']$ such that $V(Q_n) = V(P[x,x']) \cup V(P[y,y'])$.

Lemma 6 [23]. If $Q_n(n \geq 3)$ has at most $2n-5$ fault edges and each vertex is incident to at least two fault-free edges, then for any two vertices u and v with different parity, there exists a fault-free Hamilton path between u and v.

Lemma 7 [16]. Let $Q_n(n \geq 3)$ be an n–cube with $|F_e| + |F_v| \leq n-2$. Then for any arbitrary fault-free nodes u and v with different parity, Q_n contains a fault-free path $P[u,v]$ with length at least $2^n - 2|F_v| - 1$.

In the conditional fault tolerant model, the following lemmas are useful. F_v and FF_e denote the set of faulty nodes and set of faulty links in networks respectively.

Lemma 8 [18]. $FQ_n - F_v - FF_e$ for $n \geq 3$ contains a fault-free cycle of length at least $2^n - 2F_v$ if $|F_v| + |FF_e| \leq 2n-4$ and $|FF_e| \leq n-1$.

Lemma 9 [20]. $FQ_n - F_v - FF_e$ for $n \geq 3$ contains a fault-free cycle of length at least $2^n - 2F_v$ if (1) $|F_v| + |FF_e| \leq 2n-4$ and $|FF_e| \geq n$ and (2) every node in FQ_n is incident to at least two fault-free links.

On lower bound of longest fault-free cycle in Q_n, Du et al. [24] obtained the result summarized in Lemma 10.

Lemma 10 [24]. $Q_n - F_v - FF_e$ for $n \geq 3$ contains a fault-free cycle of length at least $2^n - 2F_v$ if (1) $|F_v| + |FF_e| \leq 2n-4$ and $|FF_e| \leq 2n-5$ and (2) every node in Q_n is incident to at least two fault-free links.

Lemma 11 [2]. $Q_{n,k}(n \geq 3)$ with $|FF_e| \leq 2n-3$, in which each node is incident to at least two fault-free links, contains fault-free cycles of every length from 4 to 2^n when n and k have the same parity.

Because $FQ_n = Q_{n,k}$ when $k = 1$, the following corollary can be directly derived.

Lemma 12 FQ_n for $n \geq 3$, in which each node is incident to at least two fault-free links, contains a fault-free cycle of every length from 4 to 2^n when $|FF_e| \leq 2n - 3$.

3 The Cycle Embedding

To show the fault tolerance of FQ_n, it is essential to prove that FQ_n is excellent in keeping as many nodes connected as possible when faults happen. This section will prove that, under certain conditions which are looser than before, FQ_n is able to keep almost all nodes connected except a very small number of nodes in case of faults happening.

Theorem 1. Let $FQ_n(n \geq 5)$ be an n-dimensional folded hypercube with $|f_e| + |F_v| \leq 2n - 4$, and $|f_e| \geq 1$ in which any node is incident to at least three critical edges. Then FQ_n contains a fault-free cycle of length at least $2^n - 2|F_v|$.

Proof: By lemma 2, FQ_n can be partitioned into two $(n-1)$-dimensional hypercubes such that $|(E_i \cup E_c) \cap f_e| \geq 1$, and each vertex in Q^{i0}_{n-1} (respectively, Q^{i1}_{n-1}) is incident to at least two critical edges in Q^{i0}_{n-1} (respectively, Q^{i1}_{n-1}). Without loss of generality, assume that $|F^0| \geq |F^1|$. Hence, $2|F^1| \leq |F^0| + |F^1| \leq 2n - 5$, which implies that $|F^1| \leq n - 3$. Consider all different cases as following.

Case 1. $|F^0| = 2n - 5$.

Clearly, $|(E_i \cup E_c) \cap f_e| \geq 1$, $|F^1| = 0$, and $0 \leq |f^0_e| \leq 2n - 5$. Let $uu^i \in f_e$, where $u \in Q^{i0}_{n-1}$, $u^i \in Q^{i1}_{n-1}$. Note that both u, u^i are fault-free, besides uu^i, there exist at most $n - 3$ fault edges incident to u in Q^{i0}_{n-1}, that is, $|f_e * [u]| \leq n - 3$.

Case 1.1. f^0_e is not a subset of $f_e * [u]$.

If $|f^0_e| \geq |\mathcal{F}_e * [u]| + 2$, there exist two non-critical edges $l_1, l_2 \in f^0_e - f_e * [u]$. Note that $|F^0 - \{l_1, l_2\}| = |F^0| - 2 = 2n - 7 < 2n - 6$, and $|f^0_e - \{l_1, l_2\}| \leq 2n - 7$. Then $Q^{i0}_{n-1} - (F^0 - \{l_1, l_2\})$ contains a cycle of C_0 of length at least $2^{n-1} - 2|F^0_v|$ by lemma 4. According to whether C_0 contains l_1, l_2 or not, we have the following scenarios.

Case 1.1.1. l_1 and l_2 are not in C_0. Since $\{l_1, l_2\} \subset f^0_e$ and $|F^0| = 2n - 5$, we have $|F^0_v| \leq 2n-7$. Moreover, because $|(E_i \cup E_c) \cap f_e| \geq 1$, and $\frac{|C_0|}{2} \geq \frac{2^{n-1}-2(2n-7)}{2} > 2$ for $n \geq 5$, there is an edge $u_0v_0 \in C_0$ such that $u_0u^i_0$ and $v_0v^i_0$ are fault-free. Since $|F^1| = 0$, $|F^1_v| = |f^1_e| = 0$. By lemma 1, $|F_e| = 0$. Consequently Q^{i1}_{n-1} is fault-free. By lemma 3, there exists a fault-free Hamilton path $P[u^i_0, v^i_0]$ of length $2^{n-1} - 1$ connecting u^i_0 and v^i_0 in Q^{i1}_{n-1}. Then $(C_0 - u_0v_0) \oplus u_0u^i_0 \oplus P[u^i_0, v^i_0] \oplus v^i_0v_0$ forms a fault-free cycle of length $(2^{n-1} - 2|F^0_v|) - 1 + 2 + (2^{n-1} - 1) = 2^n - 2|F_v|$ in FQ_n.

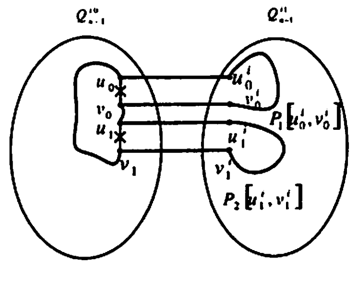

Figure 1: An illustration of Case 1.1.2 in the proof of Theorem 1.

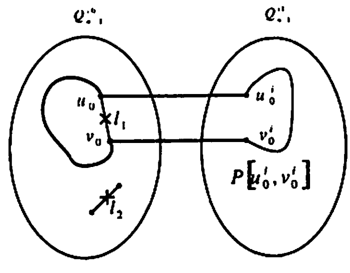

Figure 2: An illustration of Case 1.1.3 in the proof of Theorem1.

(In this paper, the symbol $\oplus$ is used to represent the path-conjunction operation to connect paths or edges.)

Case 1.1.2. l_1 and l_2 are in C_0. Let $l_1 = u_0v_0$ and $l_2 = u_1v_1$. Since $l_1, l_2 \notin f_e * [u]$, four edges $u_0u_0^i$, $v_0v_0^i$, $u_1u_1^i$, and $v_1v_1^i$ are fault-free (because $|F^0| = 2n-5$ and $uu^i \in f_e$). Because $|F^1| = 0$, by lemma 5, there exist two paths $P_1[u_0^i, v_0^i]$ and $P_2[u_1^i, v_1^i]$ such that $V(Q_{n-1}^{i0}) = V(P_1[u_0^i, v_0^i]) \cup P_2[u_1^i, v_1^i]$. For convenience, suppose nodes $u_0, v_0, u_1, andv_1$ occur in C_0 clockwise. Then we partition C_0 into four sections $C_0 = u_0v_0 \cup C_1[v_0, u_1] \cup u_1v_1 \cup C_2[v_1, u_0]$ where $C_1[v_0, u_1]$ and $C_2[v_1, u_0]$ are the sections of cycle C_0 from v_0 to u_1 and from v_1 to u_0 clockwise, respectively. Then $u_0u_0^i \oplus P_1[u_0^i, v_0^i] \oplus v_0^iv_0 \oplus C_1[v_0, u_1] \oplus u_1u_1^i \oplus P_2[u_1^i, v_1^i] \oplus v_1^iv_1 \oplus C_2[v_1, u_0]$ forms a cycle of length $(2^{n-1} - 2|F_v^0|) - 2 + 4 + (2^{n-1} - 2) = 2^n - 2|F_v^0| = 2^n - 2|F_v|$ in FQ_n.

Case 1.1.3. l_1 or l_2 but not both is in C_0, say l_1 is in C_0. Let $l_1 = u_0v_0$. Since $l_1 \notin f_e * [u]$, $u_0u_0^i$ and $v_0v_0^i$ are fault-free, and $|F^1| = 0$. There exists a Hamilton path $P[u_0^i, v_0^i]$ in Q_{n-1}^{i1}]. Then $(C_0 - u_0v_0) \oplus u_0u_0^i \oplus P[u_0^i, v_0^i] \oplus v_0^iv_0$ forms a fault-free cycle of length $(2^{n-1} - 2|F_v^0|) - 1 + 2 + (2^{n-1} - 1) = 2^n - 2|F_v|$ in FQ_n.

Case 1.2. $f_e^0 \subseteq f_e * [u]$.

Because $f_e * [u] \subseteq f_e^0$, we have $f_e^0 = f_e * [u]$ and $|f_e^0| = |f_e * [u]| \leq n-3$.

Case 1.2.1. F_v^0 is not a subset of $F_v * [u]$. There exists a fault node $d \in$

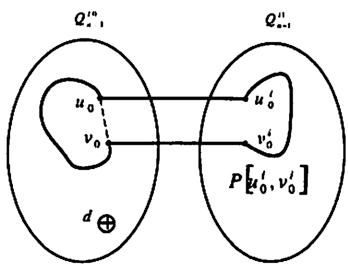

Figure 3: An illustration of Case 1.2.1.1 in the proof of Theorem1.

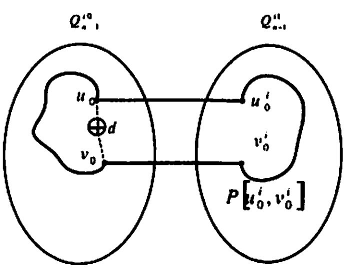

Figure 4: An illustration of Case 1.2.1.2 in the proof of Theorem1.

$F_v^0 - F_v * [u]$. Because $|F^0 - \{d\}| = |F^0| - 1 = 2n - 6$ and $|f_e^0| \leq n - 3 \leq 2n - 7$ for $n \geq 5$, by lemma 4, $Q_{n-1}^{i0} - (F^0 - \{d\})$ contains a cycle C_0 of length at least $2^{n-1} - 2(|F_v^0| - 1) = 2^{n-1} - 2|F_v^0| + 2$. According to whether C_0 contains node d or not, we have the following scenarios.

Case 1.2.1.1. Node d is not in C_0. Because $|(E_i \cup E_c) \cap f_e| \geq 1$, and $\frac{|C_0|}{2} \geq \frac{2^{n-1}-2|F_v^0|+2}{2} \geq \frac{2^{n-1}-2(2n-5)+2}{2} > 2$ for $n \geq 5$, there exists an edge $u_0v_0 \in C_0$ such that $u_0u_0^i$ and $v_0v_0^i$ are fault-free. Because Q_{n-1}^{i1} is fault-free, then by lemma 3, it contains a fault-free Hamilton path $P[u_0^i, v_0^i]$ with length $2^{n-1} - 1$. Therefore, $(C_0 - u_0v_0) \oplus u_0u_0^i \oplus P[u_0^i, v_0^i] \oplus v_0^iv_0$ forms a cycle with length $2^{n-1} - 2|F_v^0| + 2 - 1 + 2 + (2^{n-1} - 1) = 2^n - 2|F_v^0| + 2 > 2^n - 2|F_v|$ in FQ_n.

Case 1.2.1.2. Node d is in C_0. Let u_0 and v_0 are two nodes adjacent to d in C_0. Since Q_{n-1}^{i1} is fault-free, by lemma 3, it contains a path $P[u_0^i, v_0^i]$ of length $2^{n-1} - 1$. Therefore $(C_0 - u_0d - dv_0) \oplus u_0u_0^i \oplus P[u_0^i, v_0^i] \oplus v_0^iv_0$ is a cycle of length $(2^{n-1} - 2|F_v^0| + 2) - 2 + 2 + (2^{n-1} - 2) = 2^n - 2|F_v|$ in FQ_n.

Case 1.2.2. $F_v^0 \subseteq F_v * [u]$. By the assumption that $f_e^0 \subseteq f_e * [u]$ and $F_v^0 \subseteq F_v * [u]$, $F^0 = (F_v^0 \cup f_e^0) \subseteq (F_v * [u] \cup f_e * [u])$ holds. However, $F_v * [u] \cup f_e * [u] \subseteq F_v^0 \cup f_e^0$ also holds. This leads to $|F^0| = 2n - 5 = |(F_v * [u] \cup f_e * [u]| \leq n - 1$, which implies that $n \leq 4$, contradicting to $n \geq 5$.

Case 2. $|F^0| = |f_e^0| = 2n - 6$.

Because $|f_e| \le 2n-4$ and $|(E_i \cup E_c) \cap f_e| \ge 1$, then $|f_e^1| \le (2n-4)-1-(2n-6) = 1$ and $|F_v^1| \le (2n-4)-(2n-6)-1-1 = 0$. Let uu^i be the non-critical edge between Q_{n-1}^{i0} and Q_{n-1}^{i1}. By lemma 4, Q_{n-1}^{i0} contains a cycle C_0 of length $2^{n-1}-2|F_v^0|$. Select $u_0v_0 \in C_0$ such that $u_0u_0^i$ and $v_0v_0^i$ are fault-free. Since $|f_e^1| \le 1$ and $|F_v^1| = 0$, by lemma 6, Q_{n-1}^{i1} contains a fault-free Hamilton path $P[u_0^i, v_0^i]$. Then $(C_0 - u_0v_0) \oplus u_0u_0^i \oplus P[u_0^i, v_0^i] \oplus v_0^iv_0$ is a cycle of length $(2^{n-1}-2|F_v^0|)-1+2+(2^{n-1}-1) = 2^n-2|F_v|$ in FQ_n.

Case 3. $|F^0| \le 2n-6$ and $|f_e^0| \le 2n-7$.

By lemma 4, Q_{n-1}^{i0} contains a cycle C_0 with length $2^{n-1}-2|F_v^0|$. Because $|F^0| \ge |F^1|$ and $|(E_i \cup E_c) \cap f_e| \ge 1$, we have that $2|F^1| \le |F^0| + |F^1| \le 2n-5$. Thus $|F^1| \le n-3$. Select $u_0v_0 \in C_0$ such that $u_0, v_0, u_0u_0^i, and v_0v_0^i$ are fault-free. By lemma 7, $Q_{n-1}^{i1} - F^1$ contains a path $P[u_0^i, v_0^i]$ with length at least $2^{n-1}-2|F_v^1|-1$. Therefore, $(C_0 - u_0v_0) \oplus u_0u_0^i \oplus P[u_0^i, v_0^i] \oplus v_0^iv_0$ forms a cycle of length at least $(2^{n-1}-2|F_v^0|)-1+2+(2^{n-1}-2|F_v^1|-1) = 2^n-2|F_v|$ in FQ_n.

The above cases have covered all situation so have proved the theorem.

Corollary. Let $FQ_n (n \ge 5)$ be an n-dimensional folded hypercube with $|f_e| \le 2n-4$ and $|F_v| + |f_e| \le 2n-4$ in which each node is incident to at least three critical edges. Then FQ_n contains a fault-free cycle of length at least $2^n - 2|F_v|$.

Proof: If $|f_e| = 0$, then $|F_v| \le 2n-4$. Since $V(Q_n) = V(FQ_n)$, by lemma 4, Q_n contains a fault-free cycle of length at least $2^n - 2|F_v|$, which means the corollary is true. If $|f_e| \ne 0$, the theorem above guarantees the corollary. The proof has been completed.

A node is called k-free if it incident to exact k fault-free links in a faulty network.

As the second major topic of this paper, we now present a result to improve the results shown in Lemma 8 and Lemma 9. We will prove that the lower bound of longest fault-free cycle in FQ_n is $2^n - 2F_v$ on the constrains that (1) $|F_v| + |FF_e| \le 2n-3$ and $|FF_e| \ge 1$ and (2) every node in FQ_n is incident to at least two fault-free links. Before the analysis, the following lemmas need to be established.

Lemma 13. If $|F_v| + |FF_e| \le 2n-3$, then there exists at most two 2-free nodes in $FQ_n (n \ge 3)$.

Proof: Assume that FQ_n contains at least three 2-free nodes. The least total number of faulty elements is $(n-1)+(n-1)+(n-1)-2 = 3n-5 > 2n-3$, which contradicts to the condition $|F_v| + |FF_e| \le 2n-3$. Thus, FQ_n contains at most two 2-free nodes.

Lemma 14. If $|F_v| + |FF_e| \le 2n-3$ and $FQ_n (n \ge 3)$ contains exact two 2-free nodes u, v, then $d(u, v) = 1$ and $uv \in FF_e$.

Proof: If $d(u,v) > 1$, then the least number of total fault links is $(n-1)+(n-1) = 2n-2 > 2n-3$. If $uv \notin FF_e$, then the least number of fault elements is $(n-1)+(n-1) = 2n-2 > 2n-3$. It is contradict to the assumption.

Theorem 2. There exists a fault-free cycle of length at least $2^n - 2F_v$ in FQ_n for $n \geq 3$ if (1) $|F_v| + |FF_e| \leq 2n-3$ and $|FF_e| \geq 1$ and (2) every node is incident to at least two fault-free links whose other end nodes are fault-free.

Proof: When $|F_v| + |FF_e| \leq 2n-4$, Lemma 8 and Lemma 9 guarantee the theorem to be true.

When $|F_v| = 0$, $|FF_e| = 2n-3$, Lemma 12 implies the truth of the theorem. So we need only to discuss the case of $|F_v| + |FF_e| = 2n-3$ and $|F_v| \geq 1$ and $1 \leq |FF_e| \leq 2n-4$. For the sack of clearance, we consider the following three cases according to the number of 2-free nodes.

Case 1. FQ_n contains no 2-free nodes. That is, every node in FQ_n is incident to at least three fault links. In this case, we choose some $j \in \{1,2,\cdots,n\} \cup \{c\}$ such that $E_j \cap FF_e \neq \emptyset$ ($\emptyset$ denotes the empty set). $FQ_n - E_j$ is isomorphic to Q_n, and in the subgraph Q_n, there are at most $(2n-4)-1 = 2n-5$ fault links, and the total number of fault elements is at most $(2n-3)-1 = 2n-4$. Since every node in $FQ_n - E_j$ is incident to at least two fault-free links, by Lemma 10, there exists a fault-free cycle of length at least $2^n - 2|F_v|$ in the subgraph $FQ_n - E_j$. Therefore, $FQ_n - F_v - FF_e$ contains a fault-free cycle of length at least $2^n - 2|F_v|$.

Case 2. There is a unique 2-free node u in FQ_n, then very node of $FQ_n - u$ is incident to at least three fault-free links. In this case, the number of faulty links incident to u is $n-1$. We can choose some $j \in \{1,2,\cdots,n\} \cup \{c\}$ such that uu^j (or $u\bar{u}$) is faulty. Then $FQ_n - E_j$ is isomorphic to Q_n. With the same argument as Case 1, $FQ_n - F_v - FF_e$ contains a fault-free cycle of length at least $2^n - 2|F_v|$.

Case 3. There are two 2-free nodes u and v in FQ_n. By Lemma 14, $uv \in FF_e$. Let $uv \in E_j$, then $FQ_n - E_j$ contains at most $2n-4-1 = 2n-5$ faulty links, and at most $2n-3-1 = 2n-4$ faulty elements. By Lemma 10, there exists a fault-free cycle with length of $2^n - 2|F_v|$ in $FQ_n - E_j$. Hence the theorem is true.

4 Conclusion

Network topology is an important issue in the design of computer networks since it is crucial to many key properties such as the efficiency and fault tolerance. Every component in a computer network may have reliability problems, so it is important to consider the fault tolerance properties of networks. In this paper, we focus on the cycle embedding in the n-dimensional folded hypercube networks FQ_n (which is an important network topology for parallel processing computer systems) with node-fault-tolerance and edge-fault-tolerance. Using the efficient fault-tolerance properties of Q_n, based on the partition of n-dimensional folded hypercube networks FQ_n into two $(n-1)$-dimensional hypercubes Q_{n-1}, it is proved that when $|f_e| + |F_v| \leq 2n - 4$, FQ_n contains a fault-free cycle of length at least $2^n - 2|F_v|$ in which any node is incident to at least three critical edges. Meanwhile, FQ_n contains a fault-free cycles with length at least $2^n - 2|F_v|$, under the constraints that (1) The number of both faulty nodes and faulty edges is no more than $2n - 3$ and there is at least one faulty edge; (2) every node in FQ_n is incident to at least two fault-free links.

These properties imply that the reliability, efficiency and fault tolerance of FQ_n are better than hypercube Q_n. The mathematical proof in this paper theoretically shows that interconnection networks modeled by folded hypercube are extremely robust, which makes the folded hypercube an excellent choice of network topology for parallel processing computer systems.

Our future work will be focusing on the longest cycle embedding in enhanced hypercube $Q_{n,k}$ when there are many faulty edges and faulty vertices occur simultaneously. The folded hypercube FQ_n is a special type of $Q_{n,k}$ $(1 \leq k \leq n-1)$ when $k = 1$. Some researches have had preliminary results on the fault tolerance properties of $Q_{n,k}$. For instance, [2] shows that $Q_{n,k}$ $(1 \leq k \leq n-1)$ contains a fault-free cycle of every even length from 4 to 2^n when $n(\geq 3)$ and k have the same parity, and a fault-free cycle of every odd length from $n - k + 2$ to $2^n - 1$ when $n(\geq 2)$ and k have the different parity, and each vertex of $Q_{n,k}$ $(1 \leq k \leq n-1)$ is incident to at least two fault-free edges. [27] shows that $Q_{n,k} - F_v$ contains a fault-free cycle of every even length from 4 to $2^n - 4$ where $n(n \geq 3)$ and k have the same parity, and contains a fault-free cycle of every even length from 4 to $2^n - 4$, simultaneously, contains a cycle of every odd length from $n - k + 2$ to $2^n - 3$ where $n(\geq 3)$ and k have the different parity when $|F_v| = 2$. Furthermore, when $|F_v| = f_v \leq n - 2$, there exists the longest fault-free cycle, which is of even length $2^n - 2f_v$ whether $n(n \geq 3)$ and k have the same parity or not; and there exists the longest fault-free cycle, which is of odd length $2^n - 2f_v + 1$ in $Q_{n,k} - F_v$ where $n(\geq 3)$ and k have the different parity. [28] proves that $Q_{n,k} - F_v$ contains a fault-free cycle of length at least $2^n - 2|F_v|$ if $|F_v| + |f_e| \leq 2n - 4$ and each node in $Q_{n,k} - F_v$ is incident with at least two fault-free edges. It is natural to extend the excellent fault tolerance results of this paper to the wider range of networks $Q_{n,k}$.

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