Almost Arbitrary Supersubdivision of Every Graph is Cordial

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Abstract

Let G be a graph with q edges. A graph G^* is called an arbitrary supersubdivision of G if G^* is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i}, such a way that the end vertices of each e_i are identified with the two vertices of the 2-vertices part of K_{2,m_i} after removing the edge e_i from G, where m_i of K_{2,m_i} may vary arbitrarily for each edge e_i, $1 \leq i \leq q$. As recognition of cordial graph is an NP-complete, it is interesting and significant to find the graphs whose arbitrary supersubdivision graphs are cordial. In this paper, we show that arbitrary supersubdivision of every bipartite graph is cordial. This result is obtained as a corollary of the general result that "Almost arbitrary supersubdivision of every graph is cordial", where almost arbitrary supersubdivision is a relaxation of arbitrary supersubdivision graph. Let G be a graph with edge set $E(G) = E_1 \cup E_2$ and $E_1 \cap E_2 = \phi$. A graph $\hat{G}$ is called an almost arbitrary supersubdivision graph of G if $\hat{G}$ is obtained from G by replacing every edge $e_i \in E$ by a complete bipartite graph K_{2,m_i}; such a way that the end vertices of each e_i are merged with the two vertices of the 2-vertices part of K_{2,m_i} after

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removing the edge e_i from G, where m_i is chosen as an arbitrary positive integer if $e_i \in E_1$ or else m_i is chosen as an arbitrary even positive integer if $e_i \in E_2$.

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1 Introduction

At the smolenice symposium in 1963, Ringel [18] conjectured that K_{2m+1}, the complete graph on $2m+1$ vertices can be decomposed into $2m+1$ isomorphic copies of a given tree with m edges. In an attempt to solve Ringel's conjecture, in 1967 Rosa [19] introduced an hierarchical series of labeling ρ, σ, β and α-valuations and used these valuations for a cyclic decomposition of K_{2m+1} into trees with m edges. Later Golomb [10] called β-valuation as graceful.

A function f is called a graceful labeling of G with m edges, if f is an injection from the vertices of G to the set $\{0,1,2,\ldots,m\}$, such that when each edge uv is assigned the label $|f(u) - f(v)|$ then the resulting edge labels are distinct.

Harmonious labeling was introduced by Graham and Sloane [11] in connection with their study on error correcting code. A function f is called harmonious labeling of a graph G with m edges, if f is an injection from the vertices of G to the group of integers modulo m, such that when each edge uv is assigned the label $(f(u)+f(v)) \pmod{m}$ then the resulting edge labels are distinct.

In 1987, Cahit [4] introduced cordial labeling as a variation of both graceful and harmonious labeling. Let f be a function from the vertices of G to $\{0,1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. Call f a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1 and the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1.

Number of special families of graphs are shown to be cordial, Refer [1–3, 5–8,14–16,20,22–24]. For an exhaustive survey on cordial labeling refer the excellent survey on Graph Labeling by Gallian [13]. Niall Cairnie and Keith Edwards [17] have proved the problem of deciding whether or not a graph G is cordial is an NP-complete. Niall and Keith result motivates to construct or recognize the cordial graphs in a general approach or algorithmically.

In [21] Sethuraman and Selvaraju have introduced a graph operation called arbitrary supersubdivision graph. Let G be a graph with q edges. A graph G^* is called an arbitrary supersubdivision of G if G^* is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i}, such a way that the end vertices of each e_i are identified with the two vertices of the 2-vertices part of the K_{2,m_i} after removing the edge e_i from G, where m_i of K_{2,m_i} may vary arbitrarily for each edge e_i, $1 \leq i \leq q$. Here we introduce a relaxation on the arbitrary supersubdivision graph of a graph called an almost arbitrary supersubdivision.

Let G be a graph with edge set $E(G) = E_1 \cup E_2$ and $E_1 \cap E_2 = \phi$. A graph $\hat{G}$ is called an almost arbitrary supersubdivision graph of G if $\hat{G}$ is obtained from G by replacing every edge $e_i \in E$ by a complete bipartite graph K_{2,m_i}; such a way that the end vertices of each e_i are merged with the two vertices of the 2-vertices part of K_{2,m_i} after removing the edge e_i from G, where m_i is chosen as an arbitrary positive integer if $e_i \in E_1$ or else m_i is chosen as an arbitrary even positive integer if $e_i \in E_2$.

In this paper, we show that almost arbitrary supersubdivision of every graph is cordial. The proof of our result would naturally imply that arbitrary supersubdivision of every bipartite graph is cordial. This result generalizes the results of Vaidya et al. [25, 26] that arbitrary supersubdivision of special classes of bipartite graphs, trees, $P_m \times P_n$, $C_{2n} \odot P_m$ are cordial. Finally we discuss a related open problem.

Main Result

In this section we prove our main result that almost arbitrary supersubdivision graph of every graph is cordial. First, we give an algorithm to construct an almost arbitrary supersubdivision graph from a given graph G.

The following Algorithm 1 that constructs almost arbitrary supersubdivision graph from a given graph G uses the Breadth First Search algorithm in Step 1. The Breadth First Search algorithm (BFS) is a fundamental graph algorithm. When the BFS is run on a connected graph G, then it finds a spanning tree T of G as its output. For more details about the BFS refer [9, 12].

Let G be a connected graph. Run the BFS on G and obtain the BFS spanning tree T. A vertex of G is said to lie in a *level* ℓ of G if it lies in the level ℓ of the BFS spanning tree T of G. An edge of G is called the *same level edge* if the end vertices of the edge lie in the same level ℓ. Similarly an edge of G is called *different level edge* if the end vertices of the edge lie in two different levels of G.

Algorithm 1 (Construction of $\hat{G}$ from a given graph G)

Input. A graph $G = (V, E)$.
Step 1. If G is a connected graph, then run the BFS on G, obtain the BFS spanning tree T of G and determine all the same level edges and all the

different level edges of G. Let E_1 be the set of all different level edges of G and let E_2 be the set of all same level edges of G. Then $E(G) = E_1 \cup E_2$ and $E_1 \cap E_2 = \phi$.
Step 2. Let $e = uv$ be an edge of G, if the edge $e \in E_1$, that is, e is a different level edge of G, then choose a complete bipartite graph $K_{2,m}$; where m is an arbitrary positive integer or else if the edge $e \in E_2$, that is, e is a same level edge of G, then choose a complete bipartite graph $K_{2,m}$; where m is an arbitrary even positive integer. Replace the edge $e = uv$ by a complete bipartite graph $K_{2,m}$ in such a way that the end vertices u, v of e are merged with the two vertices of the 2-vertices part of the $K_{2,m}$, the graph thus obtained is an almost arbitrary supersubdivision graph of G. Let $\hat{G}$ denote the almost arbitrary supersubdivision graph of G obtained by the above process.
Step 3. If G is not connected, then find all the connected components of G. Let $G_1, G_2, \ldots, G_t$ be the connected components of G. Then for each connected components G_i, $1 \leq i \leq t$, obtain the supersubdivision graph $\hat{G}_i$, by applying Steps 1 and 2. Then obtain $\hat{G}_i = \hat{G}_1 \cup \hat{G}_2 \cup \cdots \cup \hat{G}_t$, which is an almost arbitrary supersubdivision graph of G.

Observation 1. If the graph G is bipartite then observe that there does not exist any same level edge in G with respect to the BFS spanning tree T of G (otherwise, odd cycle will be induced in G). If the graph G is not a bipartite then there exists at least one same level edge with respect to the BFS spanning tree T of G. Thus, if the graph G is bipartite, then the almost arbitrary supersubdivision graph of G, $\hat{G}$ constructed by Algorithm 1 is nothing but the arbitrary supersubdivision graph G^* of G.

Remark 1. A vertex u of an arbitrary supersubdivision graph $\hat{G}$ of a graph G is called a *base vertex* if u is the vertex of G that appear as a vertex of the 2-vertices part of the $K_{2,m}$, which replaces an edge $e = uv$ of G in obtaining $\hat{G}$. A vertex w of $\hat{G}$ is called a *non-base vertex* if w is not a base vertex of $\hat{G}$. Observe that every non-base vertex of $\hat{G}$ always lie in the m vertices part of the $K_{2,m}$ which replaces an edge e of G in obtaining $\hat{G}$.

Theorem 1.1. *Almost arbitrary supersubdivision graph $\hat{G}$ of any graph G is cordial.*

Proof. Let G be a graph.
Case 1. G is connected.
Let G be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $E(G) = \{e_1, e_2, \ldots, e_q\}$. Let E_1 denote the set of all different level edges of G and E_2 denote the set of all same level edges of G. Consider an almost arbitrary supersubdivision graph $\hat{G}$ of G, constructed by Algorithm 1. Then, $\hat{G}$ has $p + \sum_{i=1}^{q} m_i$ vertices and $2\sum_{i=1}^{q} m_i$ edges, where the p vertices of

$\hat{G}$ are originally the vertices of G, that is the base vertices of $\hat{G}$ and the remaining $\sum_{i=1}^{q} m_i$ vertices are the non-base vertices of $\hat{G}$. We denote the set of all base vertices of $\hat{G}$ by $B(\hat{G})$ and we denote the set of all non-base vertices of $\hat{G}$ by $NB(\hat{G})$.

Step 1. 0-1 labeling of the base vertices of $\hat{G}$

For the base vertex v of $\hat{G}$ considering v as the vertex in G, find the level of the vertex v with respect to the spanning tree T of G. If the level of the base vertex v is even then assign the label 0 or else if the level of the base vertex v is odd then assign the label 1.

Let $B_0(\hat{G})$ denote the set of all base vertices of $\hat{G}$ having the label 0 and let $B_1(\hat{G})$ denote the set of all base vertices of $\hat{G}$ having the label 1.
If $|B_1(\hat{G})| > |B_0(\hat{G})|$ then find $r_1 = |B_1(\hat{G})| - |B_0(\hat{G})|$.
If $|B_0(\hat{G})| > |B_1(\hat{G})|$ then find $r_0 = |B_0(\hat{G})| - |B_1(\hat{G})|$.

Step 2. 0-1 labeling of non-base vertices of $\hat{G}$

Let $NB(\hat{G}_{E_1})$ denote the set of all non-base vertices of $\hat{G}$ which are obtained by replacing every edge $e_i \in E_1$ of G by K_{2,m_i}, that is, e_i is a different level edge of G, where m_i is an arbitrary positive integer.

Similarly, let $NB(\hat{G}_{E_2})$ denote the set all non-base vertices of $\hat{G}$ which are obtained by replacing every edge $e_j \in E_2$ of G by K_{2,m_j}, that is, e_j is a same level edge of G, where m_j is an arbitrary even positive integer.

Thus the set of all non-base vertices of $\hat{G}$, $NB(\hat{G}) = NB(\hat{G}_{E_1}) \cup NB(\hat{G}_{E_2})$. Also note that $NB(\hat{G}_{E_1}) \cap NB(\hat{G}_{E_2}) = \phi$.

If r_0 exists, then there exist r_0 different edges in E_1 having one of the end vertices labeled with 0.

Observe that in constructing $\hat{G}$ these r_0 edges of G in E_1 is replaced by a complete bipartite graph $K_{2,m}$, where $m \geq 1$. Thus $\hat{G}$ always has at least r_0 non-base vertices.

If r_1 exists, by similar argument, it follows that $\hat{G}$ always has at least r_1 non-base vertices.

First, we give 0-1 labels to the non-base vertices of $NB(\hat{G}_{E_2})$ and we give 0-1 labels to the non-base vertices of $NB(\hat{G}_{E_1})$.

Step 2.1. 0-1 labeling of non-base vertices of $NB(\hat{G}_{E_2})$

For each j, $1 \leq j \leq q_2$, consider the complete bipartite graph K_{2,m_j} which is the replacement of the same level edge $e_j \in E_2$ in constructing $\hat{G}$. Then by the Algorithm 1, m_j is chosen as an arbitrary even positive integer. For each j, $1 \leq j \leq q_2$, assign the label 0 to $\frac{m_j}{2}$ vertices of m_j-part in K_{2,m_j} of $NB(\hat{G}_{E_2})$ and assign the label 1 to the other $\frac{m_j}{2}$ vertices of m_j-part in K_{2,m_j} of $NB(\hat{G}_{E_2})$.

Consequently $\frac{|NB(\hat{G}_{E_2})|}{2}$ non-base vertices get the label 0 and $\frac{|NB(\hat{G}_{E_2})|}{2}$ non-base vertices get the label 1.

Step 2.2. 0-1 labeling of the non-base vertices of $NB(\hat{G}_{E_1})$

Step 2.2.1. r_0 exists

Then assign 1 to the first set of r_0 non-base vertices in $NB(\hat{G}_{E_1})$.

Consider all the remaining $\sum_{i=1}^{q} m_i - r_0 - NB(\hat{G}_{E_2})$ non-base vertices in $NB(\hat{G}_{E_1})$. Assign 0 to $\left\lfloor \frac{\sum_{i=1}^{q} m_i - r_0 - NB(\hat{G}_{E_2})}{2} \right\rfloor$ non-base vertices in $NB(\hat{G}_{E_1})$ and assign 1 to the other $\left\lceil \frac{\sum_{i=1}^{q} m_i - r_0 - NB(\hat{G}_{E_2})}{2} \right\rceil$ non-base vertices in $NB(\hat{G}_{E_1})$.

Let $V_0(\hat{G})$ denote the set of all vertices of $\hat{G}$ assigned the label 0 and let $V_1(\hat{G})$ denote the set of all vertices of $\hat{G}$ assigned the label 1.

From Step 1, Step 2.1 and Step 2.2.1, we observe that, in $\hat{G}$, the number of vertices labeled 0 is equal to

$$|B_0(\hat{G})| + \left\lfloor \frac{\sum_{i=1}^{q} m_i - r_0 - NB(\hat{G}_{E_2})}{2} \right\rfloor + \frac{|NB(\hat{G}_{E_2})|}{2} \qquad (1)$$

and the number of vertices labeled 1 is equal to

$$|B_1(\hat{G})| + \left\lceil \frac{\sum_{i=1}^{q} m_i - r_0 - NB(\hat{G}_{E_2})}{2} \right\rceil + \frac{|NB(\hat{G}_{E_2})|}{2} + r_0 \qquad (2)$$

Since $|B_0(\hat{G})| = |B_1(\hat{G})| + r_0$, from (1) and (2) we have

$$\Big| |V_0(\hat{G})| - |V_1(\hat{G})| \Big| \leq 1.$$

Step 2.2.2. r_1 exists

Then assign 0 to the first set of r_1 non-base vertices in $NB(\hat{G}_{E_1})$.

Consider all the remaining $\sum_{i=1}^{q} m_i - r_1 - NB(\hat{G}_{E_1})$ non-base vertices in $NB(\hat{G}_{E_1})$. Assign 0 to $\left\lfloor \frac{\sum_{i=1}^{q} m_i - r_1 - NB(\hat{G}_{E_2})}{2} \right\rfloor$ non-base vertices in $NB(\hat{G}_{E_1})$ and assign 1 to the other $\left\lceil \frac{\sum_{i=1}^{q} m_i - r_1 - NB(\hat{G}_{E_2})}{2} \right\rceil$ non-base vertices in $NB(\hat{G}_{E_1})$.

From Step 1, Step 2.1 and Step 2.2.2, we observe that, in $\hat{G}$, the number of vertices labeled 0 is equal to

$$|B_0(\hat{G})| + \left\lfloor \frac{\sum_{i=1}^{q} m_i - r_1 - NB(\hat{G}_{E_2})}{2} \right\rfloor + \frac{|NB(\hat{G}_{E_2})|}{2} + r_1 \qquad (3)$$

and the number of vertices labeled 1 is equal to

$$|B_1(\hat{G})| + \left\lceil \frac{\sum_{i=1}^{q} m_i - r_1 - NB(\hat{G}_{E_2})}{2} \right\rceil + \frac{|NB(\hat{G}_{E_2})|}{2} \qquad (4)$$

Since $|B_1(\hat{G})| = |B_0(\hat{G})| + r_1$, from (3) and (4) we have

$$\Big| |V_0(\hat{G})| - |V_1(\hat{G})| \Big| \leq 1.$$

Step 2.2.3. Neither r_0 nor r_1 exists

Then we have $|B_0(\hat{G})| = |B_1(\hat{G})|$. Assign 0 to $\left\lfloor \frac{\sum_{i=1}^{q} m_i - NB(\hat{G}_{E_2})}{2} \right\rfloor$ non-base vertices in $NB(\hat{G}_{E_1})$ and assign 1 to the other remaining $\left\lceil \frac{\sum_{i=1}^{q} m_i - NB(\hat{G}_{E_2})}{2} \right\rceil$ non-base vertices in $NB(\hat{G}_{E_1})$.

From Step 1, Step 2.1 and Step 2.2.3, we observe that, in $\hat{G}$, the number of vertices labeled 0 is equal to

$$|B_0(\hat{G})| + \left\lfloor \frac{\sum_{i=1}^{q} m_i - NB(\hat{G}_{E_2})}{2} \right\rfloor + \frac{|NB(\hat{G}_{E_2})|}{2} \qquad (5)$$

and the number of vertices labeled 1 is equal to

$$|B_1(\hat{G})| + \left\lceil \frac{\sum_{i=1}^{q} m_i - NB(\hat{G}_{E_2})}{2} \right\rceil + \frac{|NB(\hat{G}_{E_2})|}{2} \qquad (6)$$

Since $|B_0(\hat{G})| = |B_1(\hat{G})|$, from (5) and (6) we have

$$\Big| |V_0(\hat{G})| - |V_1(\hat{G})| \Big| \leq 1.$$

By the definition of $\hat{G}$ observe that each complete bipartite graph K_{2,m_i} which replaces the edge e_i of G can be considered as an edge induced subgraph of $\hat{G}$ for $1 \leq i \leq q$. Thus, $\hat{G}$ is an edge disjoint union of complete bipartite graphs K_{2,m_i}'s, $1 \leq i \leq q$.

Hence, $E(\hat{G}) = \bigcup_{i=1}^{q} E(K_{2,m_i})$.

Since $E(\hat{G}) = E_1 \cup E_2$ and $E_1 \cap E_2 = \phi$, where E_1 consists of all the different level edges of G and E_2 consists of all the same level edges of G.

Thus, we have

$$\begin{aligned} E(\hat{G}) &= \bigcup_{i=1}^{q} E(K_{2,m_i}) \\ &= \left[\bigcup_{i=1}^{q_1} E(K_{2,m_i}) \right] \cup \left[\bigcup_{j=1}^{q_2} E(K_{2,m_j}) \right], \end{aligned}$$

where K_{2,m_i} is the complete bipartite graph which replaces the edge $e_i \in E_1$ and K_{2,m_j} is the complete bipartite graph which replaces the edge $e_j \in E_2$, and $|E_1| = q_1$ and $|E_2| = q_2$.

For each i, $1 \leq i \leq q_1$, consider K_{2,m_i} which is the complete bipartite graph replacement of the edge $e_i \in E_1$ in constructing $\hat{G}$, then the base vertices of K_{2,m_i} are the two vertices of the 2-vertices part of K_{2,m_i} always have different labels. Among the m_i non-base vertices of K_{2,m_i}, either all may be labeled with 0 or all may be labeled with 1 or some may be labeled with 0 and the remaining may be labeled with 1. These labeling situations are shown in Figure 1. Consequently the two edges incident at every non-base vertex will have other ends labeled with two different labels 0 and 1. Thus, the two edges incident at every non-base vertex will have two different labels 0 and 1. Hence, the complete bipartite graph K_{2,m_i} has exactly m_i edges with label 0 and m_i edges with label 1. Therefore, $\sum_{i=1}^{q_1} m_i$ edges of $\bigcup_{i=1}^{q_1} E(K_{2,m_i})$ get the label 0 and $\sum_{i=1}^{q_1} m_i$ edges of $\bigcup_{i=1}^{q_1} E(K_{2,m_i})$ get the label 1.

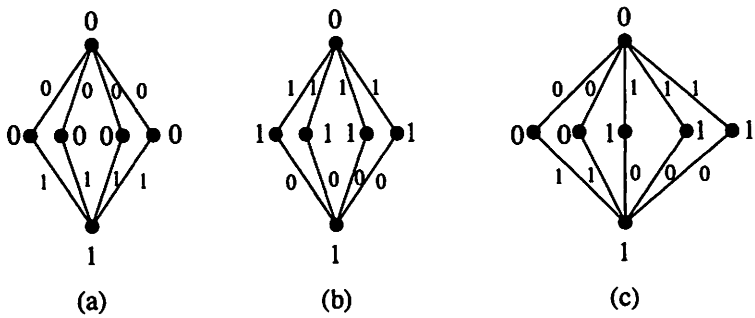

Figure 1: Possible 0-1 labeling of the complete bipartite graph replacement K_{2,m_i} of an edge $e_i \in E_1$

For each j, $1 \le j \le q_2$, consider K_{2,m_j} which is the complete bipartite graph replacement of the edge $e_j \in E_2$ in constructing $\hat{G}$. Then by the above labeling, the base vertices in K_{2,m_j} always have the same labels either 0 or 1 and among the m_j non-base vertices in K_{2,m_j}, $\frac{m_j}{2}$ non-base vertices are labeled as 0 and the remaining $\frac{m_j}{2}$ non-base vertices are labeled as 1. These labeling situations are shown in Figure 2.

Suppose both the base vertices of K_{2,m_j} have the label 0 then from the $\frac{m_j}{2}$ non-base vertices of K_{2,m_j} having the label 0 will induce the edge label 0 on m_j edges of K_{2,m_j} and from the $\frac{m_j}{2}$ non-base vertices of K_{2,m_j} having the label 1 will induce the edge label 1 on m_j edges of K_{2,m_j}. Thus the complete bipartite graph K_{2,m_j} contains m_j edges having the label 0 and m_j edges having the label 1.

Suppose both the base vertices of K_{2,m_j} have the label 1, then from the $\frac{m_j}{2}$ non-base vertices of K_{2,m_j} having the label 0 will induce the edge label 1 on the m_j edges of K_{2,m_j} and from $\frac{m_j}{2}$ non-base vertices of K_{2,m_j} having the label 1 will induce the edge label 0 on the m_j edges of K_{2,m_j}. Thus the complete bipartite graph K_{2,m_j} contains m_j edges having the label 0 and m_j edges having the label 1.

Thus, $\sum_{j=1}^{q_2} m_j$ edges of $\bigcup_{i=1}^{q_2} E(K_{2,m_j})$ gets the label 0 and $\sum_{j=1}^{q_2} m_j$ edges of $\bigcup_{i=1}^{q_2} E(K_{2,m_j})$ gets the label 1.

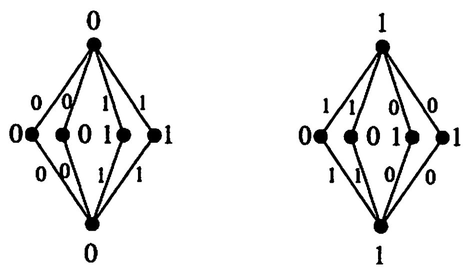

Figure 2: 0-1 labeling of K_{2,m_j}, which is the complete bipartite graph replacement of an edge $e_j \in E_2$

Let $E_0(\hat{G})$ denote the set of all edges of $\hat{G}$ gets the label 0 and let $E_1(\hat{G})$ denote the set of all edges of $\hat{G}$ gets the label 1.

Then from the above observation, we have

$$|E_0(\hat{G})| = \sum_{j=1}^{q_1} m_i + \sum_{j=1}^{q_2} m_j \tag{7}$$

$$|E_1(\hat{G})| = \sum_{j=1}^{q_1} m_i + \sum_{j=1}^{q_2} m_j \tag{8}$$

From (7) and (8), we have

$$|E_0(\hat{G})| = |E_1(\hat{G})|.$$

Hence, $\hat{G}$ is cordial.

Case 2. G is not connected.

Find all the connected components of G. Let $G_1, G_2, \ldots, G_t$ be the connected components of G. Then by the definition of almost arbitrary supersubdivision of a graph, we can consider $\hat{G} = \hat{G}_1 \cup \hat{G}_2 \cup \cdots \cup \hat{G}_t$, where $\hat{G}_i$ is an almost arbitrary supersubdivision graph of G_i, for $1 \leq i \leq t$. Let $G_i = (V_i, E_i)$ with $|V_i| = p_i$ and $|E_i| = q_i$ and let $\alpha = q_1 + q_2 + \cdots + q_t$ and $\beta = p_1 + p_2 + \cdots + p_t$.

For the convenience we order the edges of G as $e_1, e_2, \ldots, e_{q_1}$, e_{q_1+1}, e_{q_1+2}, $\ldots$, $e_{q_1+q_2}$, $e_{q_1+q_2+1}$, $e_{q_1+q_2+2}$, $\ldots$, $e_{q_1+q_2+q_3}$, $\ldots$, $e_{q_1+q_2+\cdots+q_{t-1}+1}$, $e_{q_1+q_2+\cdots+q_{t-1}+2}$, $\ldots$, $e_{q_1+q_2+\cdots+q_t}$, where $e_1, e_2, \ldots, e_{q_1} \in E(G_1)$ and $e_{q_1+q_2+\cdots+q_{i-1}+1}$, $e_{q_1+q_2+\cdots+q_{i-1}+2}$, $\ldots$, $e_{q_1+q_2+\cdots+q_{i-1}+q_i} \in E(G_i)$ for $2 \leq i \leq t$.

Thus $\hat{G}$ has $\beta + \sum_{i=1}^{\alpha} m_i$ vertices and $2\sum_{i=1}^{\alpha} m_i$ edges, where m_i denotes the cardinality of m_i-part of K_{2,m_i} which replaces the edge e_i of G in the construction of $\hat{G}$, for $1 \leq i \leq \alpha$.

Let $B(\hat{G})$ denote the set of all base vertices of $\hat{G}$. Thus $\hat{G}$ has $|B(\hat{G})| = \beta$. Let $NB(\hat{G})$ denote the set of all non base vertices of $\hat{G}$. Thus $|NB(\hat{G})| = \sum_{i=1}^{\alpha} m_i$.

Consider $\hat{G}_1$ of $\hat{G}$. As $\hat{G}_1$ is connected, assign the label 0 or 1 to the vertices of $\hat{G}_1$ as done in Case 1. Then, we have $\Big||V_0(\hat{G}_1)| - |V_1(\hat{G}_1)|\Big| \leq 1$ and $|E_0(\hat{G}_1)| = |E_1(\hat{G}_1)|$.
Therefore $\hat{G}_1$ is cordial.

If $|V_0(\hat{G}_1)| = |V_1(\hat{G}_1)| + 1$, then assign 1 to any one of the non-base vertices in $NB(\hat{G}_{2E_{21}})$ of $\hat{G}_2$, where E_{21} is the set of all different level edges of G_2, and $NB(\hat{G}_{2E_{21}})$ is the set of all non-base vertices obtained from the K_{2,m_i}'s which replace the edges of E_{21} of G_2 in defining $\hat{G}_2$.

If $|V_1(\hat{G}_1)| = |V_0(\hat{G}_1)| + 1$, then assign 0 to any one of the non-base vertices in $NB(\hat{G}_{2E_{21}})$ of $\hat{G}_2$.

For the remaining $|V(\hat{G}_2)| - 1$ unlabeled vertices of $\hat{G}_2$ assign 0-1 label as done in Case 1. Then, we have $\Big||V_0(\hat{G}_1 \cup \hat{G}_2)| - |V_1(\hat{G}_1 \cup \hat{G}_2)|\Big| \leq 1$ and $|E_0(\hat{G}_1 \cup \hat{G}_2)| = |E_1(\hat{G}_1 \cup \hat{G}_2)|$.

If $|V_0(\hat{G}_1)| = |V_1(\hat{G}_1)|$, then assign 0-1 label to the vertices of $\hat{G}_2$ as done in Case 1. Then, we have $\Big||V_0(\hat{G}_1 \cup \hat{G}_2)| - |V_1(\hat{G}_1 \cup \hat{G}_2)|\Big| \leq 1$ and $|E_0(\hat{G}_1 \cup \hat{G}_2)| = |E_1(\hat{G}_1 \cup \hat{G}_2)|$.

Thus $\hat{G}_1 \cup \hat{G}_2$ is cordial.

Continue this process of 0-1 labeling to the vertices of $\hat{G}_i$ after completing the 0-1 labeling of $\hat{G}_1 \cup \hat{G}_2 \cup \cdots \cup \hat{G}_{i-1}$, for $3 \leq i \leq t$.

This implies that $\hat{G} = \hat{G}_1 \cup \hat{G}_2 \cup \cdots \cup \hat{G}_t$ is cordial. □

Corollary 1.1. *If G is bipartite then arbitrary supersubdivision graph G^* of G is cordial.*

Proof. If G is bipartite, then there does not exist any same level edge in G, then by the construction of $\hat{G}$, every edge e_i of G is replaced by K_{2,m_i}, where m_i is an arbitrary positive integer. Hence almost arbitrary supersubdivision graph $\hat{G}$ constructed by Algorithm 1 is nothing but the arbitrary supersubdivision graph G^* of G. It follows from the proof of Theorem 1.1, G^* is cordial. □

Remark 2. Vaidya et al. [25, 26] proved that arbitrary supersubdivision of special classes of bipartite graphs, trees, $P_m \times P_n$, $C_{2n} \odot P_m$ are cordial. Our result generalizes these three results.

Discussion

Here we have shown that if G is a bipartite graph then arbitrary supersubdivision graph G^* of G is cordial. It is interesting to find the graphs different from bipartite graphs whose arbitrary supersubdivision are also cordial. Thus, we ask the following question.

"Are there graphs apart from bipartite graphs whose arbitrary supersubdivision graphs are cordial?"

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