Conjecture on Odd Graceful Graphs

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Abstract

A graph $G = (V, E)$ with p vertices and q edges is said to be odd graceful if there is an injection f from the vertex set of G to $\{0, 1, 2, \ldots, 2q - 1\}$ such that when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct and induce the set $\{1, 3, 5, \ldots, 2q - 1\}$. In 2009, Barrientos conjectured that every bipartite graph is odd graceful. In this paper, we partially solve Barrientos conjecture by showing that the following graphs are odd graceful: 1) Finite union of paths, stars and caterpillars; 2) Finite union of ladders; 3) Finite union of paths, bistars and caterpillars; 4) The coronas $K_{m,n} \odot rK_1$, and 5) Finite union of graphs obtained by one end point union of odd number of paths of uniform length.

Keywords: Graceful Graphs, Odd Graceful Graphs.

Mathematics subject classification: 05C78.

1 Introduction

Let $G = (V, E)$ be a graph with p vertices and q edges. A *caterpillar* is a tree such that the removal of pendant vertices results in a path. The *union of graphs* $G_1, G_2, \ldots, G_k$ is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$. A *bipartite graph* is the graph whose vertex set can be partitioned into two subsets X and Y so that each edge has one end in X and other end in Y. A *complete bipartite* graph is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. A *star* is a complete bipartite graph $K_{1,r}$. A *bistar* is a graph obtained by joining centre vertices of two stars by an edge. The *corona* $G_1 \odot G_2$ of two graphs is obtained by taking a copy of G_1 and $|G_1|$ copies of G_2, and joining the ith vertex of G_1 to every vertex in the ith copy of G_2. A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain conditions. Graph labelings were first introduced by Rosa [10] in 1967 in the name of *β - valuation* which is defined as an injection f from the vertex set of G to the set $\{0, 1, 2, \ldots, q\}$ such that when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Rosa introduced β - valuation as well as number of other labelings as tools for decomposing the complete graph into isomorphic subgraphs. In particular β - valuation orginated as a means of attacking the conjecture of Ringel [9] which states that K_{2n+1} can be decomposed into $2n + 1$ subgraphs which are isomorphic to a given tree with n edges. There were so many labelings introduced in the past years.

Gnanajothi [4] defined a graph G with q edges to be odd graceful if there is an injection f from $V(G)$ to $\{0, 1, 2, \ldots, 2q - 1\}$ such that when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $1, 3, \ldots, 2q - 1$. She proved that the following graphs are odd graceful: P_n; C_n if and only if n is even; $K_{m,n}$; combs $P_n \odot K_1$; books; crowns $C_n \odot K_1$ if and only if n is even; disjoint union of copies of C_4; the one point union of copies of C_4; $C_n \times K_2$ if and only if n is even; caterpillars; rooted trees of height 2; the graphs obtained from $P_n \times P_2$ ($n \geq 3$) by adding exactly two leaves at each vertex of degree 2 of P_n; the graphs obtained from $P_n \times P_2$ by deleting an edge that joins to end points of the path P_n; $K_{1,n} \odot P_3$; $K_{1,n} \odot P_4$.

Also she conjectures that all trees are odd graceful and proves the conjecture for all trees of order upto 10. Barrientos [1] and [2] extended this result to trees of order upto 12. Eldergil [3] generalised Gnanajothi's result on stars by showing that the graphs obtained by joining one end point of odd number of paths of equal length is odd graceful and also proved that the one point union of any number of C_6 is odd graceful. Kathiresan [7] has shown that the ladders and the graphs obtained from them by subdividing each step exactly once are odd graceful. Yan [12] proved that $P_m \times P_n$ is odd graceful. Sekar [11] has shown that the following graphs are odd graceful: $C_m \odot P_n$ when $n \geq 3$ and m is even; C_n^t; $C_{6,n}$ and $C_{8,n}$; the splitting graph of C_n, when n is even; the splitting graph of P_n; lobsters, banana trees and regular bamboo trees. Gao [5] proved the following graphs are odd graceful: the union of any number of stars; the union of any number of paths; the union of any number of stars and paths; $C_m \bigcup P_n$; $C_m \bigcup C_n$ and the union of any number of cycles each of which has order divisible by 4. Later Barrientos [2] conjectured that every bipartite graph is odd graceful and he proved the conjecture for the following graphs: Forest whose components are caterpillars; Tree with diameter atmost five; and Disjoint union of caterpillars. But the conjecture is open in general. In this paper we have obtained some results which prove the above conjecture affirmatively for some classes of graphs. In fact we have proved that the following graphs are odd graceful:

1) Finite union of paths, stars and caterpillars;

2) Finite union of $P_{n_j} \times P_2$ where $n_1, n_2, \ldots, n_r$ are positive integers greater than 1 (not necessarily distinct);

3) Finite union of paths, bistars and caterpillars;

4) The coronas $K_{m,n} \odot rK_1$ and

5) Finite union of graphs obtained by one end point union of odd number of paths of uniform length.

As Paths, Stars, Bistars and Caterpillars are bipartite graphs, for our convenience we view them with bipartition as given below.

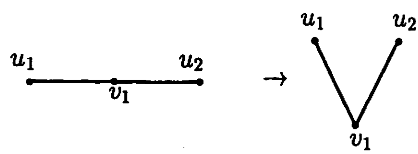


Figure 1: Path P_3

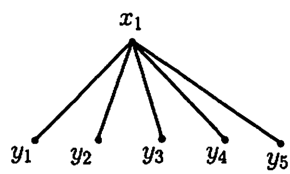


Figure 2: Star $S_6 = K_{1,5}$

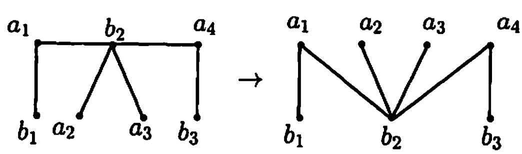


Figure 3: Caterpillar F_7

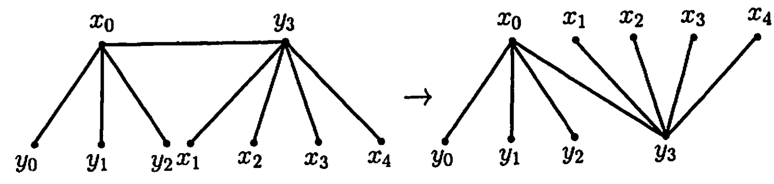


Figure 4: Bistar $B_{3,4}$

Whenever we consider paths, stars, bistars and caterpillars, we mean that the vertices of the paths, stars, bistars and caterpillars are arranged in the order as above.

2 Odd graceful labeling of bipartite graphs

Theorem 1. *Finite union of paths, stars and caterpillars is odd graceful.*

Proof. Let $G = (\bigcup_{i=1}^{p} P_{r_i}) \bigcup (\bigcup_{i=1}^{k} K_{1,a_i}) \bigcup (\bigcup_{i=1}^{s} F_{t_i})$ where r_i, a_i, t_i are positive integers (not necessarily distinct) greater than 1.

Since the path P_{r_i} is a bipartite graph we partition $V(P_{r_i})$ into (U_i, V_i) where $U_i = \{u_1^i, u_2^i, u_3^i \dots, u_{m_i}^i\}$, $V_i = \{v_1^i, v_2^i, \dots, v_{n_i}^i\}$ such that $m_i + n_i = r_i, i = 1, 2, \dots, p$ and $|m_i - n_i| \leq 1$.

Similarly, as star K_{1,a_i} is a bipartite graph we partition $V(K_{1,a_i})$ into (X_i, Y_i) where $X_i = \{x^i\}$, $Y_i = \{y_1^i, y_2^i, \dots, y_{a_i}^i\}, i = 1, 2, \dots, k$. Also as caterpillar F_{t_i} is a bipartite graph we partition the vertex set of F_{t_i} into (A_i, B_i) where $A_i = \{a_1^i, a_2^i, \dots, a_{p_i}^i\}$ and $B_i = \{b_1^i, b_2^i, \dots, b_{q_i}^i\}$ suchthat $p_i + q_i = t_i, i = 1, 2, \dots, s$.

Now we assign the labels to the vertices of G by defining an one-one map as follows:

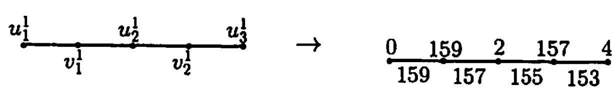
$f : V(G) \to \{0, 1, 2, \dots, 2q - 1\}$ by

$$
\begin{aligned}
f(u_i^1) &= 2i - 2, i = 1, 2, \dots, m_1, \\
f(v_i^1) &= 2q - 1 - (2i - 2), i = 1, 2, \dots, n_1, \\
f(u_i^2) &= f(u_{m_1}^1) + (2i - 1), i = 1, 2, \dots, m_2, \\
f(v_i^2) &= f(v_{n_1}^1) - (2i - 1), i = 1, 2, \dots, n_2, \cdots \\
f(u_i^{j+1}) &= f(u_{m_j}^j) + (2i - 1), i = 1, 2, \dots, m_{j+1}, \\
f(v_i^{j+1}) &= f(v_{n_j}^j) - (2i - 1), i = 1, 2, \dots, n_{j+1}, j = 1, 2, \dots, (p - 1),
\end{aligned}
$$

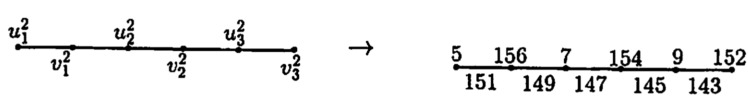
$$
\begin{aligned}
f(x^1) &= f(u^p_{m_p}) + 1,\\
f(y^1_i) &= f(v^p_{n_p}) - (2i-1), i = 1,2,\ldots,a_1,\cdots\\
f(x^{j+1}) &= f(x^j) + 1,\\
f(y^{j+1}_i) &= f(y^j_{a_j}) - (2i-1), i = 1,2,\ldots,a_{j+1}, j = 1,2,\ldots,(k-1),\\
f(a^1_i) &= f(x^k) + (2i-1), i = 1,2,\ldots,p_1,\\
f(b^1_i) &= f(y^k_{a_k}) - (2i-1), i = 1,2,\ldots,q_1,\cdots\\
f(a^{j+1}_i) &= f(a^j_{p_j}) + (2i-1), i = 1,2,\ldots,p_{j+1},\\
f(b^{j+1}_i) &= f(b^j_{p_j}) - (2i-1), i = 1,2,\ldots,q_{j+1}, j = 1,2,\ldots,(s-1),
\end{aligned}
$$

Now the edge induced function $\phi : E(G) \to \{1,3,\ldots,2q-1\}$ defined by $\phi(uv) = |f(u) - f(v)|, \forall uv \in E(G)$induce the distinct edge labels $1,3,5,\ldots,2q-1$ in G and hence the graph is odd graceful.

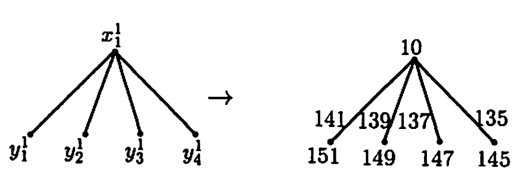
Example. *Odd graceful labeling of* $G{=}P_5 \bigcup P_6 \bigcup K_{1,4} \bigcup K_{1,6} \bigcup K_{1,7} \bigcup F_{12} \bigcup F_{11} \bigcup F_{16} \bigcup F_{19}$ *is shown below.*



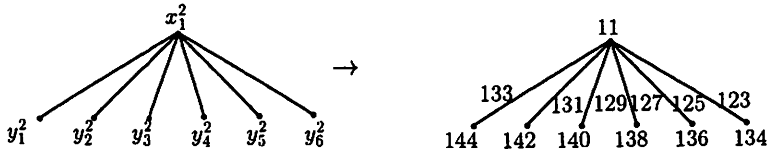
(a) Labeling of P_5



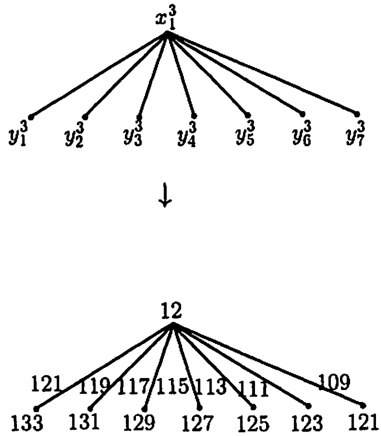
(b) Labeling of P_6



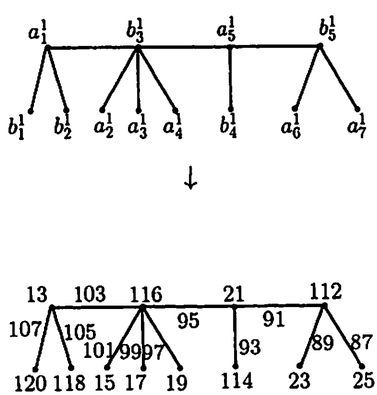
(c) Labeling of $K_{1,4}$



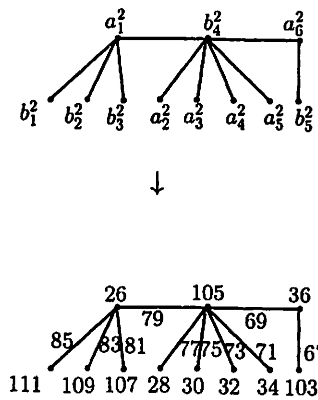
(d) Labeling of $K_{1,6}$



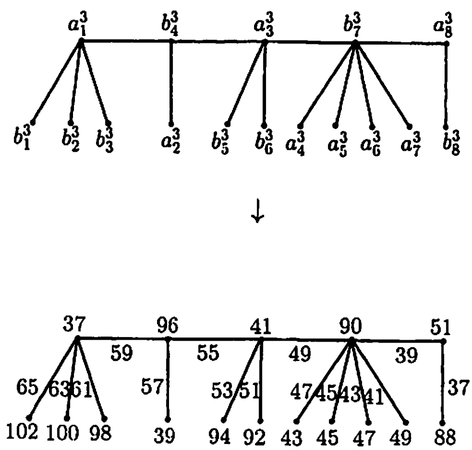
(e) Labeling of $K_{1,7}$



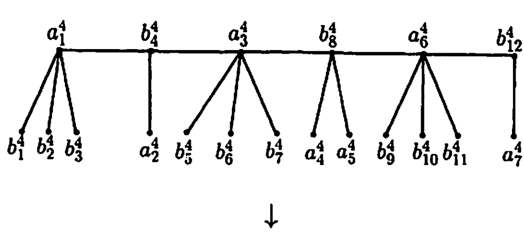
(f) Labeling of F_{12}

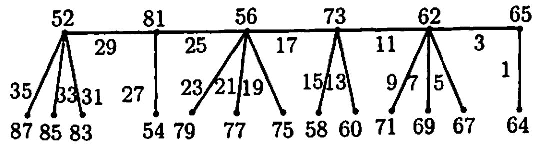


(g) Labeling of F_{11}



(h) Labeling of F_{16}





(i) Labeling of F_{19}

Figure 5: Odd graceful labeling of $G = P_5 \bigcup P_6 \bigcup K_{1,4} \bigcup K_{1,6} \bigcup K_{1,7} \bigcup F_{12} \bigcup F_1$ $F_{16} \bigcup F_{19}$.

□

Theorem 2. *The graph* $G = \bigcup_{j=1}^{r} P_{n_j} \times P_2$ *is odd graceful where* $n_1, n_2, \ldots, n_r$ *are positive integers (not necessarily distinct).*

Proof. We know that $P_{n_j} \times P_2$ is a bipartite graph. For our convenience we view $P_{n_j} \times P_2$ as follows : By twisting the vertices of the alternate partite sets of the graph of Fig 6(a) we get the graph of Fig 6(b) .

For the convenience of labeling process, we consider $P_{n_j} \times P_2$ as a bipartite graph with bipartition (U_j, V_j) where $U_j = \{u_1^j, u_2^j, \ldots, u_{n_j}^j\}$, $V_j = \{v_1^j, v_2^j, \ldots, v_{n_j}^j\}$.

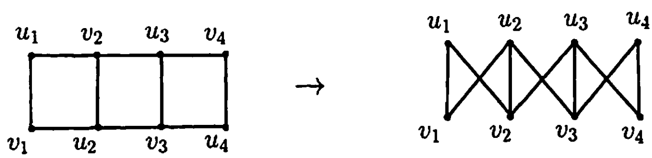


Fig 6(a) Fig 6(b)

Figure 6 : $P_4 \times P_2$

To find the odd graceful labeling of the graph $G = \bigcup_{j=1}^{r} P_{n_j} \times P_2$, we define an one-one map $f : V(G) \to \{0, 1, 2, \ldots, 2q-1\}$ as the recurrence formula as follows:

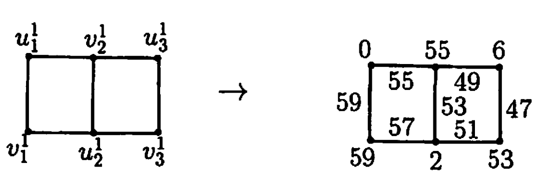
$$f(u_{2i+1}^1) = 6i, i = 0, 1, 2, \ldots, \frac{n_1 - 1}{2}, n_1 \text{ is odd,}$$
$$i = 0, 1, 2, \ldots, \frac{n_1}{2} - 1, n_1 \text{ is even,}$$

$$
\begin{aligned}
f(u^1_{2i+2}) &= 6i+2, i=0,1,2,\ldots,\frac{n_1-3}{2}, n_1 \text{ is odd,}\\
&\qquad i=0,1,2,\ldots,\frac{n_1}{2}-1, n_1 \text{ is even,}\\
f(v^1_{2i+1}) &= 2q-1-6i, i=0,1,2,\ldots,\frac{n_1-1}{2}, n_1 \text{ is odd,}\\
&\qquad i=0,1,2,\ldots,\frac{n_1}{2}-1, n_1 \text{ is even}\\
f(v^1_{2i+2}) &= 2q-5-6i, i=0,1,2,\ldots,\frac{n_1-3}{2}, n_1 \text{ is odd,}\\
&\qquad i=0,1,2,\ldots,\frac{n_1}{2}-1, n_1 \text{ is even}\\
f(u^{j+1}_{2i+1}) &= f(u^j_{n_j})+6i+1, i=0,1,2,\ldots,\frac{n_{j+1}-1}{2}, n_{j+1} \text{ is odd,}\\
&\qquad i=0,1,2,\ldots,\frac{n_{j+1}}{2}-1, n_{j+1} \text{is even,}\\
f(u^{j+1}_{2i+2}) &= f(u^j_{n_j})+6i+3, i=0,1,2,\ldots,\frac{n_{j+1}-3}{2}, n_{j+1} \text{ is odd,}\\
&\qquad i=0,1,2,\ldots,\frac{n_{j+1}}{2}-1, n_{j+1} \text{ is even,}\\
f(v^{j+1}_{2i+1}) &= f(v^j_{n_j})-(6i+1), i=0,1,2,\ldots,\frac{n_{j+1}-1}{2}, n_{j+1} \text{ is odd,}\\
&\qquad i=0,1,2,\ldots,\frac{n_{j+1}}{2}-1, n_{j+1} \text{ is even}\\
f(v^{j+1}_{2i+2}) &= f(v^j_{n_j})-(6i+5), i=0,1,2,\ldots,\frac{n_{j+1}-3}{2}, n_{j+1} \text{ is odd,}\\
&\qquad i=0,1,2,\ldots,\frac{n_{j+1}}{2}-1, n_{j+1} \text{ is even}
\end{aligned}
$$

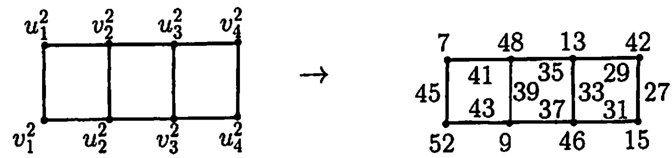
where $j = 1,2,\ldots,r-1$.

Now the edge induced function $\phi : E(G) \to \{1,3,\ldots,2q-1\}$ defined by $\phi(uv) = |f(u) - f(v)|, \forall uv \in E(G)$, induce the distinct edge labels $1,3,\ldots,2q-1$ in G and hence the graph is odd graceful.

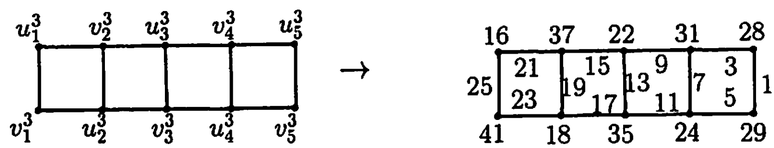
Example. *Odd graceful labeling of $G=(P_3\times P_2)\bigcup(P_4\times P_2)\bigcup(P_5\times P_2)$ is shown below.*



(a) Labeling of $P_3 \times P_2$



(b) Labeling of $P_4 \times P_2$



(c) Labeling of $P_5 \times P_2$

Figure 7 : Odd graceful labeling of $G=(P_3 \times P_2)\bigcup(P_4 \times P_2)\bigcup(P_5 \times P_2)$.

□

Theorem 3. *Finite union of paths, bistars and caterpillars is odd graceful.*

Proof. Let $G = (\bigcup_{i=1}^{p} P_{r_i})\bigcup(\bigcup_{i=1}^{k} B_{a_i,b_i})\bigcup(\bigcup_{i=1}^{s} F_{t_i})$ where a_i, b_i, t_i, r_i are positive integers (not necessarily distinct) greater than 1.

Since path P_{r_i} is a bipartite graph, its vertex set can be partitioned into (U_i, V_i) with $U_i = \{u_1^i, u_2^i, \ldots, u_{m_i}^i\}, V_i = \{v_1^i, v_2^i, \ldots, v_{n_i}^i\}$ such that $m_i + n_i = r_i, i = 1, 2, \ldots, p$ and $|m_i - n_i| \le 1$.

Similarily, we partition the vertex set of the bistar B_{a_i,b_i} into (X_i, Y_i) where $X_i=\{x_0^i, x_1^i, x_2^i, \ldots, x_{a_i}^i\}, Y_i = \{y_0^i, y_1^i, y_2^i, \ldots, y_{b_i}^i\}$, where i varies from 1 to k; and the vertex set of the caterpillar F_{t_i} into (A_i, B_i) where $A_i = \{a_1^i, a_2^i, \ldots, a_{p_i}^i\}$ and $B_i = \{b_1^i, b_2^i, \ldots, b_{q_i}^i\}$ such that $p_i + q_i = t_i, i = 1, 2, \ldots, s$.

Now we assign the labels to the vertices of G by defining an one-one map as follows:

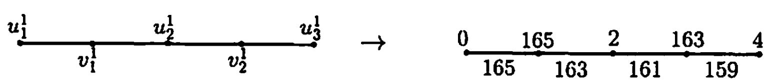
$f : V(G) \to \{0, 1, 2, 3, \ldots, 2q - 1\}$ by

$$\begin{aligned} f(u_i^1) &= 2i - 2, i = 1, 2, \ldots, m_1, \\ f(v_i^1) &= 2q - 1 - (2i - 2), i = 1, 2, \ldots, n_1, \end{aligned}$$

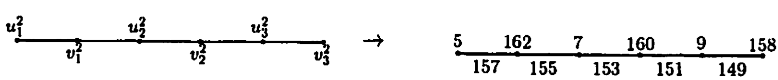
$$
\begin{aligned}
f(u_i^2) &= f(u_{m_1}^1) + (2i-1), i = 1,2,\ldots,m_2,\\
f(v_i^2) &= f(v_{n_1}^1) - (2i-1), i = 1,2,\ldots,n_2,\cdots\\
f(u_i^{j+1}) &= f(u_{m_j}^j) + (2i-1), i = 1,2,\ldots,m_{j+1},\\
f(v_i^{j+1}) &= f(v_{n_j}^j) - (2i-1), i = 1,2,\ldots,n_{j+1}, j = 1,2,\ldots,(p-1),\\
f(x_i^1) &= f(u_{m_k}^k) + (2i+1), i = 0,1,2,\ldots,a_1,\\
f(y_i^1) &= f(v_{n_k}^k) - (2i+1), i = 0,1,2,\ldots,b_1,\\
f(x_i^2) &= f(x_{a_1}^1) + (2i+1), i = 0,1,2,\ldots,a_2,\\
f(y_i^2) &= f(y_{b_1}^1) - (2i+1), i = 0,1,2,\ldots,b_2,\cdots\\
f(x_i^{j+1}) &= f(x_{a_j}^j) + (2i+1), i = 0,1,2,\ldots,a_{j+1},\\
f(y_i^{j+1}) &= f(y_{b_j}^j) - (2i+1), i = 0,1,2,\ldots,b_{j+1}, j = 1,2,\ldots,(k-1),\\
f(a_i^1) &= f(x_{a_p}^p) + (2i-1), i = 1,2,\ldots,p_1,\\
f(b_i^1) &= f(y_{b_p}^p) - (2i-1), i = 1,2,\ldots,q_1,\\
f(a_i^2) &= f(a_{p_1}^1) + (2i-1), i = 1,2,\ldots,p_2,\\
f(b_i^2) &= f(b_{q_1}^1) - (2i-1), i = 1,2,\ldots,q_2,\cdots\\
f(a_i^{j+1}) &= f(a_{p_j}^j) + (2i-1), i = 1,2,\ldots,p_{j+1},\\
f(b_i^{j+1}) &= f(b_{q_j}^j) - (2i-1), i = 1,2,\ldots,q_{j+1}, j = 1,2,\ldots,(s-1),
\end{aligned}
$$

Now the edge induced function $\phi : E(G) \to \{1,3,\ldots,2q-1\}$ defined by $\phi(uv) = |f(u) - f(v)|, \forall uv \in E(G)$ induce the distinct edge labels $1,3,\ldots,2q-1$ in G and hence the graph is odd graceful.

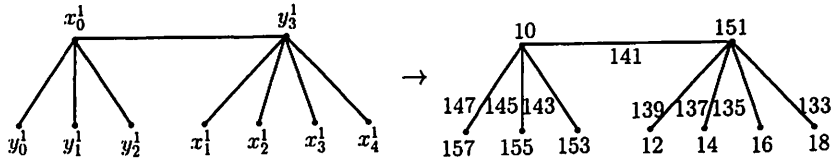
Example. *Odd graceful labeling of* $G = P_5 \bigcup P_6 \bigcup B_{3,4} \bigcup F_{14} \bigcup F_{23} \bigcup F_{32}$ *is shown below.*



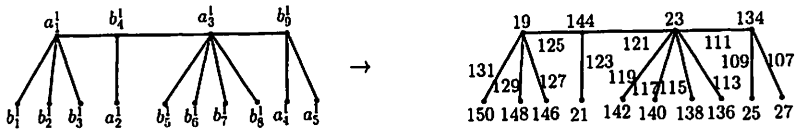
(a) Labeling of P_5



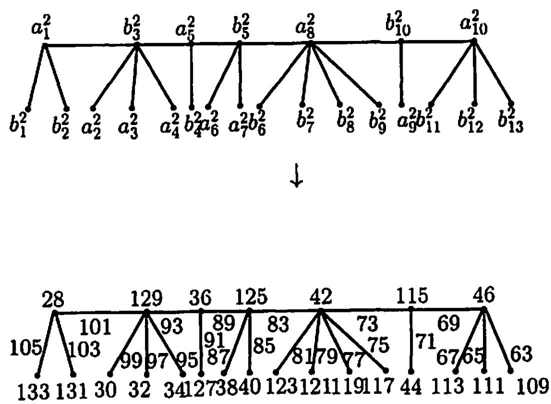
(b) Labeling of P_6



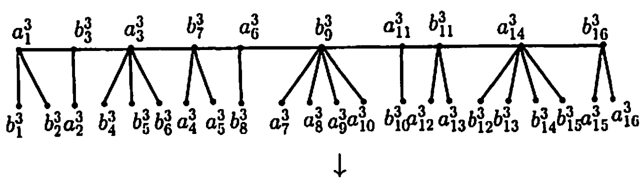
(c) Labeling of $B_{3,4}$

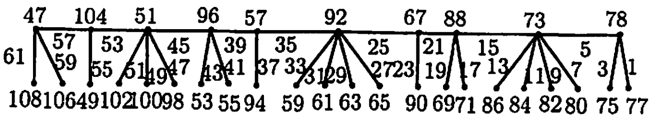


(d) Labeling of F_{14}



(e) Labeling of F_{23}





(f) Labeling of F_{32}

Figure 8: Odd graceful labeling of $G=P_5 \bigcup P_6 \bigcup B_{3,4} \bigcup F_{14} \bigcup F_{23} \bigcup F_{32}$

□

Theorem 4. *The graph $G = K_{m,n} \odot rK_1$ is odd graceful.*

Proof. Let $G = K_{m,n} \odot rK_1$ where m, n, r are positive integers.

Since $K_{m,n}$ is a complete bipartite graph, let the partition of $K_{m,n}$ be (U, V) with $U = \{u_1, u_2, \ldots, u_m\}$, $V = \{v_1, v_2, \ldots, v_n\}$.

For our convenience we name the pendant vertices of the graph $K_{m,n} \odot rK_1$ which are connected with u_j as $x_j^1, x_j^2, \ldots, x_j^r$ and the pendant vertices of the graph $K_{m,n} \odot rK_1$ which are connected with v_j as $y_j^1, y_j^2, \ldots, y_j^r$ respectively.

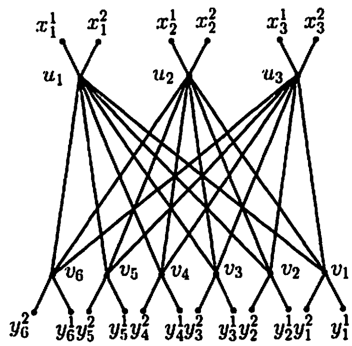
Now we assign the labels to the vertices by defining an one-one map as follows:

$f : V(G) \to \{0, 1, 2, \ldots, 2q-1\}$ by

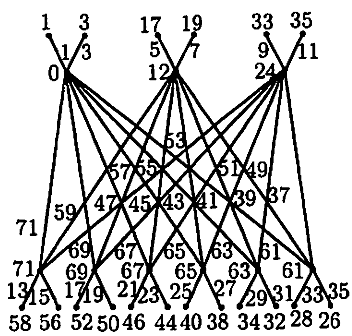
$$
\begin{aligned}
f(u_j) &= 2n(j-1), j = 1, 2, \ldots, m, \\
f(v_j) &= 2q - 2n + 2j - 1, j = 1, 2, \ldots, n, \\
f(x_1^i) &= f(u_1) + 2(i-1) + 1, i = 1, 2, \ldots, r, \\
f(x_2^i) &= f(x_1^r) + 2n + 2(i+1) - 2, i = 1, 2, \ldots, r, \cdots \\
f(x_j^i) &= f(x_{j-1}^r) + 2n + 2(i+1) - 2, i = 1, 2, \ldots, r, j = 1, 2, \ldots, m, \\
f(y_1^i) &= f(u_m) + 2i, i = 1, 2, \ldots, r, \\
f(y_2^i) &= f(y_1^r) + 2r + 2(i-1), i = 1, 2, \ldots, r, \cdots \\
f(y_j^i) &= f(y_{j-1}^r) + 2r + 2(i-1), i = 1, 2, \ldots, r, j = 1, 2, \ldots, n,
\end{aligned}
$$

Now the edge induced function $\phi : E(G) \to \{1, 3, \ldots, 2q-1\}$ defined by $\phi(uv) = |f(u) - f(v)|, \forall uv \in E(G)$ induce the distinct edge labels $1, 3, \ldots, 2q-1$ in G and hence the graph is odd graceful.

Example. *Odd graceful labeling of $G=K_{3,6} \odot 2K_1$ is shown below.*



(a) $G = K_{3,6} \odot 2K_1$



(b) Labeling of $G = K_{3,6} \odot 2K_1$

Figure 9: Odd graceful labeling of $G=K_{3,6} \odot 2K_1$.

□

Theorem 5. *Finite union of graphs obtained by one end point union of odd number of paths of uniform length is odd graceful.*

Proof. Let $G=\bigcup_{i=1}^{n} G_{r_i}^{m_i}$, where $G_{r_i}^{m_i}$ is the graph obtained by one end point union of m_i (≥ 3 is odd) paths and each path with r_i vertices.

For our convenience, we name the vertices of $G_{r_1}^{m_1}, G_{r_2}^{m_2} \ldots, G_{r_n}^{m_n}$ as follows:

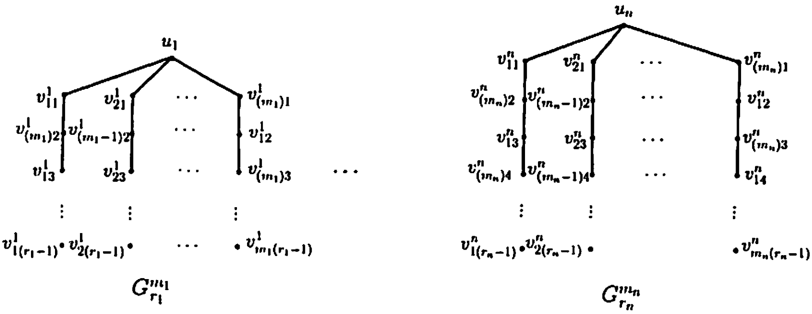


Fig 10 : $G=\bigcup_{i=1}^{n} G_{r_i}^{m_i}$

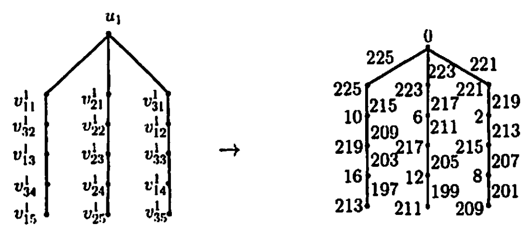
Now we assign the labels to the vertices by defining an one-one map
$f : V(G) \to \{0, 1, 2, \ldots, 2q-1\}$ by

$$
\begin{aligned}
f(u_1) &= 0,\\
f(u_2) &= \begin{cases} f(v^1_{m_1(r_1-2)}) + 1, & r_1 \text{ is even, and } m_1 \text{ is odd}\\ f(v^1_{m_1(r_1)}) + 1, & r_1 \text{ is odd, and } m_1 \text{ is odd}\end{cases}\\
f(u_i) &= \begin{cases} f(v^{i-1}_{m_{i-1}((r_{i-1})-2)}) + 1, & i = 1, 2, \ldots, n, \text{ when } r_{i-1} \text{ is even}\\ f(v^{i-1}_{m_{i-1}(r_{i-1})}) + 1, & i = 1, 2, \ldots, n, \text{ when } r_{i-1} \text{ is odd}\end{cases}\\
f(v^1_{j1}) &= 2q - 2j + 1,\ j = 1, 2, \ldots, m_1,\\
f(v^1_{j3}) &= f(v^1_{11}) - 2m_1 - 2(j-1),\ j = 1, 2, \ldots, m_1,\\
f(v^i_{jk}) &= f(v^i_{1(k-2)}) - 2m_i - 2(j-1),\ i = 1,\ j = 1, 2, \ldots, m_i,\\
&\qquad k = 1, 3, \ldots, r_i, \text{ when } r_i \text{ is odd, and}\\
&\qquad k = 1, 3, \ldots, r_i - 1, \text{ when } r_i \text{ is even}\\
f(v^1_{j2}) &= f(u_1) + 4(j-1) + 2,\ j = 1, 2, \ldots, m_1,\\
f(v^1_{j4}) &= f(v^1_{12}) + 2m_1 + 4(j-1),\ j = 1, 2, \ldots, m_1,\\
f(v^i_{jk}) &= f(v^i_{1(k-2)}) + 2m_i + 4(j-1),\ i = 1,\ j = 1, 2, \ldots, m_i,\\
&\qquad k = 2, 4, \ldots, r_i - 1, \text{ when } r_i \text{ is odd, and}\\
&\qquad k = 2, 4, \ldots, r_i - 2, \text{ when } r_i \text{ is even}\\
f(v^i_{j1}) &= \begin{cases} f(v^{(i-1)}_{1(r_{i-1})}) - 1 - 2(j-1), & i = 2,\ j = 1, 2, \ldots, m_2, \text{ when } r_{i-1} \text{ is odd}\\ f(v^{(i-1)}_{1((r_{i-1})-1)}) - 1 - 2(j-1), & i = 2,\ j = 1, 2, \ldots, m_2, \text{ when } r_{i-1} \text{ is even}\end{cases}\\
f(v^i_{j3}) &= f(v^i_{11}) - 2m_i - 2(j-1),\ i = 2,\ j = 1, 2, \ldots, m_i
\end{aligned}
$$

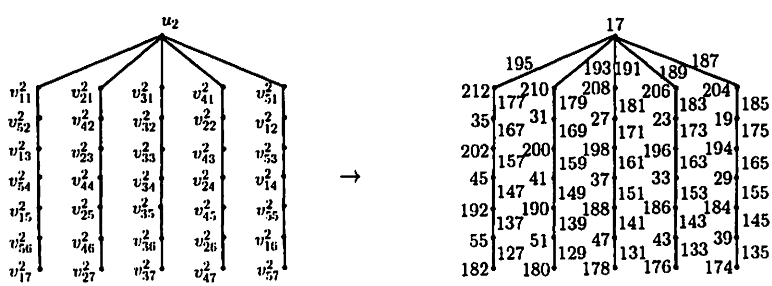
$$
\begin{aligned}
f(v^i_{jk}) &= f(v^i_{1(k-2)}) - 2m_i - 2(j-1), i = 2,3,\ldots,n, j = 1,2,\ldots,m_i,\\
&\qquad k = 1,3,\ldots,r_i, \text{ when } r_i \text{ is odd, and}\\
&\qquad k = 1,3,\ldots,r_i - 1, \text{ when } r_i \text{ is even}\\
f(v^i_{j2}) &= f(u_i) + 4(j-1) + 2, i = 2,3,\ldots,n, j = 1,2,\ldots,m_i,\\
f(v^i_{j4}) &= f(v^i_{12}) + 2m_i + 4(j-1), i = 2,3,\ldots,n, j = 1,2,\ldots,m_i\\
f(v^i_{jk}) &= f(v^i_{1(k-2)}) + 2m_i + 4(j-1), i = 2,3,\ldots,n, j = 1,2,\ldots,m_i,\\
&\qquad k = 2,4,\ldots,r_i - 1, \text{ when } r_i \text{ is odd, and}\\
&\qquad k = 2,4,\ldots,r_i - 2, \text{ when } r_i \text{ is even.}
\end{aligned}
$$

Now the edge induced function $\phi : E(G) \to \{1,3,\ldots,2q-1\}$ defined by $\phi(uv) = |f(u) - f(v)|, \forall uv \in E(G)$ induce the distinct edge labels $1,3,\ldots,2q-1$ in G and hence the graph is odd graceful.

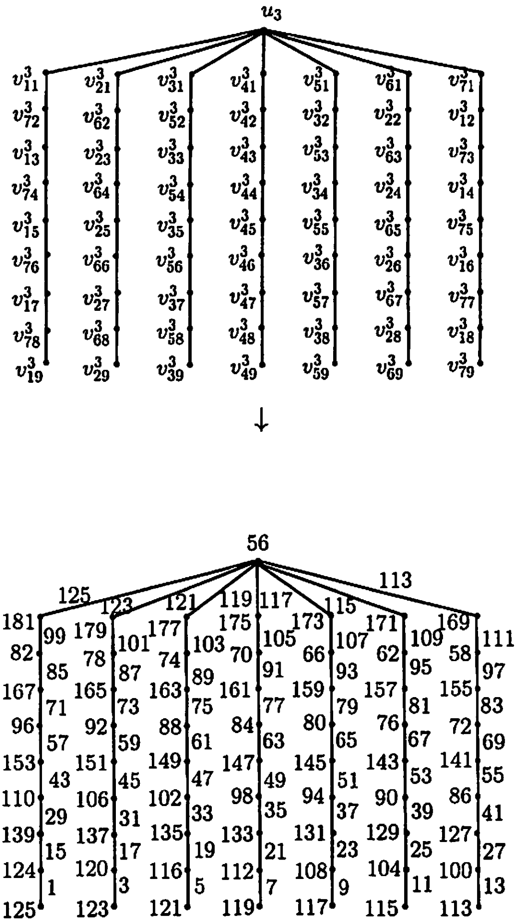
Example. *Odd graceful labeling of* $G = G^3_6 \cup G^5_8 \cup G^7_{10}$, *is shown below.*



(a) Labeling of G^3_6



(b) Labeling of G^5_8



(c) Labeling of G_{10}^7

Figure 11: Odd graceful labeling of $G = G_6^3 \cup G_8^5 \cup G_{10}^7$.

□

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