Eternal Domination Numbers of $5 \times n$ Grid Graphs

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Abstract

Eternal domination of a graph requires the positioning of guards to protect against an infinitely long sequence of attacks where, in response to an attack, each guard can either remain in place or move to a neighbouring vertex, while keeping the graph dominated. This paper investigates the m-eternal domination numbers for $5 \times n$ grid graphs. The values, previously known for $1 \leq n \leq 5$, are determined for $6 \leq n \leq 12$, and lower and upper bounds derived for $n > 12$.

1 Introduction

A dominating set for a graph can be thought of as a positioning of guards on vertices so that every vertex can be monitored from a distance of at most one. The smallest size of such a set for a graph is the graph's domination number. An eternal dominating family for a graph can be thought of as a collection of such sets of positions resulting from having to move guards to respond to an arbitrary infinitely long sequence of individual attacks at various vertices. As a result of an attack, a neighbouring guard moves to the point of the attack, and each of the remaining guards can either remain in place or move to a neighbouring vertex to keep the graph dominated. The domination number for the family where the sets have the smallest such number is the eternal domination number. This has been referred to as the "all guards move model" or "eternal m-security" [11] and as "m-eternal domination" [8]. The m-eternal domination number for members of the family of $5 \times n$ grid graphs is the focus of this paper.

The domination numbers for $m \times n$ grid graphs are known for all values of m and n. The first results from Jacobson and Kinch [10] appeared thirty years ago and the latest results by Gonçalves, Pinlou, Rao, and Thomassé [9] appeared three years ago. The m-eternal domination numbers for all n are known for $2 \times n$ grid graphs (see Goldwasser, Klostermeyer, and Mynhardt [8]) and $4 \times n$ grid graphs (see Beaton, Finbow, and MacDonald [2]). The m-eternal domination numbers for $3 \times n$ grid graphs have been investigated (see Goldwasser et al. [8] and Finbow, Messinger, and van Bommel [6]); however, the values for $5 \times n$ grid graphs are largely unknown.

The organization of the paper is as follows. Formal definitions are provided in the next section. Previous work on the domination number of the $5 \times n$ grid graph by Chang and Clark [4], along with several statements giving restrictions on the domination sets, are outlined in Section 3. Extensions of these statements that prove useful in establishing the bounds of the m-eternal domination number are presented in Section 4. A lower bound is proven in Section 5, exact values for $1 \leq n \leq 12$ are provided in Section 6, and Section 7 derives an upper bound for all n. Finally, Section 8 summarizes the results.

2 Definitions

A *dominating set* of a graph $G = (V, E)$ is a subset $D \subseteq V$ whose closed neighbourhood is V; that is, for every vertex $u \in (V - D)$ there exists $v \in D$ adjacent to u. Each vertex $v \in D$ is called a *guard* and, for each vertex x in the closed neighbourhood of v, we say v *defends* x, and x is *defended* if there exists such a v, otherwise x is *undefended.* The *domination number* $\gamma(G)$ of a graph G is the cardinality of a smallest dominating set.

Let $\mathbb{D}_q(G)$ be the set of all dominating sets of G which have cardinality q. Let $D, D' \in \mathbb{D}_q(G)$. We will say D *transforms* to D' if $D = \{v_1, v_2, \dots v_q\}$, $D' = \{u_1, u_2, \dots u_q\}$ and $u_i \in N[v_i]$ for $i = 1, 2, \dots q$. Such a transformation can be described as a *move*, or the movement of each guard from its position in D to its corresponding (adjacent or identical) position in D'.

In the "eternal domination game" a defender is given q guards to protect the graph from an infinite series of attacks on single vertices made by an attacker. An *m-eternal dominating family of* G is a subset $\mathcal{E} \subseteq \mathbb{D}_q(G)$ for some q so that for every $D \in \mathcal{E}$ and every possible attack $v \in V(G)$, there is a dominating set $D' \in \mathcal{E}$ so that $v \in D'$ and D transforms to D'. The transformation to D' protects the graph in response to the attack. When the value of q in the above definition is known we will refer to this family as an m-eternal dominating family with q guards. A set $D \in \mathbb{D}_q(G)$ is an *m-eternal dominating set* if it is a member of some m-eternal dominating family. Note that the set of all m-eternal dominating sets for some q is an m-eternal dominating family. The *m-eternal domination number* $\gamma_{\mathrm{m}}^{\infty}(G)$ of a graph G is the minimum cardinality of the dominating sets that constitute an m-eternal dominating family for G.

The $m \times n$ *grid graph* is denoted $P_m \square P_n$ as it is the Cartesian product of two path graphs P_m and P_n. The *Cartesian product* of graphs G and H is denoted by $G \square H$. The vertex set of $G \square H$ is $V(G \square H) = \{(u, v) | u \in V(G), v \in V(H)\}$, and two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. The vertices of P_m (respectively P_n) are labeled in their usual ordering $u_1, u_2, \ldots u_m$ (resp. $v_1, v_2, \ldots v_n$).

The following definition borrowed from Chang and Clark [4] defines s_j to be the number of guards in column j.

Definition 1. Let $C_j = \{(1, j), (2, j), \ldots, (n, j)\}$ denote the vertices in column j of $P_k \square P_n$. A sequence $(s_1, s_2, \ldots, s_n)$ of non-negative integers is called a **dominating sequence** for $P_k \square P_n$ if there is a dominating set S for $P_k \square P_n$ such that $s_j = |S \cap C_j|$ for $j = 1, 2, \ldots, n$.

In this paper, we discuss the m-eternal domination numbers of $m \times n$ grid graphs with $m = 5$. Each copy of P_5, corresponding to a vertex of P_n, is referred to as a column. We refer to each of the columns as the first column, second column, third column, etc. and as column 1, column 2, column 3, etc. starting from one the columns corresponding to a leaf of P_n and proceeding consecutively. We refer to specific vertices using pairs consisting of the row and column number; for example, the bottom vertex of column 2 is referred to as vertex $(5, 2)$.

In constructing m-eternal dominating families we make use of the symmetries of the $5 \times n$ grid graph. Given a dominating set $D \in \mathbb{D}_q(P_5 \square P_n)$, a vertical reflection of D (about the horizontal line of symmetry) is denoted D_v, while a horizontal reflection (about the vertical line of symmetry) is denoted D_h. A rotation of a dominating set D by 180° (which is the same as both the vertical reflection of D_h and the horizonal reflection of D_v) is denoted D_r. When we wish to discuss an arbitrary symmetry of a dominating set D, we denote it D_s.

For example, in the m-eternal dominating family for $P_5 \square P_6$ illustrated in Figure 1, there are six dominating sets, or one set A and its horizontal reflection A_h ($A_v \equiv A$ and $A_r \equiv A_h$), and another set and its three symmetries. A vertex containing a guard is denoted by a bullet. Each vertex not

A

	1	2	3	4	5	6
	•	A_h	A_h	•	•	A_h
	B	B_v	B_r	B_v	B_r	B
A	A_h	•	•	A_h	A_h	•
	B_v	B	B_h	B	B_h	B_v
	•	A_h	A_h	•	•	A_h

	1	2	3	4	5	6
	A	•	•	A	A	•
	B	B_v	B_r	B_v	B_r	B
A_h	•	A	A	•	•	A
	B_v	B	B_h	B	B_h	B_v
	A	•	•	A	A	•

	B_r	B_v	•	•	B_r	B_v
	•	B_v	B_r	B_v	B_r	•
B	A_h	A	B_h	•	A_h	A
	B_v	•	B_h	•	B_h	B_v
	B_h	•	B_v	B_v	B_h	•

	B_r	B_v	•	•	B_r	B_v
	•	B_v	B_r	B_v	B_r	•
B_h	A_h	A	•	B	A_h	A
	B_r	B	•	B	•	B_r
	•	B	B_r	B_r	•	B

	B_r	•	B	B	B_r	•
	B	•	B_r	•	B_r	B
B_v	A_h	A	B_r	•	A_h	A
	•	B	B_h	B	B_h	•
	B_h	B	•	•	B_h	B

	•	B_v	B_h	B_h	•	B_v
	B_h	B_v	•	B_v	•	B_h
B_r	A_h	A	•	B_v	A_h	A
	•	B	B_h	B	B_h	•
	B_h	B	•	•	B_h	B_v

Figure 1: An m-eternal dominating family for $P_5 \square P_6$ with 9 guards.

containing a guard is labeled with the name of a set to which to transform in order to protect against an attack on that vertex. In this example, any of the sets can transform to any of the other sets; thus the protection against an attack on vertex $(3,4)$ of set A by transformation to A_h could also be accomplished by a transformation to B or B_v.

Three fundamental theorems are used throughout this paper. Their proofs are obvious from the definitions, and all appear in some form in previous works (see for example [6, 7, 12]).

Theorem 2. *Given dominating sets $D, E \in \mathbb{D}_q(P_m \square P_n)$ and any arbitrary symmetry s resulting from a reflection or rotation, D transforms to E if and only if D_s transforms to E_s.*

Theorem 3. *For any G, $\gamma(G) \leq \gamma_{\mathrm{m}}^{\infty}(G)$.*

Theorem 4. *If $\gamma_{\mathrm{m}}^{\infty}(P_m \square P_n) = j$ and $\gamma_{\mathrm{m}}^{\infty}(P_m \square P_q) = k$, then*

$$\gamma_{\mathrm{m}}^{\infty}(P_m \square P_{n+q}) \leq j + k.$$

n	$\gamma(P_5 \square P_n)$
1	2
2	3
3	4
4	6
5	7

n	$\gamma(P_5 \square P_n)$
6	8
7	9
8	11
9	12
10	13

n	$\gamma(P_5 \square P_n)$
11	14
12	16
13	17
14	18
15	19

Table 1: $\gamma(P_5 \square P_n)$ for $n \leq 15$.

3 Previous Results

We use the formula for the domination number of $P_5 \square P_n$ graphs given by Chang and Clark [4]:

$$\gamma(P_5 \square P_n) = \begin{cases} \lfloor \frac{6n+6}{5} \rfloor & \text{if } n = 2, 3, 7 \\ \lfloor \frac{6n+8}{5} \rfloor & \text{otherwise for } n \geq 1. \end{cases}$$

Domination numbers of $P_5 \square P_n$ for $n \leq 15$ are listed in Table 1. In Table 2, we list the previously known m-eternal domination numbers for $P_5 \square P_n$ graphs, along with a reference to the source of each result.

n	$\gamma_m^\infty(P_5 \square P_n)$	Reference
1	3	[7]
2	4	[8]
3	5	[8]
4	6	[2]
5	7	[3]

Table 2: Known values for $\gamma_m^\infty(P_5 \square P_n)$ and their references.

The following statements, due to Chang and Clark [4], used to establish some restrictions on the domination number, assume that $(s_1, s_2, \ldots, s_n)$ is a dominating sequence for $P_5 \square P_n$.

S 5. *If $(s_1, s_2, \ldots, s_n)$ is a dominating sequence, then so is the reversed sequence $(s_n, s_{n-1}, \ldots, s_1)$. (The horizontal reflection of a dominating set is a dominating set.)*

S6. $0 \leq s_j \leq 5$. *(Each column contains between 0 and 5 guards, inclusive.)*

S7. $\sum_{t=1}^{j} s_t \geq \gamma(P_5 \square P_{j-1})$ *for* $j \geq 2$.

S8. $\sum_{t=j}^{n} s_t \geq \gamma(P_5 \square P_{n-j})$ *for* $j < n$.

S9. $\sum_{t=i}^{j} s_t \geq \gamma(P_5 \,\Box\, P_{j-i-1})$ *for* $j \geq i+2$.

S10. $s_1 = 0 \implies s_2 = 5$. *(No guards in column 1 implies five in 2).*

S11. $s_j = 0 \implies s_{j-1} + s_{j+1} \geq 5$ *for* $2 \leq j \leq n-1$. *(If a column contains no guard then the adjacent columns contain a total of at least five guards.)*

S12. $s_1 \geq 1$ *and* $s_2 \geq 1 \implies s_1 + s_2 \geq 3$. *(The first two columns cannot each contain exactly one guard.)*

S13. $s_j \geq 1$ *for each* $j, k \leq j \leq k+4 \implies \sum_{j=k}^{k+4} s_j \geq 6$. *(Five adjacent columns cannot each contain exactly one guard.)*

S14. $s_j \geq 1$ *for each* $j, 1 \leq j \leq 6 \implies \sum_{j=1}^{6} s_j \geq 8$. *(If each of the first six columns contain at least one guard, then the first six columns contain at least eight guards.)*

S15. *If for some* j, $2 \leq j \leq n-1$, $s_j = 0$, *and for all* $i < j$, $s_i \geq 1$, *then*

$$\sum_{i=1}^{j+1} s_i \geq \begin{cases} \gamma(P_5 \,\Box\, P_{j-1}) + 4 & \text{if } j = 4, 8 \\ \gamma(P_5 \,\Box\, P_{j-1}) + 3 & \text{otherwise.} \end{cases}$$

S16. *If* $n = 5$ *and* $s_1 \geq 3$ *then* $\sum_{j=1}^{5} s_j > 7$. *(If there are three or more guards in column 1, then there are more than seven guards in the* $P_5 \,\Box\, P_5$*.)*

S17. *If* $n = 9$ *and* $s_1 \geq 3$ *then* $\sum_{j=1}^{9} s_j > 12$. *(If there are three or more guards in column 1, then there are more than twelve guards in the* $P_5 \,\Box\, P_9$*.)*

The following lemma, due to Chang and Clark [4], is also helpful.

Lemma 18. *[4] Let* S *be a dominating set for* $P_5 \,\Box\, P_n$ *and assume that each of columns* i, $i+1$, $i+2$, $i+3$ *contains exactly one element of* S*. Then* $2 \leq i$, $i+3 \leq n-1$, *and there are only the two possible configurations for* S *in columns* i, $i+1$, $i+2$, *and* $i+3$ *shown in Figure 2. It follows that the guards indicated by circles in the columns* $i-1$ *and* $i+4$ *must lie in* S.

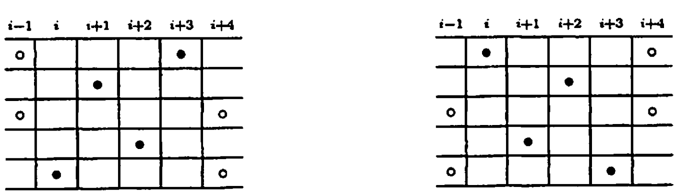

Figure 2: $P_5 \,\Box\, P_n$ with one guard in four adjacent columns.

4 Preliminary Results

The following statements used to establish some additional restrictions on the domination number assume that $(s_1, s_2, \ldots, s_n)$ is a dominating sequence for $P_5 \square P_n$.

S19. $s_1 = 1 \implies s_2 \geq 2$. *(If column 1 contains one guard, then column 2 contains at least two.)*

Proof. The one guard in column 1 defends at most three of the vertices in that column, leaving two vertices to be defended by guards in column 2, so $s_2 \geq 1$, and $s_1 + s_2 \geq 3$ by **S12**. $\square$

S 20. $s_1 + s_2 \leq 2 \implies s_1 = 2$ *and* $s_3 \geq 3$. *(If the first two columns contain at most two guards, then column 1 contains two guards and column 3 contains at least three guards.)*

Proof. If $s_1 = 0$, then $s_2 = 5$ by **S10**. If $s_1 = 1$, then $s_2 \geq 2$ by **S19**. Thus $s_q + s_2 \leq 2$ implies $s_1 = 2$ and $s_2 = 0$, and $s_1 + s_3 \geq 5$ by **S11**, so $s_3 \geq 3$. $\square$

S21. *If for some* j, $2 \geq j \geq n-3$, $s_j = 0$, *and for all* $i > j$, $s_i \geq 1$, *then*

$$\sum_{i=j-1}^{n} s_i \geq \left\lfloor \frac{6(n-j)+8}{5} \right\rfloor + 3.$$

Proof. By **S15** and **S5**, we have

$$\sum_{i=j-1}^{n} s_i \geq \begin{cases} \gamma(P_5 \square P_{n-j}) + 4 = \left\lfloor \frac{6(n-j)+6}{5} \right\rfloor + 4 & \text{if } j = n-3 \text{ or } n-7 \\ \gamma(P_5 \square P_{n-j}) + 3 = \left\lfloor \frac{6(n-j)+8}{5} \right\rfloor + 3 & \text{otherwise} \end{cases}$$

$$\geq \left\lfloor \frac{6(n-j)+8}{5} \right\rfloor + 3 \qquad \square$$

Lemma 22. *Let* S *be a dominating set for* $P_5 \square P_n$, $n > 3$, *and assume that the first three columns contain at most three guards. Then there are three guards in the first two columns and at least six in the first four columns, and there are only the two possible configurations for* S *in the first three columns as shown in Figure 3. It follows that the guards indicated by circles in the figure must lie in* S.

Figure 3: $P_5 \square P_n$ with fewer than four guards in the first three columns.

Proof. If $s_1 = 0$, then $s_2 = 5$ by **S10**. If $s_1 = 1$, then $s_2 \geq 2$ by **S19**, so $s_2 = 2$, $s_3 = 0$, and column 4 would require at least three guards to have all vertices in column 3 dominated. The only possible configuration of these guards is shown on the left of Figure 3. If $s_2 = 0$, then $s_1 + s_3 \geq 5$ by **S11**. Finally, if $s_1 = 2$ and $s_2 = 1$, then $s_3 = 0$ and column 4 would require at least four guards to have all vertices in column 3 dominated. The only possible configuration of these guards is shown on the right of Figure 3. □

Lemma 23. *Let S be a dominating set for $P_5 \square P_n$, $n > 4$, and assume that the first four columns contain at most four guards. Then there are three guards in the first two columns, one in column 3, zero in column 4, and at least four in column 5, giving at least eight in the first five columns. Furthermore, the only possible configuration for the four guards is as shown in Figure 4.*

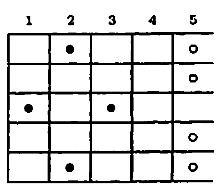

Figure 4: $P_5 \square P_n$ with fewer than five guards in the first four columns.

Proof. If $s_1 + s_2 \leq 2$, then $s_3 \geq 3$ by **S20**. If $s_1 + s_2 = 4$, then at least one vertex in column 3 is undefended. Thus $s_1 + s_2 = 3$, Lemma 22 implies $s_3 = 1$, only one vertex in column 4 is defended, and column 5 would need four guards to defend the others. Exhaustive search reveals the only possible configuration of the guards is shown in Figure 4. □

5 A Lower Bound

A lower bound on the m-eternal domination number for the $P_5 \square P_n$ family of graphs can be established by examining the size of the dominating sets containing a guard at vertex $(3, 2)$; that is, those elements in the m-eternal dominating sets which result after an attack on that vertex. We define a notation for these sets and examine some of their general properties.

Definition 24. Let $\gamma^*(P_5 \square P_n)$ be the minimum size of a dominating set of the $5 \times n$ grid graph containing a guard at vertex $(3, 2)$.

Lemma 25. *In any dominating set for $P_5 \square P_n$, $n > 3$, with a guard at vertex $(3, 2)$:*

1. *the first two columns contain at least three guards,*
2. *if the first two columns contain exactly three guards, then column 1 contains two guards and column 2 the other,*
3. *if the first three columns contain exactly three guards, then column 4 contains at least four guards and the first four columns contain at least seven guards, and*
4. *if the first four columns each contain at least one guard, then the first four columns contain at least six guards.*

Proof. A guard at vertex $(3, 2)$ defends the middle vertex of column 1 and the middle three vertices in column 2. To defend the top two vertices of column 1 requires either one guard at one of these vertices in column 1 or at least two guards placed elsewhere in the first two columns. The same is true for the bottom two vertices in column 1. If there are exactly three guards in the first two columns, then two must be located on vertices in column 1. If there are exactly three guards in the first three columns, then two must be located on vertices in column 1, and these three only defend one vertex in column three, thus four guards are required in column four. If the first four columns contain at most five guards and each column contains at least one guard, then columns 3 and 4 each contain one guard, and at least one vertex in column 3 is undefended. □

Lemma 26. $\gamma^*(P_5 \square P_6) \geq 9$.

Proof. Consider any dominating set for $P_5 \square P_6$ with a guard at vertex $(3, 2)$. By Lemma 25 (1.), there are at least three guards in the first two columns. If the last three columns contain at most three guards, then by Lemma 22, the last four columns contain at least six guards. If the last three columns contain four guards, then by **S20**, the last two columns contain at least three guards, and column 4 contains at most one guard. If the first two columns contain three guards, then Lemma 25 (2.) shows only one guard is in column 2, and two additional guards are needed to defend column three. If the first two columns contain four guards, then at most three are in column 2, so at least one vertex of column 3 is undefended. Finally, if the last three columns contain five guards, then the last two columns have at least two guards, so column 4 has at most three guards, thus at least one vertex in column 3 in undefended. □

Lemma 27. $\gamma^*(P_5 \square P_7) \geq 10$.

Proof. Consider any dominating set for $P_5 \square P_7$ with a guard at vertex $(3, 2)$. By Lemma 25, there are at least three guards in the first two columns and, if exactly three, then two are in column 1. If the last four columns contain at most four guards, then by Lemma 23, the last five columns contain at least eight guards. If the last four column contain five guards, then by Lemma 22, the last three columns contain at least four guards, so column 4 contains at most one guard. If the first two columns contain three guards, then two guards are needed to defend column 3, and if the first two columns contain four guards, then at most three are in column 2, so at least one vertex of column 3 is undefended. Finally, if the last four columns contain six guards, at least three are in the last three columns by **S20**, so at most three are in column 4, thus at least one vertex in column 3 is undefended. □

Lemma 28.

$$\gamma^*(P_5 \square P_2) = 3, \quad \gamma^*(P_5 \square P_3) = 5, \quad \gamma^*(P_5 \square P_4) = 6,$$

$$\gamma^*(P_5 \square P_5) = 7, \quad \gamma^*(P_5 \square P_6) = 9, \quad \gamma^*(P_5 \square P_7) = 10.$$

Proof. We establish sufficiency with Figure 5. Necessity is given by the domination number for $\gamma^*(P_5 \square P_2)$, $\gamma^*(P_5 \square P_4)$, and $\gamma^*(P_5 \square P_5)$. Next, for $\gamma^*(P_5 \square P_3)$, with a guard at $(3, 2)$, we have necessity as unique guards are needed to defend $(1, 1)$, $(2, 3)$, $(4, 1)$, and $(5, 3)$. Finally, for $\gamma^*(P_5 \square P_6)$ and $\gamma^*(P_5 \square P_7)$, necessity is given by Lemmas 26 and 27, respectively. □

Lemma 29. *If* $(s_1, s_2, \ldots, s_n)$ *is a dominating sequence for* $P_5 \times P_n$ *such that* $s_i \geq 1$ *for all* i *and* $(3, 2)$ *contains a guard, then for* $n \geq 6$,

$$\sum_{i=1}^{n} s_i \geq \left\lfloor \frac{6n + 10}{5} \right\rfloor.$$

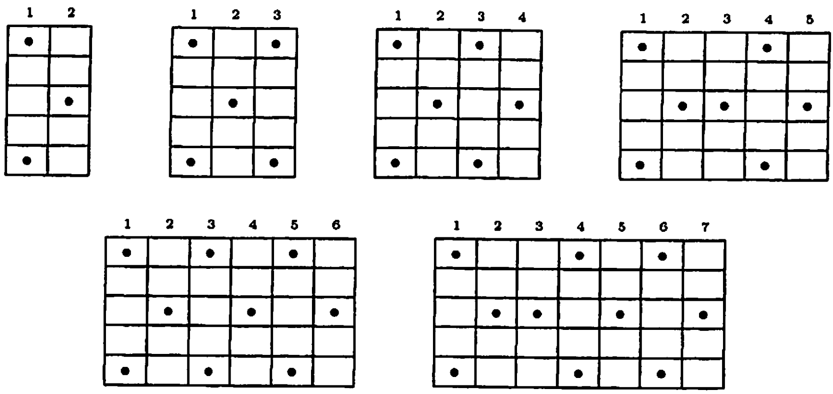

Figure 5: Dominating sets of $P_5 \square P_n$, $2 \le n \le 7$, with a guard at $(3,2)$.

Proof. We use repeatedly in this proof the fact that if $s_i \ge 1$ for $k \le i \le l$ and $m = l - k + 1$, then the number of terms in $\sum_{i=k}^{l} s_i$, is divisible by 5, then by **S13** we have

$$\sum_{i=k}^{l} s_i \ge 6\left(\frac{l-k+1}{5}\right). \tag{1}$$

Let $n = 5q + r$, $0 \le r \le 4$. We consider two cases.

$r = 0$: By (1), Lemma 25 (4.), and **S14**,

$$\sum_{i=1}^{n} s_i = \sum_{i=1}^{4} s_i + \sum_{i=5}^{n-6} s_i + \sum_{i=n-5}^{n} s_i \ge 6 + 6\left(\frac{n-10}{5}\right) + 8 = \left\lfloor \frac{6n+10}{5} \right\rfloor.$$

$r \ne 0$: By (1), Lemma 25 (4.), and **S12**,

$$\begin{aligned}\sum_{i=1}^{n} s_i &= \sum_{i=1}^{4} s_i + \sum_{i=5}^{n-r-1} s_i + \sum_{i=n-r}^{n-2} s_i + \sum_{i=n-1}^{n} s_i \\ &\ge 6 + 6\left(\frac{n-r-5}{5}\right) + r - 1 + 3 = \frac{6n-r+10}{5} = \left\lfloor \frac{6n+10}{5} \right\rfloor.\end{aligned}$$

□

Theorem 30. $\gamma^*(P_5 \square P_n) \ge \begin{cases} 3 & \textit{for } n = 2 \\ 7 & \textit{for } n = 5 \\ \lfloor \frac{6n+10}{5} \rfloor & \textit{otherwise.} \end{cases}$

Proof. For $n \leq 7$ the theorem follows from Lemma 28. Thus, we may assume $n \geq 8$. We proceed by induction, letting $n \geq 8$ and assuming the theorem holds for $2 \leq k < n$.

If $s_i \geq 1$ for $1 \leq i \leq n$, then by Lemma 29, the result holds. So we can assume $s_i = 0$ for at least one i. Let $j := \max\{i | s_i = 0\}$. We verify the bound for each value of j. For $3 \geq j \geq n-3$, we apply **S21** and find $\sum_{i=1}^{n} s_i = \sum_{i=1}^{j-2} s_i + \sum_{i=j-1}^{n} s_i \geq \gamma^*(P_5 \square P_{j-3}) + \left\lfloor \frac{6(n-j)+8}{5} \right\rfloor + 3$. We also note by Lemma 28 that $\gamma^*(P_5 \square P_n) = 7 = \lfloor \frac{6n+9}{5} \rfloor$ and so we can also assume by induction that $\gamma^*(P_5 \square P_n) \geq \lfloor \frac{6n+9}{5} \rfloor$ for $3 \leq k < n$, which is sufficient to verify most cases.

$j = 1$: By **S10**, $s_2 = 5$. Thus, by **S8**,

$$\begin{aligned}\sum_{i=1}^{n} s_i &= s_2 + \sum_{i=3}^{n} s_i \\ &\geq 5 + \gamma(P_5 \square P_{n-3}) \\ &\geq 5 + \left\lfloor \frac{6(n-3)+6}{5} \right\rfloor = \left\lfloor \frac{6n+13}{5} \right\rfloor \geq \left\lfloor \frac{6n+10}{5} \right\rfloor.\end{aligned}$$

$j = 2$: By the definition of $\gamma^*(P_5 \square P_n)$, $s_2 \neq 0$.

$j = 3$: By **S21**,

$$\begin{aligned}\sum_{i=1}^{n} s_i &= s_1 + \left\lfloor \frac{6(n-3)+8}{5} \right\rfloor + 3 \\ &\geq 1 + \left\lfloor \frac{6(n-3)+8}{5} \right\rfloor + 3 = \left\lfloor \frac{6n+10}{5} \right\rfloor.\end{aligned}$$

$j = 4$: By **S12** and **S21**,

$$\begin{aligned}\sum_{i=1}^{n} s_i &= \sum_{i=1}^{2} s_i + \left\lfloor \frac{6(n-4)+8}{5} \right\rfloor + 3 \\ &\geq 3 + \left\lfloor \frac{6(n-4)+8}{5} \right\rfloor + 3 = \left\lfloor \frac{6n+14}{5} \right\rfloor \geq \left\lfloor \frac{6n+10}{5} \right\rfloor.\end{aligned}$$

$j = 5$: There are two subcases:

$s_3 = 0$: We first show the first four columns cannot be defended with six guards. We proceed by contradiction. By **S20**, the first two columns have at least three guards. Since columns 3 and 5 have no guards, column 4 has at least two guards. If the first two columns have exactly three guards, then column 2 has one guard, so column 4 contains four guards. If the first two columns have exactly four guards, then column 4 has two guards. Column 2 has at most three guards, and if it has exactly three, then they must be in the top three vertices (up to symmetry). But then the two guards in column 4 must be in the bottom two vertices to defend column 3, but then at least one vertex in column 4 is undefended. Thus, the first four columns have at least seven guards, so

$$\sum_{i=1}^{n} s_i = \sum_{i=1}^{4} s_i + \sum_{i=6}^{n} s_i$$
$$\geq 7 + \left\lfloor \frac{6(n-5)+6}{5} \right\rfloor = \left\lfloor \frac{6n+11}{5} \right\rfloor \geq \left\lfloor \frac{6n+10}{5} \right\rfloor .$$

$s_3 \geq 1$: By Lemma 25 (1.),

$$\sum_{i=1}^{n} s_i \geq \sum_{i=1}^{2} s_i + s_3 + \left\lfloor \frac{6(n-5)+8}{5} \right\rfloor + 3$$
$$\geq 3 + 1 + \left\lfloor \frac{6(n-5)+8}{5} \right\rfloor + 3$$
$$\geq \left\lfloor \frac{6n+10}{5} \right\rfloor .$$

$j = 8$: There are two subcases:

$n = 10$: Suppose the grid can be defended with thirteen guards. By Lemma 27, the first seven columns contain at least ten guards and by **S12**, the last two columns contain at least three guards. Thus, the first seven columns contain ten guards and the last two guards contain three guards. By Lemma 26, the first six columns contain at least nine guards, so column 7 contains at most one guard. Column 9 contains at most two guards, so column 7 contains at least three guards, which is a contradiction. Thus, the grid requires at least 14 guards.

$n \neq 10$: By Lemma 28,

$$\sum_{i=1}^{n} s_i \geq \gamma^*(P_5 \square P_5) + \left\lfloor \frac{6(n-8)+8}{5} \right\rfloor + 3$$
$$\geq 7 + \left\lfloor \frac{6(n-8)+8}{5} \right\rfloor + 3 = \left\lfloor \frac{6n+10}{5} \right\rfloor .$$

$j = n-2$: By **S7**,

$$\begin{aligned}\sum_{i=1}^{n} s_i &= \sum_{i=1}^{n-4} s_i + \sum_{i=n-3}^{n} s_i \\ &\geq \gamma^*(P_5 \Box P_{n-5}) + \gamma(P_5 \Box P_2) + 3 \\ &\geq \left\lfloor \frac{6(n-5)+10}{5} \right\rfloor + 3 + 3 = \left\lfloor \frac{6n+10}{5} \right\rfloor.\end{aligned}$$

$j = n-1$: By **S11** and **S7**,

$$\begin{aligned}\sum_{i=1}^{n} s_i &= \sum_{i=1}^{n-3} s_i + s_{n-2} + s_n \\ &\geq \gamma^*(P_5 \Box P_{n-4}) + 5 \\ &\geq \left\lfloor \frac{6(n-4)+9}{5} \right\rfloor + 5 = \left\lfloor \frac{6n+10}{5} \right\rfloor.\end{aligned}$$

$j = n$: By **S10**, $s_{n-1} = 5$. Thus by **S7**,

$$\begin{aligned}\sum_{i=1}^{n} s_i &= \sum_{i=1}^{n-2} s_i + s_{n-1} \\ &\geq \gamma^*(P_5 \Box P_{n-3}) + 5 \\ &\geq \left\lfloor \frac{6(n-3)+9}{5} \right\rfloor + 5 \geq \left\lfloor \frac{6n+10}{5} \right\rfloor.\end{aligned}$$

Otherwise:

$$\begin{aligned}\sum_{i=1}^{n} s_i &\geq \gamma^*(P_5 \Box P_{j-3}) + \left\lfloor \frac{6(n-j)+8}{5} \right\rfloor + 3 \\ &\geq \left\lfloor \frac{6(j-3)+10}{5} \right\rfloor + \left\lfloor \frac{6(n-j)+8}{5} \right\rfloor + 3 \\ &\geq \left\lfloor \frac{6n}{5} \right\rfloor + 2 = \left\lfloor \frac{6n+10}{5} \right\rfloor.\end{aligned}$$

□

The result in Theorem 30 leads directly to our lower bound.

Corollary 31. $\gamma_{\mathrm{m}}^{\infty}(P_5 \Box P_n) \geq \begin{cases} \left\lfloor \frac{6n+6}{5} \right\rfloor & \textit{for } n = 2 \textit{ or } 5 \\ \left\lfloor \frac{6n+10}{5} \right\rfloor & \textit{otherwise.} \end{cases}$

Proof. Let $\mathcal{E}$ be an m-eternal dominating family of $P_5 \,\square\, P_n$ and consider each $D \in \mathcal{E}$ required to protect a graph in response to an attack on vertex $(3,2)$. The result follows. $\square$

The lower bound is not tight, as can be shown for $n = 9$.

Lemma 32. $\gamma_{\text{m}}^{\infty}(P_5 \,\square\, P_9) \geq 13$.

Proof. Let $\mathcal{E}$ be an m-eternal dominating family of $P_5 \,\square\, P_9$ which uses twelve guards, and consider a $D \in \mathcal{E}$ resulting after an attack on vertex $(3,2)$. Exhaustive search reveals there are only three possible configurations of twelve guards which dominate the $P_5 \,\square\, P_9$ grid with a guard located at vertex $(3,2)$. Two are illustrated in Figure 6, the third is the vertical reflection of grid A.

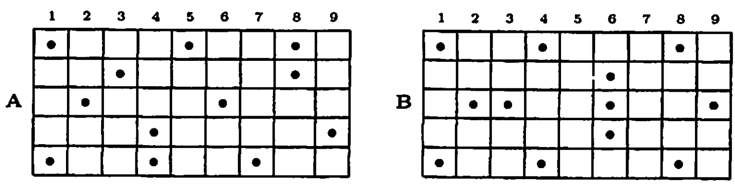

Figure 6: The dominating sets for $P_5 \,\square\, P_9$ with a guard in vertex $(3,2)$.

Both grid A and its vertical reflection can be forced to transform to the configuration of grid B by an attack, targeting vertex $(5,6)$ in grid A and the corresponding vertex $(1,6)$ of its reflection. The response to either of these attacks is the grid which is the horizontal reflection of grid B. From this point a sequence of attacks cannot be defended. The first attack is on vertex $(1,7)$ of grid B. The response to this attack leads to the transformation to one of the four configurations illustrated in Figure 7.

The next attack is on vertex $(3,5)$. Grid C is unable to defend this attack. Grids D and E are forced to move to the configuration shown as grid Y in Figure 8, grid F is forced to transform to either the vertical reflection of grid Y or to grid Z in Figure 8.

Grid Y and Z cannot recover from attacks on vertices $(1,3)$ and $(5,3)$, respectively. $\square$

6 M-eternal domination numbers for $n \leq 12$

Table 3 lists the m-eternal domination numbers for $P_5 \,\square\, P_n$ for $1 \leq n \leq 12$, repeating those for $n \leq 5$ from Table 2.

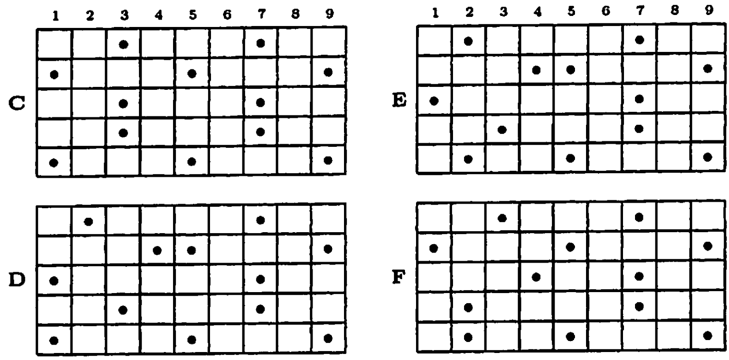

Figure 7: Dominating sets after attacking $(1,7)$ of grid B in Figure 6.

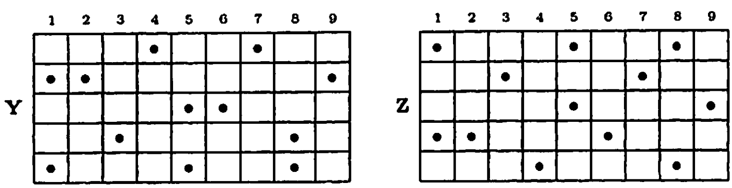

Figure 8: Dominating sets after attacking $(3,5)$ of grids in Figure 7.

n	$\gamma_{\mathrm{m}}^{\infty}(P_5 \square P_n)$
1	3
2	4
3	5
4	6
5	7
6	9
7	10
8	11
9	13
10	14
11	15
12	16

Table 3: $\gamma_{\mathrm{m}}^{\infty}(P_5 \square P_n)$ for $1 \leq n \leq 12$.

Theorem 33. $\gamma_{\mathrm{m}}^{\infty}(P_5 \square P_n) = \begin{cases} \lfloor \frac{6n+11}{5} \rfloor & \textit{for } n = 9, 10 \\ \lfloor \frac{6n+9}{5} \rfloor & \textit{otherwise for } 1 \leq n \leq 12. \end{cases}$

Proof. The values for $1 \leq n \leq 5$ are equivalent to those from the references shown in Table 2 [2, 3, 7, 8]. Necessity for $n = 9$ is given by Lemma 32. Necessity for $n = 6, 7, 8, 10, 11, 12$ is given by the lower bound in Corollary 31. We establish sufficiency for $n = 6$ in Figure 1, and for $n = 7, 8, 11, 12$ with Figures 9 to 11. Sufficiency for $n = 9, 10$ can be established by combining smaller sets as per Theorem 4, using copies of the m-eternal dominating sets for $P_5 \square P_4$ and $P_5 \square P_5$. □

7 An Upper Bound

In order to derive an upper bound, we observe from the values in Table 3 that for $2 \leq n \leq 11$, $\gamma_{\mathrm{m}}^{\infty}(P_5 \square P_n) \leq \lfloor \frac{4n+4}{3} \rfloor$. Combining smaller sets as per Theorem 4 as shown in Table 4 derives upper bounds which, combined with the lower bounds defined in Corollary 31, gives the intervals for the m-eternal dominating numbers for the grids shown in the table. The value $\gamma_{\mathrm{m}}^{\infty}(P_5 \square P_{12}) = 16 = \frac{4(12)}{3}$ extends the range for our upper bound for the m-eternal domination number.

Theorem 34. *For* $n \geq 2$, $\gamma_{\mathrm{m}}^{\infty}(P_5 \square P_n) \leq \lfloor \frac{4n+4}{3} \rfloor$.

8 Conclusion

We have proven bounds for the m-eternal domination number of $5 \times n$ grid graphs, $n \geq 2$, as

$$\left\lfloor \frac{6n+9}{5} \right\rfloor \leq \gamma_{\mathrm{m}}^{\infty}(P_5 \square P_n) \leq \left\lfloor \frac{4n+4}{3} \right\rfloor.$$

We have already established the lower bound is not tight, and the same is true for the upper bound, as shown by the cases where $n = 5, 8, 11$, and 12. Further investigation is warranted.

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n	Combine	$\gamma_{\mathrm{m}}^{\infty}(P_5 \square P_n)$
13	$5+8$	[17..18]
14	$7+7$	[18..20]
15	$7+8$	[20..21]
16	$8+8$	[21..22]
17	$5+12$	[22..23]
18	$6+12$	[23..25]

n	Combine	$\gamma_{\mathrm{m}}^{\infty}(P_5 \square P_n)$
19	$7+12$	[24..26]
20	$8+12$	[26..27]
21	$9+12$	[27..29]
22	$10+12$	[28..30]
23	$11+12$	[29..31]
24	$12+12$	[30..32]

Table 4: Intervals for $\gamma_{\mathrm{m}}^{\infty}(P_5 \square P_n)$ for $13 \le n \le 24$.

	1	2	3	4	5	6	7
	$\bullet$	C	C_v	$\bullet$	C	C_h	$\bullet$
	C_v	C	C_r	B	C_v	C_h	C
A	B	$\bullet$	$\bullet$	C	$\bullet$	$\bullet$	B
	C	C_v	C_h	B	C	C_r	C_v
	$\bullet$	C_v	C	$\bullet$	C_v	C_r	$\bullet$

	1	2	3	4	5	6	7
	C_r	$\bullet$	C_v	A	C	$\bullet$	C_v
	C_v	C	C_r	$\bullet$	C_v	C_h	C
B	$\bullet$	$\bullet$	C_h	C	C	$\bullet$	$\bullet$
	C	C_v	C_h	$\bullet$	C	C_r	C_v
	C_h	$\bullet$	C	A	C_v	$\bullet$	C

	1	2	3	4	5	6	7
	C_r	$\bullet$	C_v	A	$\bullet$	C_h	C_v
	C_v	$\bullet$	C_r	B	C_v	C_h	$\bullet$
C	B	A	C_h	$\bullet$	$\bullet$	A	B
	$\bullet$	C_v	C_h	B	$\bullet$	C_r	C_v
	C_h	C_v	$\bullet$	A	C_v	C_r	$\bullet$

Figure 9: An m-eternal dominating family for $P_5 \square P_7$ with 10 guards.

	1	2	3	4	5	6	7	8
	E_h	$\bullet$	G	C_v	C	$\bullet$	C_h	A_v
	C_h	C_h	F_v	$\bullet$	C	C_v	C	$\bullet$
A	$\bullet$	E_h	C	$\bullet$	F	C_h	$\bullet$	C_h
	F_v	F_v	C_h	$\bullet$	C_v	C	C_v	A_v
	C_h	$\bullet$	G_v	C	C_v	$\bullet$	C_h	$\bullet$

	1	2	3	4	5	6	7	8
	B_r	B_v	$\bullet$	B_r	$\bullet$	D_v	B_r	B_v
	$\bullet$	B_v	B_r	D_v	D_r	B_v	$\bullet$	$\bullet$
B	D_v	D_r	D_r	$\bullet$	B_r	C_r	D_v	C_r
	B_v	$\bullet$	D_v	C_r	G_v	$\bullet$	B_v	B_v
	D_r	$\bullet$	B_v	B_r	$\bullet$	B_r	C_r	$\bullet$

	1	2	3	4	5	6	7	8
	B_r	$\bullet$	A_h	C_h	$\bullet$	A	C_h	A_v
	C_h	C_h	B_r	C_h	$\bullet$	D_h	$\bullet$	$\bullet$
C	$\bullet$	A_h	$\bullet$	A	A_h	C_h	A	C_h
	B_r	B_r	C_h	A	A_h	$\bullet$	F	A_v
	C_h	$\bullet$	A_h	$\bullet$	C_h	A	C_h	$\bullet$

	B_r	$\bullet$	G	B_r	$\bullet$	D_v	B_r	$\bullet$
	C_h	B_v	$\bullet$	D_v	G	B_v	B_r	F_v
D	$\bullet$	E_h	C_v	B_v	B_r	$\bullet$	$\bullet$	C_h
	B_v	B_r	D_v	$\bullet$	C_v	F_v	B_v	B_v
	C_h	$\bullet$	B_v	B_r	D_v	$\bullet$	C_h	$\bullet$

	E_h	$\bullet$	A_h	E_v	$\bullet$	A	E_h	$\bullet$
	A_h	C_h	B_r	E_h	$\bullet$	E_v	B_r	A
E	$\bullet$	E_h	$\bullet$	A	A_h	E_h	$\bullet$	E_h
	B_r	B_r	E_h	A	E_v	$\bullet$	C_v	A_v
	E_h	$\bullet$	A_h	$\bullet$	E_v	A	E_h	$\bullet$

	F_v	A	A_h	$\bullet$	F_v	A	A_h	$\bullet$
	$\bullet$	$\bullet$	F_v	A	A_h	$\bullet$	F_v	F_v
F	A	A_h	C	F_h	$\bullet$	D_v	A	A_h
	F_v	F_v	$\bullet$	A	A_h	F_v	$\bullet$	$\bullet$
	$\bullet$	A	A_h	F_v	$\bullet$	A	A_h	F_v

	G_v	A	$\bullet$	E_h	G_v	A	A_h	$\bullet$
	$\bullet$	B_v	G_v	A	$\bullet$	$\bullet$	G_v	G_v
G	A	A_h	$\bullet$	A	A_h	D	A	A_h
	G_v	B	$\bullet$	A	G_v	G_v	$\bullet$	$\bullet$
	$\bullet$	A	G_v	E_r	$\bullet$	A	A_h	G_v

Figure 10: An m-eternal dominating family for $P_5 \square P_8$ with 11 guards.

	1	2	3	4	5	6	7	8	9	10	11
	F_h	•	A_h	F_r	•	B_v	A_h	F_h	•	A_h	E
	A_h	•	B_v	B_v	A_h	F_r	•	B_v	E	A_h	•
A	B_v	B_h	E	•	F_h	B_v	G	•	E	B_v	B_v
	•	•	E	B_v	H_h	•	E	B_v	B_h	•	A_h
	E	A_h	H_h	•	A_h	B_v	•	A_h	E	•	F_h

	E_h	•	A_h	A_v	E_h	•	A_v	J	E_h	•	E
	A_v	A_v	E_h	•	A_h	A_v	E_h	•	E	A_v	E_h
B	•	H_h	E	A_v	H_h	•	C	A_v	E	•	•
	E_h	A_v	•	•	E_h	A_h	A_v	•	E_h	A_h	A_v
	E	•	E_h	J	A_v	•	E_h	A_h	A_v	•	E_h

	D_h	•	E	F_r	D_h	•	E	F_h	D_h	•	E
	E	F_r	B_v	•	E	F_r	D_h	•	E	G_r	D_h
C	•	J	E	•	D_h	B	•	D_h	E	•	•
	E_h	F_h	B	•	E_h	F_h	D_r	•	E_h	G_h	D_r
	D_r	•	E_h	F_h	D_r	•	E_h	F_h	D_r	•	E_h

	D_h	D_v	•	F	D_h	D_v	•	F_h	D_h	D_v	•
	•	F_r	D_h	D_v	•	F_r	D_h	D_v	•	G_r	D_h
D	•	C_h	F	•	D_h	J	•	D_h	F_h	F_h	•
	D_v	F_h	F_r	•	D_v	F	F_r	•	D_v	F	F
	F	•	D_v	F	F_r	•	D_v	F	G_r	•	D_v

	E_h	A	•	J	E_h	B	•	A_r	E_h	B	•
	•	A	E_h	B	•	A_r	E_h	B	•	A_r	E_h
E	B	B_h	•	A	C_h	•	C	A	•	B	B
	E_h	A	•	B	E_h	A	•	B	E_h	A	•
	•	B	E_h	A	•	B	E_h	J	•	A	E_h

	•	A_r	A_h	•	F_h	C_h	•	F_v	D_h	A_h	•
	A_h	A_r	F_h	C_h	•	F_v	F_h	C_h	•	F_v	F_v
F	C_h	•	•	A_h	F_h	J	•	A_h	F_h	F_h	C_h
	F_h	F_h	D_r	C_h	F_v	•	D_r	C_h	F_v	•	•
	•	A_h	A_r	•	A_h	C_h	F_v	•	D_r	A_r	F_v

	G_h	A	•	F	G_h	C_h	•	F_v	G_h	A_h	•
	•	A	G_h	C_h	•	F_v	G_h	C_h	•	A_h	G_h
G	C_h	C_h	F	•	G_h	J	•	G_h	F_h	F_h	C_h
	G_h	•	D_r	C_h	F_v	•	D_r	C_h	F_v	•	•
	•	A_h	H_h	•	A_h	C_h	A	•	D_r	A	G_h

	E_h	H_v	•	F	H_v	B_h	•	F_h	H_v	A_h	•
	•	H_v	E_h	B_h	•	J	H_v	B_h	•	A_h	E_h
H	B_h	B_h	E	•	D_r	B_h	•	A_h	E	•	B_h
	H_v	•	D_r	B_h	H_v	A_h	•	B_h	H_v	A_h	A_h
	D_r	•	H_v	F	•	B_h	H_v	A_h	•	B_h	•

	•	B	D	•	D_h	B	D	•	D_h	B	•
	D	F_r	B_v	B	D	•	D_h	B	B_r	F_v	D_h
J	B	•	•	C	C_h	•	C	C_h	•	•	B
	D_v	F_h	B	B	D_v	•	D_r	B	B_h	F	D_r
	•	B	D_v	•	D_r	B	D_v	•	D_r	B	•

(a) $P_5 \square P_{11}$ with 15 guards.

	1	2	3	4	5	6	7	8	9	10	11	12
	F_h	•	H	D_h	C	•	D_h	F_h	A_v	•	F_h	A_v
	D_h	D_h	C	•	H	F_h	C	•	A_v	G_r	C	•
A	•	F_h	F_h	•	D_h	F_h	•	G_v	D_h	F_h	•	F_h
	G_v	G_h	D_h	•	C	F_h	D_h	A_v	•	C	D_h	A_v
	D_h	•	G_v	F_h	D_h	•	C	D_h	•	A_v	D_h	•

	B_r	•	B_v	B_r	•	B_v	B_r	F_h	B_v	•	F_h	B_v
	B_v	•	C	B_r	H	B_v	•	•	B_v	G_r	C	•
B	C	B_r	F_h	•	D_h	F_h	C	D	B_r	F_h	•	F_h
	•	B_v	D_h	C	B_r	•	B_v	B_v	•	C	B_r	B_v
	D_h	B_v	•	F_h	B_v	•	C	B_r	•	B_v	B_r	•

	D_r	•	G	E_h	•	A	C_v	D_v	•	A	D_r	C_v
	G	B	•	C_v	C_v	D_v	•	A	C_v	C_v	•	•
C	•	E_h	E_h	A	D_r	E_h	•	D_v	D_r	E	A	E_h
	B	D_r	C_v	•	•	B	C_v	J	•	•	C_v	C_v
	E_h	•	B	D_r	C_v	A	•	F_h	C_v	D_r	E_h	•

	D_r	D_v	•	F_h	D_v	•	A_h	D_v	•	B	D_r	D_v
	•	D_v	D_r	C_v	A_h	•	D_r	B	A_h	D_v	•	•
D	C_v	A_h	F_h	•	D_r	A_h	C_v	•	D_r	F_h	B	A_h
	D_v	•	C_v	A_h	G_v	•	D_r	J	A_h	•	D_v	D_v
	A_h	•	D_v	D_r	•	D_v	D_r	•	D_v	D_r	A_h	•

	C_r	•	G	C_h	C	•	C_v	C_h	•	G_h	C_h	•
	C_h	C_h	C	•	C_v	C_h	C	C_r	C_v	C_v	C	C
E	•	F_h	F_h	•	G_h	C_h	•	•	F_h	•	•	C_h
	C_r	C_r	C_v	•	C	C_r	C_v	C_h	C	C	C_v	C_v
	C_h	•	G_v	C_r	C_v	•	C	C_r	•	G_r	C_h	•

	A_r	•	A_h	A_r	•	C_r	A_h	B_h	•	B_r	A_h	•
	A_h	C_h	C	A_r	A_h	B_h	•	C_r	A_h	C_v	B_h	B_r
F	•	A_h	•	•	D_h	A_h	•	G	A_h	•	•	A_h
	A_r	C_r	C_v	A_h	A_r	B_r	•	C_h	A_h	C	B_r	B_h
	A_h	•	A_r	A_h	•	C_h	A_h	B_r	•	B_h	A_h	•

	G_v	A_v	•	A_r	G_v	A_v	A_h	•	G_v	B_r	A_h	•
	•	D_v	G_v	A_v	•	•	G_v	J_v	A_v	•	G_v	G_v
G	A_v	A_h	•	A_v	E_h	A_h	A_v	•	A_h	E	A_v	A_h
	G_v	B_v	•	A_v	G_v	G_v	•	A_v	A_h	G_v	•	•
	•	A_v	G_v	A_h	•	A_v	A_h	G_v	•	A_v	A_h	G_v

	H_h	A	•	A_r	H_h	A	A_h	•	A_v	H_h	A_h	•
	•	B	H_h	A	•	•	H_h	H_h	A_v	•	B_h	H_h
H	A	A_h	•	A	E_h	A_h	A	E	A_h	•	A	A_h
	H_h	B_v	•	A	H_h	H_h	•	•	A	H_h	B_r	•
	•	A	H_h	A_h	•	A	A_h	H_h	A	•	A_h	H_h

	E_h	•	D	E_h	J_v	A	•	J_v	C	A	•	C_v
	D	G_r	J_v	•	•	D	C	J_v	•	C_v	C	A
J	•	E_h	E_h	A	E_h	•	A	D	E_h	•	A	•
	G_v	D	•	J_v	J_v	D	C_v	•	J_v	C	C_v	C_v
	E_h	•	G_v	E_h	•	A	J_v	•	A	G_r	•	A

(b) $P_5 \square P_{12}$ with 16 guards.

Figure 11: M-eternal dominating families for $P_5 \square P_{11}$ and $P_5 \square P_{12}$.

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