# The Metamorphoses of Maximum Packings of $2K_n$ with Triples to Maximum Packings of $2K_n$ with 4-cycles for $n \equiv 5, 8,$ and $11$ $(mod\ 12)$

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**Abstract**

Gionfriddo and Lindner detailed the idea of the metamorphosis of 2-fold triple systems with no repeated triples into 2-fold 4-cycle systems of all orders where each system exists in [3]. In this paper, this concept is expanded to address all orders $n$ such that $n \equiv 5, 8,$ or $11$ $(mod\ 12)$. When $n \equiv 11$ $(mod\ 12)$, a maximum packing of $2K_n$ with triples has a metamorphosis into a maximum packing of $2K_n$ with 4-cycles, with the leave of a double edge being preserved throughout the metamorphosis. For $n \equiv 5$ or $8$ $(mod\ 12)$, a maximum packing of $2K_n$ with triples has a metamorphosis into a a 2-fold 4-cycle system of order $n$, except for when $n = 5$ or $8$, when no such metamorphosis is possible.

## 1 Introduction

A $\lambda$-fold $k$-cycle system of order $n$ is a pair $(X, C)$, where $C$ is a collection of edge disjoint $k$-cycles which partitions the edge set of $\lambda K_n$ with vertex set $X$ ($\lambda K_n$ denotes the graph on $n$ vertices in which each pair of vertices is joined by exactly $\lambda$ edges). It is well-known that the spectrum for 2-fold 3-cycle (or *triple*) systems is the set of all $n \equiv 0$ or $1$ $(mod\ 3)$ and the spectrum for 2-fold 4-cycle systems, i.e. the values of $n$ for which such a system exists, is the set of all $n \equiv 0$ or $1$ $(mod\ 4)$ (see [6]). A *hinge* is the multigraph comprised of 2 edge disjoint 3-cycles with exactly 2 vertices in common (see Figure 1 below); the two edges joining the common vertices are naturally called the *double edge* of the hinge. A *maximum packing* of a graph $G$ with a subgraph $C$ is an ordered triple $(V(G), T, L)$, where $T$ is a collection of copies of $C$ whose edges partition

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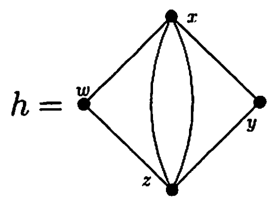
$E(G) \setminus L$. Note that if $L$ is empty, $G$ is $\lambda K_n$, and $C$ is a 3-cycle, this would correspond to a $\lambda$-fold triple system. Thus, a maximum packing of a graph $G$ with $C$ is simply a collection of edge-disjoint copies of $C$ that cover as many edges of $G$ as possible. The uncovered edges $L$ are called the *leave* of the packing.

Figure 1:




The following notation will be used throughout. Figure 1 above is a hinge, $h$, and will be denoted by $h =< x,z,y,w >$, $< x,z,w,y >$, $< z,x,y,w >$, or $< z,x,w,y >$. A double edge between vertices $x$ and $y$ will be denoted by $< x,y >$. Triples will be denoted by any cyclic shift of $(x,y,z)$, 4-cycles by any cyclic shift of $(x,y,z,w)$, and single edges by $(x,y)$ or $(y,x)$.

Let $G$ be a graph and suppose $H$ is a set of hinges. For a hinge $h =< x,z,y,w >$, let $\Delta(h) = \{(x,z,y),(x,z,w)\}$. Define $\Delta(H) = \bigcup_{h\in H} \Delta(h)$. Similarly, let $\Box(h) = \{(x,y,z,w)\}$, let $D(h) = \{< x,z >\}$, and define $\Box(H) = \bigcup_{h\in H} \Box(h)$, $D(H) = \bigcup_{h\in H} D(h)$. Suppose $(V(G), \Delta(H), L^{\Delta})$ is a maximum packing of $G$ with triples and $(V(G), \Box(H) \cup D^{\star}, L^{\Box})$ is a maximum packing of $G$ with 4-cycles, where either the edges in $D(H)$ or $D(H) \cup L^{\Delta}$ can be partitioned into the set $D^{\star}$, each element of which induces a 4-cycle. Further suppose $L^{\Delta} \subseteq L^{\Box}$ or $L^{\Box} \subseteq L^{\Delta}$. Then we will call $(V(G), H, L^{\Delta}, D^{\star}, L^{\Box})$ a *metamorphosis* of $G$.

In [3], Gionfriddo and Lindner proved the following, which connects the aforementioned systems through the use of hinges.

**Theorem 1.1.** *$2K_n$ has a metamorphosis $(X, H, \emptyset, D^{\star}, \emptyset)$ from a 2-fold triple system to a 2-fold 4-cycle system for all $n \equiv 0, 1, 4$, or $9$ $(mod\ 12)$ except $n = 4$.*

When $n \equiv 2$ $(mod\ 3)$, there exists a maximum packing of $2K_n$ with triples with only a double edge in the leave $L^{\Delta}$ (see [7]). When $n \equiv 0$ or $1$ $(mod\ 4)$, the goal is to form a metamorphosis in which these 2 edges are used in 4-cycles; otherwise, they form $L^{\Box}$. Let $G \setminus H$ denote the graph $(V(G), E(G) \setminus E')$, where $V' \subseteq V(G)$ and $(V', E')$ is isomorphic to $H$. Call $(V', E')$ the *hole* and say the hole is *on* $V'$ and the *size* of the hole is $|V'|$. Let $Q$ be a set of integers and let $H(Q)$ be a partition of $Q$ into pairwise disjoint sets, also called *holes*. A *quasigroup with holes* $H(Q)$ is a quasigroup $(Q, \circ)$ in which for each $h \in H$, $(h, \circ)$ is a subquasigroup of $(Q, \circ)$. For the purposes of this work, each $(h, \circ)$ can be thought of as a set of "forbidden" products in $(Q, \circ)$. The following theorem is used to find our desired result for $n \equiv 5, 8,$ or $11$ $(mod\ 12)$.

**Theorem 1.2.** *Let $n_0, n_1, \ldots, n_{m-1}$ be positive integers, $z = \sum_{0 \le i < m} n_i$, $\infty = \{\infty_1, \infty_2, \ldots, \infty_j\}$, $B_i = \mathbb{Z}_{n_i} \times \mathbb{Z}_3 \times \{i\}$, and $A_i = \infty \cup B_i$ for $0 \le i < m$. Let $C_j = 2K_{n_0, n_1, \ldots, n_{m-1}}$ with vertex set $\bigcup_{0 \le i < m} \mathbb{Z}_{n_i} \times \{j\} \times \{i\}$ for $0 \le j < 3$. Then there exists a metamorphosis $(X, H, L^{\Delta}, D^*, L^{\Box})$ of $2K_{j+3z}$ with vertex set $X = \bigcup_{0 \le i < m} A_i$ if*

*(i) there exists a metamorphosis $(A_0, H_0, L^{\Delta}, D_0^*, L_0)$ of $2K_{j+3n_0}$,*

*(ii) there exists a metamorphosis $(A_i, H_i, \emptyset, D_i^*, L_i)$ of $2K_{j+3n_i} \setminus 2K_j$ (where the hole is on $\infty$) for $0 < i < m$,*

*(iii) there exists a commutative quasigroup $(Q, \circ)$ of order $z$ with holes of sizes $n_0, n_1, \ldots,$ and $n_{m-1}$, and*

*(iv) if $E^* = \left( \bigcup_{0 \le i < 3} E(C_i) \right) \cup \left( \bigcup_{0 \le i < m} L_i \right)$, then there exists a set $C$ of 4-cycles which partitions $E^* \setminus L^{\Box}$, where $L^{\Box} = \emptyset$ or $L^{\Delta} \subseteq L^{\Box}$.*

*Proof.* Using $(Q, \circ)$, create another set of hinges, $H^*$, as follows: let $\mu = (u, i, x)$, $\nu = (v, i, y)$, $\mu\nu = (u \circ v, i+1, t)$, and $\nu\mu = (v \circ u, i+2, t)$. Then $H^* = \{< \mu, \nu, \mu\nu, \nu\mu >: u \in \mathbb{Z}_{n_x}, v \in \mathbb{Z}_{n_y}, x \ne y, i \in \mathbb{Z}_3\}$. Note that $D(H^*) = \bigcup_{0 \le i < 3} E(C_i)$. Let $H = H^* \cup \left( \bigcup_{0 \le i < m} H_i \right)$ and $D^* = C \cup \left( \bigcup_{0 \le i < m} D_i^* \right)$. □

It is nice that in the situation where $L^{\Delta} \ne \emptyset$, either $L^{\Delta}$ is "used up" in the metamorphosis or "preserved" as $L^{\Box}$. For the purposes of this work, $n_0 = n_1 = \cdots = n_{m-1} = n$, and $n = 2$ or $n = 4$. This means that the notation can be simplified a bit, but more importantly, it means that we will always be able to decompose $2K_{n,n,\ldots,n}$ into 4-cycles, due to a theorem of Dominique Sotteau ([9]). It also means that the quasigroups we need will always exist (except possibly for small cases). These theorems will be used in the constructions throughout this work, so they are presented here.

**Theorem 1.3.** *Necessary and sufficient conditions for the complete bipartite graph $K_{m,n}$ to be partitioned into $(2k)$-cycles are:*

*(i) $m$ and $n$ are even*

*(ii) $k \le m$ and $k \le n$, and*

*(iii) $2k | mn$.*

**Theorem 1.4.** *There exists a commutative quasigroup* $(Q, \circ)$ *with holes of size 2 if* $|Q| \equiv 0$ *(mod 2) and* $|Q| > 4$.

**Theorem 1.5.** *There exists a commutative quasigroup* $(Q, \circ)$ *with holes of size 4 if* $|Q| \equiv 0$ *(mod 4) and* $|Q| > 8$.

The quasigroups referred to in Theorem 1.4 can be constructed using pairwise-balanced designs, as in [7]. Theorem 1.5 can be proved in a similar fashion, and [1] ensures that all the necessary designs (with appropriate block sizes) exist.

# 2 Case: $n \equiv 11 \pmod{12}$

If $n = 12k + 11$, we can write $n = 5 + 3(m)(2)$, where $m$ is odd. Let us begin finding our necessary ingredients with $n = 11$ case.

**Lemma 2.1.** *There exists a metamorphosis of* $2K_{11}$.

*Proof.* Let the vertex set $V = \{\infty_1, \infty_2\} \cup (\{0,1,2\} \times \{0,1,2\})$. Let the set of hinges be $H = \{< (i,1),(i,2),\infty_1,\infty_2 >, < (i,1),(i,0),\infty_1,\infty_2 >,$ $< (i,2),(i,0),\infty_1,\infty_2 >, < (1,i),(2,i),(0,i+1),(0,i-1) >,$ $< (1,i),(0,i),(2,i{+}1),(2,i{-}1) >, < (2,i),(0,i),(1,i{+}1),(1,i{-}1) >: i = 0,1,2\}$, where all calculations are reduced modulo 3. Then $(V, \Delta(H), L)$ is maximum packing of $2K_{11}$ with triples with leave $L = \{< \infty_1, \infty_2 >\}$.

The set of 4-cycles $\square(H)$ that we get from these hinges should be obvious, and so they will not be listed in any construction that follows. The double edges stripped from the hinges form 2 copies of the cartesian product of a 3-cycle with itself; the remaining 4-cycles are just $D^\star = \{((i,j),(i,j+1),(i+1,j+1),(i+1,j)) : i,j \in Z_3\}$. Now, we have a maximum packing of $2K_{11}$ with 4-cycles $(V, \square(H) \cup D^\star, L)$ and $(V, H, L, D^\star, L)$ is a metamorphosis of $2K_{11}$. □

**Lemma 2.2.** *There exists a metamorphosis of* $2K_{11} \setminus 2K_5$.

*Proof.* We will be decomposing $2K_{11}$ with vertex set $V = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup \{1,2,3,4,5,6\}$ minus $2K_5$ on the vertex set $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ (call this a *hole* of size 5).

Let $H = \{< 1,3,\infty_1,\infty_2 >, < 2,6,\infty_1,\infty_2 >, < 4,5,\infty_1,\infty_2 >,$ $< 1,6,\infty_1,\infty_5 >, < 2,5,\infty_1,\infty_5 >, < 3,4,\infty_1,\infty_5 >, < 1,5,\infty_2,\infty_3 >,$ $< 2,4,\infty_2,\infty_3 >, < 3,6,\infty_2,\infty_3 >, < 1,2,\infty_3,\infty_4 >, < 3,5,\infty_3,\infty_4 >,$ $< 4,6,\infty_3,\infty_4 >, < 1,4,\infty_4,\infty_5 >, < 2,3,\infty_4,\infty_5 >, < 5,6,\infty_4,\infty_5 >\}$.

Now, let $D^\star = \{(2,3,6,5), (2,3,5,6), (2,5,3,6), (1,2,4,5), (1,2,4,5),$ $(1,3,4,6), (1,3,4,6)\}$ and $L = \{< 1,4 >\}$. Thus $(V, H, \emptyset, D^\star, L)$ is a metamorphosis of $2K_{11} \setminus 2K_5$. □

With the above example in hand, we can proceed to the $12n + 11 \geq 23$ Construction.

**Theorem 2.3.** *There exists a metamorphosis of* $2K_{12n+11}$ *for all* $n \geq 0$.

*Proof.* Write $12n + 11 = 3(4n + 2) + 5$. Since the $n = 0$ case is handled in Lemma 2.1, $12n + 11 \geq 23$, and thus $4n + 2 \geq 6$ (in light of Theorem 1.4, this is important). Let $\infty = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ and $Q = \{1, 2, 3, \ldots, 4n + 2\}$. Let $H(Q) = \{h_0, h_1, \ldots, h_{2n}\}$, where $h_i = \{2i + 1, 2i + 2\}$. Let $(Q, \circ)$ be a commutative quasigroup of order $4n+2$ with holes $H(Q)$ and set $V = \infty \cup (Q \times \{1, 2, 3\})$.

Proceed as in Theorem 1.2 by letting $B_i = h_i \times \{1, 2, 3\}$ for $0 \leq i \leq 2n$. Let $L^{\Delta} = L_0 = \{< \infty_1, \infty_2 >\}$ and let $L_i = \{< (2i + 1, 1), (2i + 2, 1) >\}$ for $0 < i \leq 2n$. Thus $(A_0, H_0, L_0, D_0^*, L_0)$ is a metamorphosis of $2K_{11}$ and $(A_i, H_i, \emptyset, D_i^*, L_i)$ is a metamorphosis of $2K_{11} \setminus 2K_5$ for $0 < i \leq 2n$. Cover the leaves and the double edges remaining from our hinges as follows:

1. We have $2K_{2,2,\ldots,2}$ on $Q \times \{2\}$ and $Q \times \{3\}$. We can apply the result of Sotteau repeatedly to partition the remaining edges into 4-cycles and put them in $C$.

2. We have $2K_{4n+2}$ with $< (1, 1), (2, 1) >$ removed on $Q \times \{1\}$. For $0 < i \leq n$, on $\{(4i-1, 1), (4i, 1), (4i+1, 1), (4i+2, 1)\}$, we have $\{((4i-1, 1), (4i, 1), (4i+1, 1), (4i+2, 1)), ((4i-1, 1), (4i+1, 1), (4i+2, 1), (4i, 1)), ((4i-1, 1), (4i+1, 1), (4i, 1), (4i+2, 1))\} \subseteq C$. We now have 2 copies of the complete $(n+1)$-partite graph with one partite set having size 2 and the rest having size 4 remaining, so we can, again, apply the result of Sotteau and place the 4-cycles in $C$.

$(V, H, L_0, D^*, L_0)$ is a metamorphosis of $2K_{12n+11}$, as desired. □

# 3 Case: $n \equiv 5 \pmod{12}$

We begin this section by showing the nonexistence for $n = 5$.

**Lemma 3.1.** *There does not exist a metamorphosis of* $2K_5$.

*Proof.* If there were to be a suitable packing with triples (and thus a packing with hinges), then we would have to be able to use the double edges from our hinges (along with the leave) to create two 4-cycles. Thus we would have a repeated 4-cycle. Let $x$ be the vertex that does not appear in this 4-cycle. Our 3 hinges can cover at most 6 edges incident with $x$, but $d_{2K_5}(x) = 8$. □

Before we can proceed to the $12n + 5$ Construction, we must first produce a metamorphosis of $2K_{17}$; this is because no quasigroup of order 4 with holes of size 2 exists, which would be required for the application of Theorem 1.2.

**Lemma 3.2.** *There exists a metamorphosis of* $2K_{17}$.

*Proof.* Let $V = (\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}) \cup \{\infty\}$. Our leave will be $L = < (4, 4), \infty >$.

Let $H = \{< (1, 1), (1, 2), (1, 4), \infty >, < (2, 1), (2, 2), (2, 4), \infty >,$
$< (3, 1), (3, 2), (3, 4), \infty >, < (4, 1), (4, 2), (4, 4), \infty >,$

$<(1,3),\infty,(1,1),(1,2)>,<(2,3),\infty,(2,1),(2,2)>,$
$<(3,3),\infty,(3,1),(3,2)>,<(4,3),\infty,(4,1),(4,2)>,$
$<(1,4),(2,4),(4,4),\infty>,<(3,4),\infty,(1,4),(2,4)>,$
$<(1,3),(1,4),(1,1),(1,2)>,<(2,3),(2,4),(2,1),(2,2)>,$
$<(3,3),(3,4),(3,1),(3,2)>,<(4,3),(4,4),(4,1),(4,2)>,$
$<(3,4),(4,4),(1,4),(2,4)>,<(3,3),(4,3),(1,3),(2,3)>,$
$<(3,2),(4,2),(1,2),(2,2)>,<(3,1),(4,1),(1,1),(2,1)>,$
$<(1,4),(2,3),(3,2),(4,1)>,<(1,3),(2,4),(3,1),(4,2)>,$
$<(1,1),(2,2),(3,3),(4,4)>,<(1,2),(2,1),(3,4),(4,3)>,$
$<(1,3),(2,3),(3,3),(4,3)>,<(1,2),(2,2),(3,2),(4,2)>,$
$<(1,1),(2,1),(3,1),(4,1)>,<(3,2),(4,1),(1,4),(2,3)>,$
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$<(1,3),(2,2),(3,4),(4,1)>,<(3,3),(4,2),(1,4),(2,1)>,$
$<(3,1),(4,4),(1,2),(2,3)>,<(3,2),(4,3),(1,1),(2,4)>,$
$<(3,4),(4,1),(1,3),(2,2)>,<(1,4),(2,2),(3,1),(4,3)>,$
$<(1,1),(2,3),(3,4),(4,2)>,<(1,3),(2,1),(3,2),(4,4)>,$
$<(1,2),(2,4),(3,3),(4,1)>,<(3,1),(4,3),(1,4),(2,2)>,$
$<(3,4),(4,2),(1,1),(2,3)>,<(3,2),(4,4),(1,3),(2,1)>,$
$<(3,3),(4,1),(1,2),(2,4)>\}$.

Let $D^* = \{(\infty,(4,3),(3,4),(4,4)),((3,3),\infty,(4,4),(4,3)),$
$((3,4),(3,3),(4,3),\infty),((4,4),(3,4),\infty,(3,3)),((4,3),(4,4),(3,3),(3,4)),$
$(\infty,(1,3),(2,4),(2,3)),(\infty,(1,3),(1,4),(2,3)),((1,3),(1,4),(2,4),(2,3)),$
$((1,3),(2,3),(1,4),(2,4)),((1,1),(2,3),(1,2),(2,4)),((1,1),(2,3),(1,2),(2,4)),$
$((2,1),(1,3),(2,2),(1,4)),((2,1),(1,3),(2,2),(1,4)),((3,1),(4,3),(3,2),(4,4)),$
$((3,1),(4,3),(3,2),(4,4)),((4,1),(3,3),(4,2),(3,4)),((4,1),(3,3),(4,2),(3,4)),$
$((1,1),(2,1),(2,2),(1,2)),((1,1),(2,1),(1,2),(2,2)),((1,1),(1,2),(2,1),(2,2)),$
$((3,1),(4,1),(4,2),(3,2)),((3,1),(4,1),(3,2),(4,2)),$
$((3,1),(3,2),(4,1),(4,2))\}$.

$(V,H,L,D^*,\emptyset)$ is a metamorphosis of $2K_{17}$, as desired. □

We can now proceed to the $12n+5 \geq 29$ Construction, which is simply a modification of the $12n+11 \geq 23$ Construction.

**Theorem 3.3.** *There exists a metamorphosis of* $2K_{12n+5}$ *if and only if* $12n+5 \geq 17$.

*Proof.* Write $12n+5 = 3(4n)+5$. Since Lemma 3.1 shows that $n \neq 0$ and Lemma 3.2 handles the case where $n=1$, assume $12n+5 \geq 29$, and thus $4n \geq 8$. Let $\infty = \{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$ and $Q = \{1,2,3,\ldots,4n\}$. Let $H(Q) = \{h_0,h_1,\ldots,h_{2n-1}\}$, where $h_i = \{2i+1,2i+2\}$. Let $(Q,\circ)$ be a commutative quasigroup of order $4n$ with holes $H(Q)$ and set $V = \infty \cup (Q \times \{1,2,3\})$.

Proceed as in Theorem 1.2 by letting $B_i = h_i \times \{1,2,3\}$ for $0 \leq i \leq 2n-1$. Let $L_i = \{<(2i+1,1),(2i+2,1)>\}$ for $0 \leq i < 2n$ and let $L^\Delta = L_0$. Thus $(A_0,H_0,L^\Delta,D_0^*,L^\Delta)$ is a metamorphosis of $2K_{11}$ and $(A_i,H_i,\emptyset,D_i^*,L_i)$ is a metamorphosis of $2K_{11} \setminus 2K_5$ for $0 < i < 2n$. Cover the leaves and the double edges remaining from our hinges as follows:

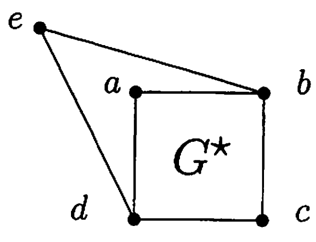
1. We have $2K_{2,2,\ldots,2}$ on $Q \times \{2\}$ and $Q \times \{3\}$. We can again apply the result of Sotteau to partition the remaining edges into 4-cycles and put them in $C$.

2. We have $2K_{4n}$ on $Q \times \{1\}$. For $0 \le i < n$, on $\{(4i+1,1),(4i+2,1),(4i+3,1),(4i+4,1)\}$, we have $\{((4i+1,1),(4i+2,1),(4i+3,1),(4i+4,1)),((4i+1,1),(4i+3,1),(4i+4,1),(4i+2,1)),((4i+1,1),(4i+3,1),(4i+2,1),(4i+4,1))\} \subseteq C$. What remains is $2K_{4,4,\ldots,4}$ on $Q\times\{1\}$, so we can, again, apply the result of Sotteau and place the 4-cycles in $C$.

$(V, H, L^{\Delta}, D^{\star}, \emptyset)$ is a metamorphosis of $2K_{12n+5}$, as desired. □

# 4 Case: $n \equiv 8 \pmod{12}$

We begin this section by showing that there does not exist a metamorphosis of $2K_8$ in Corollary 4.9, after a series of lemmas. If there were to be a suitable packing with triples (and thus a packing with hinges), then we would have to be able to use the double edges from our hinges (along with the leave) to create five 4-cycles. Let $2G$ represent the multigraph induced by $D$ and the double edge in the leave. Then clearly $2G$ has 20 edges. Let $G$ be the simple graph on 8 vertices with 10 edges formed by treating each double edge in $2G$ as a single edge. If there exists a 4-cycle system on $2G$, each edge in $G$ must be in a 4-cycle, and no vertex in $G$ can have degree 1. If $2K_8$ has a maximum packing with hinges, each vertex of $2G$ is incident with at most 3 double edges from $D$, so each vertex $v$ in $G$ has degree at most 4, with equality only if $v$ is incident with the edge corresponding to the leave. We begin with a handy Lemma concerning the graph below, $G^{\star}$.



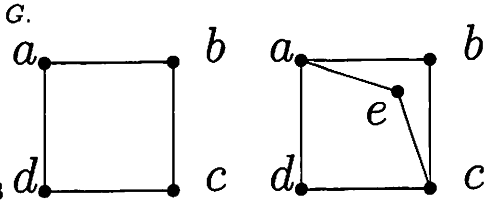

**Lemma 4.1.** *$K_4$ and $G^{\star}$ are the only simple graphs with exactly 6 edges that have a double cover by 4-cycles.*

*Proof.* A simple graph with 6 edges must have at least 4 vertices, and any 4-cycle would require 4 vertices. A connected simple graph on 6 vertices with 6 edges cannot contain any 4-cycles. It is easily verified that $G^{\star}$ is the only simple graph on 5 vertices with exactly 6 edges that contains more than one 4-cycle.

Now, $\{(a,b,c,d),(a,b,d,c),(a,d,b,c)\}$ is a double cover of $K_4$ by 4-cycles and $\{(a,b,c,d),(a,e,c,b),(a,d,c,e)\}$ is a double cover of $G^{\star}$ by 4-cycles. □

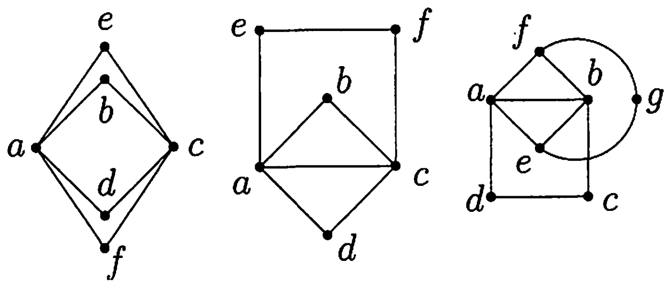
**Lemma 4.2.** *There can be no repeated 4-cycles in our 4-cycle system on 2G.*

*Proof.* Suppose $C = (a, b, c, d)$ is a repeated 4-cycle. Then none of $\{(a, b), (b, c), (c, d), (a, d)\}$ can be used in another 4-cycle. Thus the remaining three 4-cycles, $C^*$, must be a double cover of the remaining six edges in $G$. Let $k$ be the maximum number of vertices from $\{a, b, c, d\}$ incident with a particular cycle in $C^*$, and let $C'$ be a cycle where that occurs. The cases are approached by examining subgraphs of $G$.




Case: $k = 4$ or $k = 3$

Obviously, $C = C'$ for $k = 4$. Without loss of generality, suppose $C' = (a, b, c, e)$ for $k = 3$. Amongst many reasons, $(a, b)$ is used again in either situation.

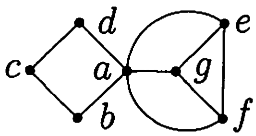



Case: $k = 2$

The first situation is easily dismissed. Suppose without loss of generality that $a$ and $c$ are the vertices in some cycle $C' \in C^*$ which are not adjacent in $C$ and not adjacent in $C'$. Since $d_G(a) = d_G(c) = 4$, $a$ and $c$ must be incident to the edge corresponding to the leave of our packing. This means $a$ and $c$ should be adjacent, but they are not.

The next situation is also easily dismissed. Suppose without loss of generality that $a$ and $c$ are the vertices in some cycle $C' \in C^*$ which are not adjacent in $C$ but *are* adjacent in $C'$. Without loss of generality, $C' = (a, e, f, c)$. Now, 2 more edges must be added to $C'$ to form $K_4$ or $G^*$ (by Lemma 4.1), but then $d_G(a) > 4$ or $d_G(c) > 4$.

Finally, suppose without loss of generality that $a$ and $b$ are the vertices in some cycle $C' \in C^*$ which are adjacent in $C$. Note that they cannot be adjacent in $C'$, since $(a, b)$ cannot be used again. Without loss of generality, $C' = (a, e, b, f)$, and the other cycles in $C^*$ are $(a, e, g, f)$ and $(b, e, g, f)$. Now, since $d_G(a) = d_G(b) = 4$, $< a, b >$ is the leave of our packing. Consider the hinges containing $a$. First, $a$ must be incident with the double edge in each of them. The hinge containing $< a, e >$ must be $< a, e, c, h >$, since $(e, g)$ cannot be used again. Similarly, the hinge containing $< a, f >$ must be $< a, f, c, h >$. Now, we still need to cover $(a, g)$ twice, but we cannot do that with the remaining hinge containing $a$.




Case: $k = 1$

Without loss of generality, this can't happen because $d_G(a) > 4$.

Case: $k = 0$

The remaining edges in $G$ form $K_4$ on $\{e, f, g, h\}$. Now, 9 of the 10 edges in $G$ are associated with hinges in our packing, so we need 36 edges from $2K_{4,4}$ to finish those hinges; unfortunately, $2|E(K_{4,4})| = 32$. □

**Corollary 4.3.** *Vertices of degree 2 in G cannot be adjacent.*

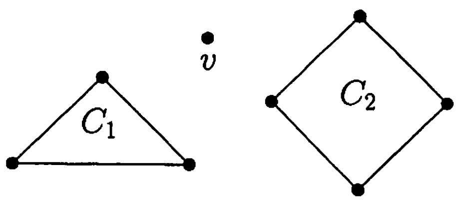
*Proof.* If there were a pair of adjacent vertices of degree 2 in $G$, they would have to be in a repeated 4-cycle in our double cover, a contradiction to Lemma 4.2. □

**Lemma 4.4.** *G contains no 7-cycle*

*Proof.* Suppose $G$ contains a 7-cycle, $C$. Let $e, f$, and $g$ be the remaining 3 edges. Now, $C$ contains no 4-cycles, so each 4-cycle must contain at least one of these edges. Without loss of generality, assume that $e$ and $f$ are both covered by some 4-cycle, since $e, f$, and $g$ must each be covered by two 4-cycles and this means that there must be some 4-cycle that covers a pair of these edges. Now, there still must be two 4-cycles that cover $g$, but that requires a repeated 4-cycle, a contradiction to Lemma 4.2. □

**Corollary 4.5.** *G has no vertex of degree 0.*

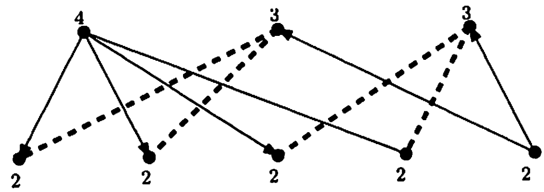
*Proof.* Suppose $d_G(v) = 0$. Note that this means that there is no other vertex of degree 0. If there were, neither our hinges from our packing nor the leave would cover the edges between $v$ and this vertex. This also means that $v$ appears in exactly 7 of the hinges in our packing to cover all edges incident with $v$ in $2K_8$. Hence, by Lemma 4.4, we must have the following graph as a subgraph of $G$:



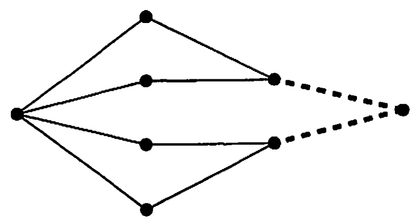

Now, all the edges in $C_1$ must be double covered by 4-cycles, and a 4-cycle can cover at most a pair of its edges. Thus the remaining 3 edges in $G$ must go from $C_1$ to $C_2$ to cover the edges in $C_1$. However, this means we must repeat $C_2$ to double cover its edges, a contradiction to Lemma 4.2. □

**Lemma 4.6.** *G cannot have* $(4, 3, 3, 2, 2, 2, 2, 2)$ *as its degree sequence.*

*Proof.* Suppose $G$ has $(4,3,3,2,2,2,2,2)$ as its degree sequence. Without loss of generality, since no vertices of degree 2 can be adjacent (Corollary 4.3), $G$ must look like this:




Now, redraw $G$ to look like this:



Note that no 4-cycle can contain the dotted edges. □

**Lemma 4.7.** *$G$ cannot have $(3,3,3,3,2,2,2,2)$ as its degree sequence.*

*Proof.* Suppose $G$ has $(3,3,3,3,2,2,2,2)$ as its degree sequence. Call an edge *pure* if it is incident with two vertices of degree 3 and call it *mixed* otherwise. Note that there are exactly 2 pure edges. By Corollary 4.3, each 4-cycle in our double cover must use an even number of mixed edges, and thus it must also use an even number of pure edges. Hence, the pure edges must share a vertex, $v$, and both of these edges must be a part of two (distinct) 4-cycles. Now, note that the remaining edge at $v$ must be mixed, but there is no way to cover it with a 4-cycle. □

**Theorem 4.8.** *$G$ does not have a double cover by 4-cycles.*

*Proof.* $G$ must have one of the following degree sequences, none of which has our desired double cover by 4-cycles:

1. $(4,4,3,3,3,3,0,0)$ (corollary 4.5)
2. $(4,4,3,3,2,2,2,0)$ (corollary 4.5)
3. $(4,4,2,2,2,2,2,2)$ (corollary 4.3)
4. $(4,3,3,3,3,2,2,0)$ (corollary 4.5)
5. $(4,3,3,2,2,2,2,2)$ (lemma 4.6)
6. $(3,3,3,3,3,3,2,0)$ (corollary 4.5)

7. $(3,3,3,3,2,2,2,2)$ (lemma 4.7)

□

This leads us to the following conclusion:

**Corollary 4.9.** *There does not exist a metamorphosis of* $2K_8$. □

Before we can begin the $12n+8$ Construction, we must consider the cases where $n \in \{1,2\}$. We begin with some intermediate results. The first comes from [8].

**Lemma 4.10.** *There exists a metamorphosis of* $2K_{10}$.

*Proof.* Let the vertex set $V = \mathbb{Z}_{10}$. Let $H = \{< 0,2,1,3 >, < 0,4,5,6 >, < 0,8,7,9 >, < 2,4,1,7 >, < 2,8,3,6 >, < 4,8,6,7 >, < 1,5,7,8 >, < 3,5,7,8 >, < 3,9,4,6 >, < 1,9,6,8 >, < 5,6,0,2 >, < 6,7,1,3 >, < 7,9,0,2 >, < 5,9,2,4 >, < 1,3,0,4 >\}$. Let $D^* = \{(0,2,8,4), (0,4,2,8), (0,8,4,2), (1,5,3,9), (1,5,3,9), (5,6,7,9), (5,6,7,9)\}$ and $L = \{< 1,3 >\}$. Then $(V, H, \emptyset, D^*, L)$ is a metamorphosis of $2K_{10}$, as desired. □

**Lemma 4.11.** *Let $G'$ be a doubled 1-factor on 8 vertices. Then there exists a metamorphosis of* $2K_8 \setminus G'$.

*Proof.* Let $\infty = \{\infty_1, \infty_2\}$. Let the vertex set $V = \infty \cup (\mathbb{Z}_2 \times \mathbb{Z}_3)$ and let $E(G') = \{< \infty_1, \infty_2 >\} \cup \{< (0,i), (1,i) >: i \in \mathbb{Z}_3\}$.

Let $H = \{< \infty_1, (0,0), (0,1), (1,1) >, < \infty_1, (1,0), (0,2), (1,2) >, < \infty_2, (0,0), (0,2), (1,2) >, < \infty_2, (1,0), (0,1), (1,1) >, < (0,1), (0,2), \infty_1, \infty_2 >, < (1,1), (1,2), \infty_1, \infty_2 >, < (0,1), (1,2), (0,0), (1,0) >, < (0,2), (1,1), (0,0), (1,0) >\}$.

Let $C = \{(\infty_1, (0,0), \infty_2, (1,0)), ((0,1), (0,2), (1,1), (1,2))\}$ and $D^* = 2C$. Then $(V, H, \emptyset, D^*, \emptyset)$ is a metamorphosis of $2K_8 \setminus G'$, as desired. □

**Lemma 4.12.** *There exists a metamorphosis of* $2K_{32}$.

*Proof.* Note that $32 = 3((5)(2)) + 2$. Let $\infty = \{\infty_1, \infty_2\}$ and $Q = \{1,2,3,\ldots,10\}$. Let $H(Q) = \{h_0, h_1, \ldots, h_4\}$ be a partition of $Q$ into pairwise disjoint sets of size 2, where $h_i = \{2i+1, 2i+2\}$. Let $(Q, \circ)$ be the antisymmetric quasigroup of order 10 with holes $H(Q)$ found below and set $V = \infty \cup (Q \times \{1,2,3\})$. For $0 \leq i \leq 4$, let $B_i = h_i \times \{1,2,3\}$ and $A_i = \infty \cup B_i$. For $0 < i \leq 3$, let $V_i = Q \times \{i\}$. Let $L = \{< \infty_1, \infty_2 >\}$.

(i) For $0 < i \leq 3$, let $(V_i, J_i, \emptyset, E_i^*, M_i)$ be a metamorphosis of $2K_{10}$, where $M_i = \{< (1,i), (2,i) >\}$ (see Lemma 4.10).

(ii) For $0 \leq i \leq 4$, let $(A_i, H_i, \emptyset, D_i^*, \emptyset)$ be a metamorphosis of $2K_8 \setminus G_i'$, where $E(G_i') = \{< \infty_1, \infty_2 >\} \cup \left( \bigcup_{0<k\leq 3} h_i \times \{k\} \right)$ and if $h_i = \{x,y\}$, then $D(H_i) = \{< \infty_1, (x,1) >, < \infty_1, (y,1) >, < \infty_2, (x,1) >, < \infty_2, (y,1) >, < (x,2), (x,3) >, < (x,2), (y,3) >, < (y,2), (x,3) >, < (y,2), (y,3) >\}$ (see Lemma 4.11).

(iii) Now, $L \cup \left( \bigcup_{0<i\leq 3} M_i \right) \cup \left( \bigcup_{0\leq i\leq 4} D(H_i) \right)$ can be decomposed into 4-cycles $(X \cup Y \cup 2Z) \subseteq D^*$, where $X$, $Y$, and $Z$ are defined as follows.

Let $X = \{(\infty_1, \infty_2, (1,1), (2,1)), (\infty_1, \infty_2, (2,1), (1,1)), (\infty_1, (1,1), \infty_2, (2,1))\}$, let $Y = \{((1,2),(2,2),(1,3),(2,3)), ((1,2),(2,2),(2,3),(1,3)), ((1,2),(1,3),(2,2),(2,3))\}$, and let $Z = \{(\infty_1, (2i+2,1), \infty_2, (2i+1,1)), ((2i+1,2),(2i+1,3),(2i+2,2),(2i+2,3)) : 0 < i \leq 4\}$.

(iv) We now need to use the edges between vertices in $B_i$ and $B_j$ for $i \neq j$. Let $H^* = \{< (x,1),(y,2),(x \circ y,3),(y \circ x,3) >, < (x,2),(y,1),(x \circ y,3),(y \circ x,3) >: x \in h_i, y \in h_j, i \neq j\}$, where $(Q, \circ)$ is found below.

| $\circ$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | 5 | 6 | 7 | 8 | 9 | 10 | 3 | 4 |
| 2 | | | 6 | 5 | 8 | 7 | 10 | 9 | 4 | 3 |
| 3 | 6 | 5 | | | 9 | 10 | 1 | 2 | 7 | 8 |
| 4 | 5 | 6 | | | 10 | 9 | 2 | 1 | 8 | 7 |
| 5 | 8 | 7 | 10 | 9 | | | 3 | 4 | 1 | 2 |
| 6 | 7 | 8 | 9 | 10 | | | 4 | 3 | 2 | 1 |
| 7 | 10 | 9 | 2 | 1 | 4 | 3 | | | 5 | 6 |
| 8 | 9 | 10 | 1 | 2 | 3 | 4 | | | 6 | 5 |
| 9 | 4 | 3 | 8 | 7 | 2 | 1 | 6 | 5 | | |
| 10 | 3 | 4 | 7 | 8 | 1 | 2 | 5 | 6 | | |

(v) After removing the double edges from our hinges in $H^*$, we have the following edges remaining to use in our metamorphosis: edges of the type $< (x,1),(y,2) >$ and edges of the type $< (x,2),(y,1) >$, where $x \in h_i$, $y \in h_j$, and $i \neq j$. Let $C = \{((x_1,1),(y_1,2),(x_2,1),(y_2,2)), ((x_1,2),(y_1,1),(x_2,2),(y_2,1)) : h_i = \{x_1,x_2\}, h_j = \{y_1,y_2\}, i \neq j\}$. Now, put $2C$ in $D^*$.

Let $H = H^* \cup \left( \bigcup_{0<i\leq 3} J_i \right) \cup \left( \bigcup_{0\leq i\leq 4} H_i \right)$. Then $(V, H, L, D^*, \emptyset)$ is a metamorphosis of $2K_{32}$, as desired. □

Now, we must consider the following examples that we will use in the $12n + 8 \geq 44$ Construction.

**Lemma 4.13.** *There exists a metamorphosis of* $2K_{20}$.

*Proof.* We will decompose $2K_{20}$ with vertex set $V = \{1,2,3,4\} \times \{1,2,3,4,5\}$. Let $L = \{< (4,1),(4,2) >\}$.

Let $H = \{< (1,1),(1,2),(2,1),(2,2) >, < (2,1),(2,2),(3,1),(3,2) >, < (3,1),(3,2),(1,1),(1,2) >, < (1,1),(4,1),(2,1),(3,1) >,$

$< (1,1),(4,2),(2,2),(3,2) >, < (1,2),(4,1),(2,2),(3,1) >,$
$< (1,2),(4,2),(2,1),(3,2) >, < (3,1),(4,2),(2,1),(2,2) >,$
$< (3,2),(4,1),(2,1),(2,2) >, < (1,3),(1,4),(1,1),(1,2) >$
$< (1,3),(1,5),(1,1),(1,2) >, < (1,4),(1,5),(1,1),(1,2) >,$
$< (2,3),(2,4),(2,1),(2,2) >, < (2,3),(2,5),(2,1),(2,2) >,$
$< (2,4),(2,5),(2,1),(2,2) >, < (3,3),(3,4),(3,1),(3,2) >,$
$< (3,3),(3,5),(3,1),(3,2) >, < (3,4),(3,5),(3,1),(3,2) >,$
$< (4,3),(4,4),(4,1),(4,2) >, < (4,3),(4,5),(4,1),(4,2) >,$
$< (4,4),(4,5),(4,1),(4,2) >, < (1,1),(3,3),(2,4),(4,4) >,$
$< (2,1),(4,3),(1,4),(3,4) >, < (1,1),(3,4),(2,5),(4,5) >,$
$< (2,1),(4,4),(1,5),(3,5) >, < (1,1),(3,5),(2,3),(4,3) >,$
$< (2,1),(4,5),(1,3),(3,3) >, < (1,2),(3,3),(2,5),(4,5) >,$
$< (2,2),(4,3),(1,5),(3,5) >, < (1,2),(3,4),(2,3),(4,3) >,$
$< (2,2),(4,4),(1,3),(3,3) >, < (1,2),(3,5),(2,4),(4,4) >,$
$< (2,2),(4,5),(1,4),(3,4) >, < (1,3),(3,1),(2,4),(4,4) >,$
$< (2,3),(4,1),(1,4),(3,4) >, < (1,3),(3,2),(2,5),(4,5) >,$
$< (2,3),(4,2),(1,5),(3,5) >, < (1,3),(3,3),(2,3),(4,3) >,$
$< (2,3),(4,3),(1,3),(3,3) >, < (1,3),(3,4),(2,1),(4,1) >,$
$< (2,3),(4,4),(1,1),(3,1) >, < (1,3),(3,5),(2,2),(4,2) >,$
$< (2,3),(4,5),(1,2),(3,2) >, < (1,4),(3,1),(2,5),(4,5) >,$
$< (2,3),(4,5),(1,2),(3,2) >, < (1,4),(3,1),(2,5),(4,5) >,$
$< (2,4),(4,1),(1,5),(3,5) >, < (1,4),(3,2),(2,3),(4,3) >,$
$< (2,4),(4,2),(1,3),(3,3) >, < (1,4),(3,3),(2,2),(4,2) >,$
$< (2,4),(4,3),(1,2),(3,2) >, < (1,4),(3,4),(2,4),(4,4) >,$
$< (2,4),(4,4),(1,4),(3,4) >, < (1,4),(3,5),(2,1),(4,1) >$
$< (2,4),(4,5),(1,1),(3,1) >, < (1,5),(3,1),(2,3),(4,3) >,$
$< (2,5),(4,1),(1,3),(3,3) >, < (1,5),(3,2),(2,4),(4,4) >,$
$< (2,5),(4,2),(1,4),(3,4) >, < (1,5),(3,3),(2,1),(4,1) >,$
$< (2,5),(4,3),(1,1),(3,1) >, < (1,5),(3,4),(2,2),(4,2) >,$
$< (2,5),(4,4),(1,2),(3,2) >, < (1,5),(3,5),(2,5),(4,5) >,$
$< (2,5),(4,5),(1,5),(3,5) >\}.$

Let $D^* = \{((3,1),(3,2),(4,1),(4,2)), ((3,1),(3,2),(4,1),(4,2)),$
$((1,1),(4,1),(1,2),(4,2)), ((1,1),(4,1),(1,2),(4,2)),$
$((1,1),(1,2),(3,4),(3,3)), ((2,1),(2,2),(4,4),(4,3)),$
$((1,1),(1,2),(3,5),(3,4)), ((2,1),(2,2),(4,5),(4,4)),$
$((1,2),(3,3),(1,1),(3,5)), ((2,2),(4,3),(2,1),(4,5)),$
$((1,2),(3,3),(3,5),(3,4)), ((2,2),(4,3),(4,5),(4,4)),$
$((3,3),(3,4),(1,1),(3,5)), ((4,3),(4,4),(2,1),(4,5)),$
$((1,3),(3,2),(1,4),(3,1)), ((2,3),(4,2),(2,4),(4,1)),$
$((1,3),(3,2),(1,5),(3,1)), ((2,3),(4,2),(2,5),(4,1)),$
$((1,4),(3,2),(1,5),(3,1)), ((2,4),(4,2),(2,5),(4,1)),$
$((1,3),(3,4),(1,4),(3,5)), ((2,3),(4,4),(2,4),(4,5)),$
$((1,4),(3,3),(1,5),(3,4)), ((2,4),(4,3),(2,5),(4,4)),$
$((1,3),(3,3),(1,5),(3,5)), ((2,3),(4,3),(2,5),(4,5)),$
$((1,3),(1,4),(1,5),(3,4)), ((2,3),(2,4),(2,5),(4,4)),$
$((1,3),(1,5),(1,4),(3,3)), ((2,3),(2,5),(2,4),(4,3)),$

$((1,3),(1,5),(3,5),(1,4)),\ ((2,3),(2,5),(4,5),(2,4))\}$.

Then $(V, H, L, D^*, \emptyset)$ is a metamorphosis of $2K_{20}$, as desired. □

**Lemma 4.14.** *There exists a metamorphosis of* $2K_{20} \setminus 2K_8$.

*Proof.* We will be decomposing $2K_{20}$ with vertex set $V = \{1,2,3,4,5\} \times \{1,2,3,4\}$ minus $2K_8$ on the vertex set $\{1,2\} \times \{1,2,3,4\}$.

Let $H = \{<(3,1),(3,2),(3,3),(3,4)>, <(3,3),(3,4),(3,1),(3,2)>,$
$<(4,1),(4,2),(4,3),(4,4)>, <(4,3),(4,4),(4,1),(4,2)>,$
$<(5,1),(5,2),(5,3),(5,4)>, <(5,3),(5,4),(5,1),(5,2)>,$
$<(3,1),(4,1),(1,1),(2,1)>, <(3,1),(5,1),(1,1),(2,1)>,$
$<(4,1),(5,1),(1,1),(2,1)>, <(3,1),(4,2),(1,3),(2,3)>,$
$<(3,1),(5,2),(1,3),(2,3)>, <(4,1),(5,2),(1,3),(2,3)>,$
$<(3,1),(4,3),(1,4),(2,4)>, <(3,1),(5,3),(1,4),(2,4)>,$
$<(4,1),(5,3),(1,4),(2,4)>, <(3,1),(4,4),(1,2),(2,2)>,$
$<(3,1),(5,4),(1,2),(2,2)>, <(4,1),(5,4),(1,2),(2,2)>,$
$<(3,2),(4,1),(1,4),(2,4)>, <(3,2),(5,1),(1,4),(2,4)>,$
$<(4,2),(5,1),(1,4),(2,4)>, <(3,2),(4,2),(1,2),(2,2)>,$
$<(3,2),(5,2),(1,2),(2,2)>, <(4,2),(5,2),(1,2),(2,2)>,$
$<(3,2),(4,3),(1,1),(2,1)>, <(3,2),(5,3),(1,1),(2,1)>,$
$<(4,2),(5,3),(1,1),(2,1)>, <(3,2),(4,4),(1,3),(2,3)>,$
$<(3,2),(5,4),(1,3),(2,3)>, <(4,2),(5,4),(1,3),(2,3)>,$
$<(3,3),(4,1),(1,2),(2,2)>, <(3,3),(5,1),(1,2),(2,2)>,$
$<(4,3),(5,1),(1,2),(2,2)>, <(3,3),(4,2),(1,4),(2,4)>,$
$<(3,3),(5,2),(1,4),(2,4)>, <(4,3),(5,2),(1,4),(2,4)>,$
$<(3,3),(4,3),(1,3),(2,3)>, <(3,3),(5,3),(1,3),(2,3)>,$
$<(4,3),(5,3),(1,3),(2,3)>, <(3,3),(4,4),(1,1),(2,1)>,$
$<(3,3),(5,4),(1,1),(2,1)>, <(4,3),(5,4),(1,1),(2,1)>,$
$<(3,4),(4,1),(1,3),(2,3)>, <(3,4),(5,1),(1,3),(2,3)>,$
$<(4,4),(5,1),(1,3),(2,3)>, <(3,4),(4,2),(1,1),(2,1)>,$
$<(3,4),(5,2),(1,1),(2,1)>, <(4,4),(5,2),(1,1),(2,1)>,$
$<(3,4),(4,3),(1,2),(2,2)>, <(3,4),(5,3),(1,2),(2,2)>,$
$<(4,4),(5,3),(1,2),(2,2)>, <(3,4),(4,4),(1,4),(2,4)>,$
$<(3,4),(5,4),(1,4),(2,4)>, <(4,4),(5,4),(1,4),(2,4)>\}$.

Let $D^* = \{((3,1),(3,2),(4,1),(4,2)), ((3,1),(3,2),(4,2),(4,1)),$
$((3,1),(4,1),(3,2),(4,2)), ((4,3),(4,4),(5,3),(5,4)),$
$((4,3),(4,4),(5,4),(5,3)), ((4,3),(5,3),(4,4),(5,4)),$
$((3,3),(3,4),(5,1),(5,2)), ((3,3),(3,4),(5,2),(5,1)),$
$((3,3),(5,1),(3,4),(5,2)), ((4,1),(5,1),(4,2),(5,2)),$
$((4,1),(5,1),(4,2),(5,2)), ((3,3),(4,3),(3,4),(4,4)),$
$((3,3),(4,3),(3,4),(4,4)), ((3,1),(5,1),(3,2),(5,2)),$
$((3,1),(5,1),(3,2),(5,2)), ((3,3),(5,3),(3,4),(5,4)),$
$((3,3),(5,3),(3,4),(5,4)), ((3,1),(4,3),(3,2),(4,4)),$
$((3,1),(4,3),(3,2),(4,4)), ((4,1),(5,3),(4,2),(5,4)),$
$((4,1),(5,3),(4,2),(5,4)), ((3,3),(4,1),(3,4),(4,2)),$
$((3,3),(4,1),(3,4),(4,2)), ((4,3),(5,1),(4,4),(5,2)),$

$((4,3),(5,1),(4,4),(5,2))$, $((3,1),(5,3),(3,2),(5,4))$,
$((3,1),(5,3),(3,2),(5,4))\}$.

Then $(V, H, \emptyset, D^*, \emptyset)$ is a metamorphosis of $2K_{20} \setminus 2K_8$, as desired. □

With the above examples in hand, we can proceed to the $12n + 8 \geq 44$ Construction.

**Theorem 4.15.** *There exists a metamorphosis of* $2K_{12n+8}$ *if and only if* $12n + 8 \geq 20$.

*Proof.* Write $12n + 8 = 3(4n) + 8$. The cases where $n = 1$ and $n = 2$ are settled in Lemmas 4.13 and 4.12, respectively, and the fact that $n$ cannot be 0 is shown in Corollary 4.9. Hence $12n + 8 \geq 44$, and $4n \geq 12$. Let $\infty = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7, \infty_8\}$ and $Q = \{1, 2, 3, \ldots, 4n\}$. Let $H(Q) = \{h_0, h_1, \ldots, h_{n-1}\}$, where $h_i = \{4i+1, 4i+2, 4i+3, 4i+4\}$. Let $(Q, \circ)$ be a commutative quasigroup of order $4n$ with holes $H(Q)$ and set $V = \infty \cup (Q \times \{1,2,3\})$.

Proceed as in Theorem 1.2 by letting $B_i = h_i \times \{1,2,3\}$ for $0 \leq i < n$. Let $L^{\Delta} = \{<\infty_1, \infty_2>\}$ and let $L_i = \emptyset$ for $0 \leq i < n$. Thus $(A_0, H_0, L^{\Delta}, D_0^*, \emptyset)$ is a metamorphosis of $2K_{20}$ and $(A_i, H_i, \emptyset, D_i^*, \emptyset)$ is a metamorphosis of $2K_{20} \setminus 2K_8$ for $0 < i < n$. What remains is $2K_{4,4,\ldots,4}$ on $Q \times \{k\}$ for $0 < k \leq 3$, so we can, again, apply the result of Sotteau and place the 4-cycles in $C$. $(V, H, L^{\Delta}, D^*, \emptyset)$ is a metamorphosis of $2K_{12n+8}$, as desired. □

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