A Study of Graphical Permutations

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Abstract

A permutation π on a set of positive integers $\{a_1, a_2, ..., a_n\}$ is said to be graphical if there exists a graph containing exactly a_i vertices of degree $\pi(a_i)$ for each i $(1 \leq i \leq n)$. It has been shown that for positive integers with $a_1 < a_2 < ... < a_n$, if $\pi(a_n) = a_n$ then the permutation π is graphical if and only if the sum $\sum_{i=1}^{n} a_i\pi(a_i)$ is even and $a_n \leq \sum_{i=1}^{n-1} a_i\pi(a_i)$. We use a criterion of Tripathi and Vijay to give a new proof of this result, as well as to provide a similar result for permutations π such that $\pi(a_{n-1}) = a_n$. We prove that such a permutation is graphical if and only if the sum $\sum_{i=1}^{n} a_i\pi(a_i)$ is even and $a_n a_{n-1} \leq a_{n-1}(a_{n-1} - 1) + \sum_{i \neq n-1} a_i\pi(a_i)$. We also consider permutations such that $\pi(a_n) = a_{n-1}$, and then, more generally, those such that $\pi(a_n) = a_{n-j}$ for some j $(1 < j < n)$.

1 Introduction

Let G be a graph with vertices $v_1, v_2, \ldots, v_n$ with degrees $d_i = deg(v_i)$ for each i $(1 \leq i \leq n)$. Then $d_1, d_2, \ldots, d_n$ is a degree sequence for G. It is standard to list the terms of a degree sequence for a graph in nonincreasing order. A sequence $s : d_1, d_2, \ldots, d_n$ of nonnegative integers is a graphical sequence if there exists a graph G whose degree sequence is s. There are

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several known characterizations of graphical sequences, but we mention the familiar result of Erdös and Gallai [1].

Theorem 1 (Erdös-Gallai Theorem) *A sequence* $s: d_1, d_2, \ldots, d_p$ $(p \geq 2)$ *of nonnegative integers with* $d_1 \geq d_2 \geq \cdots \geq d_p$ *is graphical if and only if* $\sum_{k=1}^{p} d_k$ *is even and for each integer* n *with* $1 \leq n \leq p$,

$$\sum_{k=1}^{n} d_k \leq n(n-1) + \sum_{k=n+1}^{p} \min\{n, d_k\}.$$

In [5], Tripathi and Vijay give a new result, which states that the number of values for which the inequality in the Erdös-Gallai Theorem must be verified in order to conclude that a sequence is graphical can be reduced. In fact, in the case that the degree sequence contains any values that are repeated multiple times, we must only check the inequality in the Erdös-Gallai Theorem at the end of each segment of repeated values. Because of this, we refer to the result as the EG Shortcut, stated formally below. We shall use the notation $(d)_m$ to mean m occurrences of the value d.

Theorem 2 (EG Shortcut) *Let* $s : (d_1)_{m_1}, (d_2)_{m_2}, \ldots, (d_\ell)_{m_\ell}$ *be a sequence where* $d_1 > d_2 > \cdots > d_\ell$, *and* $m_k \geq 1$ *for each* k $(1 \leq k \leq \ell)$, *with* $m_1 + m_2 + \cdots + m_\ell = p$. *For each* $k = 1, 2, \ldots, \ell$, *let* $\sigma_k = \sum_{i=1}^{k} m_i$, *and for* $1 \leq r \leq t \leq \ell$ *let* $S_{r,t} = \sum_{i=r}^{t} d_i m_i$. *Then the sequence* s *is graphical if and only if* $S_{1,\ell}$ *is even, and for each* $k = 1, 2, \ldots, \ell$,

$$S_{1,k} = \sum_{i=1}^{k} d_i m_i \leq \sigma_k(\sigma_k - 1) + \sum_{i=k+1}^{\ell} m_i \cdot \min\{\sigma_k, d_i\}.$$

Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of distinct positive integers such that $1 \leq a_1 < a_2 < \cdots < a_n$, and let π be a permutation of the elements of the set S. We say that the permutation π is a graphical permutation, or simply that π is graphical, if there exists a graph G containing exactly a_i vertices of degree $\pi(a_i)$ for each i. This concept was first introduced by Hansen and Schultz in [2]. If a permutation π on a set S as described is graphical, the resulting degree sequence is

$$(a_n)_{\pi^{-1}(a_n)}, (a_{n-1})_{\pi^{-1}(a_{n-1})}, \ldots, (a_2)_{\pi^{-1}(a_2)}, (a_1)_{\pi^{-1}(a_1)}.$$

Note that in order to keep our degree sequence listed in nonincreasing order, we need to consider π^{-1}, since each degree a_i will appear $\pi^{-1}(a_i)$ times in the sequence.

In [3], Schultz and Watson characterized all graphical permutations on sets of four elements, and a proof is given for the specific permutation denoted by $\pi_{24} = (a\ c\ b\ d)$. We would like to give a correction to the characterization given for this permutation. We do not need to consider the two cases as stated in [3]. Instead, regardless of how the size of c compares to the size of $(a+b-1)$, both of the stated inequalities must always hold in order for the permutation to be graphical. This result is proved by Thune [4]. We note that the proof given in [3] uses graph constructions and requires the consideration of six different cases. The proof stated in [4] does not require the consideration of graph constructions, but instead uses the EG Shortcut theorem, resulting in a much shorter proof.

Theorem 3 *Let $S = \{a, b, c, d\}$ be a set of positive integers such that $1 \leq a < b < c < d$. Then the permutation $\pi_{24} = (a\ c\ b\ d)$ is graphical if and only if a and b or c and d have the same parity, and $bd \leq ab + bc + ad + b(b-1)$ and $ac + bd \leq bc + ad + (a+b)(a+b-1)$.*

Next we consider a corresponding permutation on a set of six elements, and give a characterization which has a similar proof, also provided in [4].

Theorem 4 *Let $S = \{a, b, c, d, e, f\}$ be a set of positive integers such that $1 \leq a < b < c < d < e < f$. Let π be the permutation of S as follows: $\pi = (a\ d\ b\ e\ c\ f)$. Then π is graphical if and only if the sum $cf + be + ad + ce + bd + af$ is even and the following inequalities hold:*

$$(1)\ cf \leq c(c-1) + bc + ac + ce + bd + af$$
$$(2)\ cf + be \leq (b+c)(b+c-1) + ce + bd + af + a \cdot \min\{b+c, d\}$$
$$(3)\ cf + be + ad \leq (a+b+c)(a+b+c-1) + ce + bd + af$$

2 Main Results

We now consider some general results for graphical permutations on sets of n elements. We are particularly interested in the largest elements of the given set on which the permutations will act. Specifically, we prove results for permutations which send large elements to other large elements, and give necessary and sufficient conditions for when such permutations are graphical.

Suppose $S = \{a_1, a_2, \ldots, a_n\}$ is a set of positive integers satisfying $1 \leq a_1 < a_2 < \cdots < a_n$. In [3], permutations π on S such that $\pi(a_n) = a_n$, are considered, and necessary and sufficient conditions are given for π to be graphical. We state it here and mention that a proof of this result using the EG Shortcut is given by Thune in [4]. Since our next result uses a similar technique, the proof is omitted.

Theorem 5 *Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of integers such that $1 \leq a_1 < a_2 < \cdots < a_n$, and let π be a permutation of S such that $\pi(a_n) = a_n$. Then π is graphical if and only if $\sum_{i=1}^{n} a_i \pi(a_i)$ is even, and*

$$a_n \leq \sum_{i=1}^{n-1} a_i \pi(a_i).$$

A similar result holds for permutations π such that $\pi(a_{n-1}) = a_n$.

Theorem 6 *Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of integers such that $1 \leq a_1 < a_2 < \cdots < a_n$, and let π be a permutation of S such that $\pi(a_{n-1}) = a_n$. Then π is graphical if and only if $\sum_{i=1}^{n} a_i \pi(a_i)$ is even, and*

$$a_n a_{n-1} \leq a_{n-1}(a_{n-1} - 1) + \sum_{i \neq n-1} a_i \pi(a_i).$$

Proof. The sequence to be considered is as follows:

$$(a_n)_{a_{n-1}}, (a_{n-1})_{\pi^{-1}(a_{n-1})}, \ldots, (a_2)_{\pi^{-1}(a_2)}, (a_1)_{\pi^{-1}(a_1)}.$$

In order to match our current notation, the value of σ_k as stated in the EG Shortcut will be as follows throughout this proof:

$$\sigma_k = \sum_{i=n-k+1}^{n} \pi^{-1}(a_i).$$

Assume that π is graphical, and so the sequence above is graphical. Then there exists a graph G with exactly a_i vertices of degree $\pi(a_i)$ for each i $(1 \leq i \leq n)$. Since such a graph exists, clearly the sum of the degrees, $\sum_{i=1}^{n} a_i \pi(a_i)$ must be even. By the EG Shortcut, the degree sequence must satisfy the inequality in the Erdös-Gallai Theorem for each value of σ_k $(1 \leq k \leq n)$. In particular, the degree sequence must satisfy the inequality corresponding to $\sigma_1 = a_{n-1}$, which is the following:

$$a_n a_{n-1} \leq a_{n-1}(a_{n-1} - 1) + \sum_{i=1}^{n-1} \pi^{-1}(a_i) \cdot \min\{a_{n-1}, a_i\}.$$

Since our summation is taken up to $n-1$, and $a_i \leq a_{n-1}$ for each $i \leq n-1$, this is equivalent to

$$a_n a_{n-1} \leq a_{n-1}(a_{n-1} - 1) + \sum_{i \neq n} a_i \pi^{-1}(a_i).$$

Finally, since $a_n \pi^{-1}(a_n) = a_n a_{n-1} = a_{n-1}\pi(a_{n-1})$, this is equivalent to

$$a_n a_{n-1} \leq a_{n-1}(a_{n-1} - 1) + \sum_{i \neq n-1} a_i \pi(a_i),$$

as desired.

For the converse, assume that $\sum_{i=1}^{n} a_i \pi(a_i)$ is even, and

$$a_n a_{n-1} \leq a_{n-1}(a_{n-1} - 1) + \sum_{i \neq n-1} a_i \pi(a_i).$$

By the EG Shortcut, we must verify the inequality in the Erdös-Gallai Theorem for each value of σ_k $(1 \leq k \leq n)$. Thus in general, for σ_k we need to show that

$$\sum_{i=n-k+1}^{n} a_i \pi^{-1}(a_i) \leq \sigma_k(\sigma_k - 1) + \sum_{i=1}^{n-k} \pi^{-1}(a_i) \cdot \min\{\sigma_k, a_i\}.$$

Using our notation for this particular permutation, and since $a_i \leq a_{n-1}$ for each $i \leq n-1$, after some rearranging this is equivalent to

$$\begin{aligned} a_n a_{n-1} \leq\ & a_{n-1}(a_{n-1} - 1) \\ & + \left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i) \right) \left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i) + 2a_{n-1} - 1 \right) \\ & + \sum_{i=1}^{n-k} a_i \pi^{-1}(a_i) - \sum_{i=n-k+1}^{n-1} a_i \pi^{-1}(a_i). \end{aligned}$$

Recall that we are assuming that

$$a_n a_{n-1} \leq a_{n-1}(a_{n-1}-1) + \sum_{i \neq n-1} a_i \pi(a_i),$$

and so it suffices to show that

$$\begin{aligned}\sum_{i \neq n-1} a_i \pi(a_i) \leq &\left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i) + 2a_{n-1} - 1\right) \\ &+ \sum_{i=1}^{n-k} a_i \pi^{-1}(a_i) \\ &- \sum_{i=n-k+1}^{n-1} a_i \pi^{-1}(a_i). \qquad (\star)\end{aligned}$$

Since $a_{n-1}\pi(a_{n-1}) = a_{n-1}a_n = a_n \pi^{-1}(a_n)$, we note that

$$\sum_{i \neq n-1} a_i \pi(a_i) = \sum_{i=1}^{n-1} a_i \pi^{-1}(a_i) = \sum_{i=1}^{n-k} a_i \pi^{-1}(a_i) + \sum_{i=n-k+1}^{n-1} a_i \pi^{-1}(a_i),$$

by first converting from π to π^{-1}, and then separating the summation into two parts. We also note that

$$\sum_{i=1}^{n-k} a_i \pi^{-1}(a_i) + \sum_{i=n-k+1}^{n-1} a_i \pi^{-1}(a_i) \leq \sum_{i=1}^{n-k} a_i \pi^{-1}(a_i) + a_{n-1} \cdot \sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i),$$

since $a_i \leq a_{n-1}$ for each $i \leq n-1$. Thus we have shown that

$$\sum_{i \neq n-1} a_i \pi(a_i) \leq \sum_{i=1}^{n-k} a_i \pi^{-1}(a_i) + a_{n-1} \cdot \sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i).$$

Then notice that

$$\left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i) + a_{n-1} - 1\right) - \sum_{i=n-k+1}^{n-1} a_i \pi^{-1}(a_i)$$

must be positive, and so we add this expression to the right hand side to obtain

$$\sum_{i\neq n-1} a_i\pi(a_i) \leq \sum_{i=1}^{n-k} a_i\pi^{-1}(a_i) + a_{n-1}\cdot \sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i)$$
$$+\left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n-1} \pi^{-1}(a_i) + a_{n-1} - 1\right)$$
$$-\sum_{i=n-k+1}^{n-1} a_i\pi^{-1}(a_i).$$

After factoring the right hand side, this is equivalent to ($\star$) as desired. Thus we have shown that the sequence satisfies the inequality stated in the Erdös-Gallai Theorem for each value of σ_k ($1 \leq k \leq n$). Then by the EG Shortcut, the sequence is graphical, and thus the permutation π is graphical. □

Now instead of looking at which element the permutation sends to the largest element a_n, as in the previous result, we will consider which element a_n is sent to by the permutation. We will again be particularly interested in the next largest element of the set, a_{n-1}.

Theorem 7 *Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of integers such that $1 \leq a_1 < a_2 < \cdots < a_n$, and let π be a permutation of S such that $\pi(a_n) = a_{n-1}$. Then π is graphical if and only if $\sum_{i=1}^{n} a_i\pi(a_i)$ is even.*

Proof. The sequence being considered corresponding to this permutation is as follows:

$$(a_n)_{\pi^{-1}(a_n)}, (a_{n-1})_{a_n}, (a_{n-2})_{\pi^{-1}(a_{n-2})}, \ldots, (a_2)_{\pi^{-1}(a_2)}, (a_1)_{\pi^{-1}(a_1)}.$$

To match our current notation for the permutation π, the value of σ_k as stated in the EG Shortcut will be as follows throughout this proof:

$$\sigma_k = \sum_{i=n-k+1}^{n} \pi^{-1}(a_i).$$

By the EG Shortcut theorem, this sequence is graphical if and only if the sum $\sum_{i=1}^{n} a_i\pi(a_i)$ is even and the inequality stated in the Erdös-Gallai Theorem is satisfied for each value of σ_k ($1 \leq k \leq n$).

We will first consider the inequality corresponding to $\sigma_1 = \pi^{-1}(a_n)$, which is

$$a_n\pi^{-1}(a_n) \leq (\pi^{-1}(a_n))\left(\pi^{-1}(a_n) - 1\right) + \sum_{i=1}^{n-1} \pi^{-1}(a_i) \cdot \min\{a_i, \pi^{-1}(a_n)\},$$

or equivalently, since $\pi^{-1}(a_{n-1}) = a_n$,

$$a_n\pi^{-1}(a_n) \leq (\pi^{-1}(a_n))\left(\pi^{-1}(a_n) - 1\right) + a_n \cdot \min\{a_{n-1}, \pi^{-1}(a_n)\} + \sum_{i=1}^{n-2} \pi^{-1}(a_i) \cdot \min\{a_i, \pi^{-1}(a_n)\}.$$

Observe that $\pi^{-1}(a_n) \leq a_{n-1}$, and therefore $\min\{a_{n-1}, \pi^{-1}(a_n)\} = \pi^{-1}(a_n)$. Thus the inequality we need to verify is equivalent to

$$a_n\pi^{-1}(a_n) \leq (\pi^{-1}(a_n))\left(\pi^{-1}(a_n) - 1\right) + a_n\pi^{-1}(a_n) + \sum_{i=1}^{n-2} \pi^{-1}(a_i) \cdot \min\{a_i, \pi^{-1}(a_n)\},$$

which is clearly true. Therefore, the inequality corresponding to σ_1 holds.

For $k \geq 2$, notice that σ_k is a sum containing a_n, and thus $\min\{\sigma_k, a_i\} = a_i$ in each inequality corresponding to σ_k with $k \geq 2$. Knowing this, and using our particular sequence, what we need to show for each σ_k $(k \geq 2)$ is

$$\sum_{i=n-k+1}^{n} a_i\pi^{-1}(a_i) \leq \left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i) - 1\right) + \sum_{i=1}^{n-k} a_i\pi^{-1}(a_i). \qquad (\star)$$

Since $a_i \leq a_n$ for each $i \leq n$, it is easy to see that

$$\sum_{i=n-k+1}^{n} a_i\pi^{-1}(a_i) \leq a_n \cdot \sum_{i=n-k+1}^{n} \pi^{-1}(a_i).$$

Then since the expression

$$\left(\sum_{\substack{i \geq n-k+1 \\ i \neq n-1}} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i) - 1\right) - a_n + \sum_{i=1}^{n-k} a_i\pi^{-1}(a_i)$$

must be positive, we can add this to the right hand side to obtain

$$\begin{aligned}\sum_{i=n-k+1}^{n} a_i\pi^{-1}(a_i) \leq a_n \cdot &\sum_{i=n-k+1}^{n} \pi^{-1}(a_i) \\ &+ \left(\sum_{\substack{i\geq n-k+1 \\ i\neq n-1}} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i)-1\right) \\ &- a_n + \sum_{i=1}^{n-k} a_i\pi^{-1}(a_i),\end{aligned}$$

or equivalently,

$$\begin{aligned}\sum_{i=n-k+1}^{n} a_i\pi^{-1}(a_i) \leq a_n \cdot &\left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i)-1\right) \\ &+ \left(\sum_{\substack{i\geq n-k+1 \\ i\neq n-1}} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i)-1\right) \\ &+ \sum_{i=1}^{n-k} a_i\pi^{-1}(a_i)\end{aligned}$$

Then by factoring the right hand side, we obtain

$$\begin{aligned}\sum_{i=n-k+1}^{n} a_i\pi^{-1}(a_i) \leq &\left(a_n + \sum_{\substack{i\geq n-k+1 \\ i\neq n-1}} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i)-1\right) \\ &+ \sum_{i=1}^{n-k} a_i\pi^{-1}(a_i),\end{aligned}$$

which is equivalent to the inequality ($\star$). Thus the sequence satisfies the inequality corresponding to σ_k, $k \geq 2$.

We have shown that the sequence satisfies the inequality in the Erdös-Gallai Theorem for each value of σ_k $(1 \leq k \leq n)$, and so by the EG Shortcut, the sequence is graphical. Therefore the permutation π is graphical. □

Next, we will generalize this idea. We consider a permutation π such that $\pi(a_n) = a_{n-j}$ for some j $(1 < j < n)$. Notice that the previous theorem is the case when $j = 1$.

Theorem 8 *Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of integers such that $1 \leq a_1 < a_2 < \cdots < a_n$, and let π be a permutation of S such that $\pi(a_n) = a_{n-j}$ for some j $(1 < j < n)$. Let σ_k be defined as $\sigma_k = \sum_{i=n-k+1}^{n} \pi^{-1}(a_i)$. Then π is graphical if and only if $\sum_{i=1}^{n} a_i \pi(a_i)$ is even, and the inequality in the Erdös-Gallai Theorem holds for each value of σ_k $(1 \leq k \leq j)$.*

Proof. The sequence being considered for this permutation is as follows:

$$(a_n)_{\pi^{-1}(a_n)}, (a_{n-1})_{\pi^{-1}(a_{n-1})}, \ldots$$

$$\ldots, (a_{n-j+1})_{\pi^{-1}(a_{n-j+1})}, (a_{n-j})_{a_n}, (a_{n-j-1})_{\pi^{-1}(a_{n-j-1})}, \ldots$$

$$\ldots, (a_2)_{\pi^{-1}(a_2)}, (a_1)_{\pi^{-1}(a_1)}.$$

First assume that π is graphical, and thus the above sequence is graphical. Then clearly the sum $\sum_{i=1}^{n} a_i \pi(a_i)$ is even. By the EG Shortcut, the sequence must satisfy the inequality in the Erdös-Gallai Theorem for each value of σ_k $(1 \leq k \leq n)$. Therefore, the inequalities corresponding to the values $\sigma_1, \sigma_2, \ldots, \sigma_j$ must all be satisfied, as desired.

For the converse, we assume the sum $\sum_{i=1}^{n} a_i \pi(a_i)$ is even, and that the inequalities corresponding to the values $\sigma_1, \sigma_2, \ldots, \sigma_j$ are all satisfied. We show that the remaining inequalities corresponding to the values for σ_k $(j + 1 \leq k \leq n)$ are also satisfied.

We notice that for $k > j$, the value of σ_k is a summation containing a_n, since $\pi^{-1}(a_{n-j}) = a_n$. Thus the inequality corresponding to σ_k, where $j + 1 \leq k \leq n$ is

$$\sum_{i=n-k+1}^{n} a_i \pi^{-1}(a_i) \leq \left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i) \right) \left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i) - 1 \right) + \sum_{i=1}^{n-k} a_i \pi^{-1}(a_i). \quad (\star)$$

Since $a_i \leq a_n$ for each $i \leq n$, we know that

$$\sum_{i=n-k+1}^{n} a_i \pi^{-1}(a_i) \leq a_n \cdot \sum_{i=n-k+1}^{n} \pi^{-1}(a_i).$$

Observe that the expression

$$\left(\sum_{\substack{i\geq n-k+1\\ i\neq n-j}} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i) - 1\right) - a_n + \sum_{i=1}^{n-k} a_i\pi^{-1}(a_i)$$

must be positive, and so by adding this to the right hand side, we obtain

$$\begin{aligned}\sum_{i=n-k+1}^{n} a_i\pi^{-1}(a_i) \leq a_n \cdot & \sum_{i=n-k+1}^{n} \pi^{-1}(a_i) \\ & + \left(\sum_{\substack{i\geq n-k+1\\ i\neq n-j}} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i) - 1\right) \\ & - a_n + \sum_{i=1}^{n-k} a_i\pi^{-1}(a_i).\end{aligned}$$

Then notice that by factoring the right hand side, this is equivalent to

$$\begin{aligned}\sum_{i=n-k+1}^{n} a_i\pi^{-1}(a_i) \leq & \left(a_n + \sum_{\substack{i\geq n-k+1\\ i\neq n-j}} \pi^{-1}(a_i)\right)\left(\sum_{i=n-k+1}^{n} \pi^{-1}(a_i) - 1\right) \\ & + \sum_{i=1}^{n-k} a_i\pi^{-1}(a_i).\end{aligned}$$

Since $a_n = \pi^{-1}(a_{n-j})$, this inequality is equivalent to the inequality $(\star)$. Thus, we have shown that the sequence satisfies the inequality in the Erdös-Gallai Theorem for each value of σ_k $(j+1 \leq k \leq n)$. Recall that by assumption, the inequality is satisfied for values of σ_k $(1 \leq k \leq j)$. Thus, by the EG Shortcut theorem, the sequence is graphical, and thus the permutation π is graphical. □

3 Conclusions

We conclude with a restatement of the EG Shortcut theorem, specific to the notation that comes from considering graphical permutations.

Theorem 9 *Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of integers such that $1 \leq a_1 < a_2 < \cdots < a_n$. Let π be a permutation on the set S such that*

$\sum_{i=1}^{n} a_i\pi(a_i)$ *is even. Let* $S_k = \sum_{i=n-k+1}^{n} a_i\pi^{-1}(a_i)$ *and let* $T_k = \sum_{i=n-k+1}^{n} \pi^{-1}(a_i)$*. Then the permutation* π *is graphical if and only if the following holds for each value of* k $(1 \leq k \leq n)$*:*

$$S_k \leq T_k(T_k - 1) + \sum_{\substack{i \leq n-k \\ a_i \leq T_k}} a_i\pi^{-1}(a_i) + \sum_{\substack{i \leq n-k \\ a_i > T_k}} T_k\pi^{-1}(a_i).$$

We also point out that Theorem 8, which considers a permutation π such that $\pi(a_n) = a_{n-j}$ for some j $(1 < j < n)$, reduces (even more so than by using the EG Shortcut) how many different inequalities must be verified in order to conclude that π is graphical. If $\pi(a_n) = a_{n-j}$, then we must verify the inequality stated in the Erdös-Gallai Theorem for j different values. Thus it is easier to conclude a permutation π on a set S is graphical when it sends its largest element a_n to another large element of the set, which results in a smaller value of j.

While this general result still does not give us one single inequality to verify in order to conclude a permutation π is graphical, it does allow us to immediately identify a maximum number of inequalities which would need to be verified, based on how the permutation acts on the largest element a_n.

We note that this is not the best possible case, in the sense that it does not give the minimum number of inequalities which are necessary to verify. For example, recall Theorem 4, which gave a result for a particular permutation on a set of six elements, $\pi = (a\ d\ b\ e\ c\ f)$. With the EG Shortcut, there are six inequalities to verify. Since $\pi(f) = a$, or in different notation $\pi(a_6) = a_1$, for this permutation, we have $j = 5$, and by Theorem 8, we need to verify five inequalities. But, in fact, by Theorem 4, we only needed to verify three inequalities for this permutation, those corresponding to σ_1, σ_2, and σ_3.

Consider a generalization of this permutation on a set of $2n$ elements, $S = \{a_1, a_2, \ldots, a_{2n-1}, a_{2n}\}$, such that $1 \leq a_1 < a_2 < \cdots < a_{2n-1} < a_{2n}$. The permutation in question is $\pi = (a_1\ a_{n+1}\ a_2\ a_{n+2} \cdots\ a_{n-1}\ a_{2n-1}\ a_n\ a_{2n})$. This type of permutation can be defined as follows:

$$\pi(a_p) = \begin{cases} a_{p+n} & \text{for} \quad p \leq n \\ a_{p-n+1} & \text{for} \quad n < p < 2n \\ a_1 & \text{for} \quad p = 2n \end{cases}$$

We give the following conjecture for this type of permutation.

Conjecture 1 *Let* $S = \{a_1, a_2, \ldots, a_{2n-1}, a_{2n}\}$ *be a set of* $2n$ *positive integers such that* $1 \leq a_1 < a_2 < \cdots < a_{2n-1} < a_{2n}$. *Let* π *be the permutation of the set* S *as defined above, with* $\sum_{i=1}^{2n} a_i\pi(a_i)$ *even. Let* $\sigma_k = \sum_{i=(2n)-k+1}^{2n} \pi^{-1}(a_i)$. *Then* π *is graphical if and only if the inequality in the Erdös-Gallai Theorem holds for each value of* σ_k $(1 \leq k \leq n)$.

This conjecture says that for this type of permutation on a set of even cardinality, we need to verify only half of the inequalities required by the EG Shortcut theorem. It would be interesting to see if a similar method can be used to prove this conjecture for this specific type of permutation.

Note that in this case, the largest product $a_i\pi(a_i)$ occurs for $i = n$, with the product $a_n\pi(a_n) = a_n a_{2n}$, and we proposed that n inequalities need to be verified. Suppose for general permutations, instead of only considering how the permutation acts on the largest element of the set, we focus on this largest product $a_i\pi(a_i)$. This has been considered in conjectures in previous papers, including [3] and [6], however the proposed conjectures have since been shown to be false. Perhaps this idea could lead to interesting results of graphical permutations with future study.

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