DIMENSION OF A CATERPILLAR

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ABSTRACT. k-labeling of a graph is a labeling of vertices of the graph by k-tuples of non-negative integers in such a way that two vertices of G are adjacent if and only if their label k-tuples differ in each coordinate. The dimension of a graph G is the least k such that G has a k-labeling.

Lovász et al showed that for $n \geq 3$, the dimension of a path of length n is $(\log_2 n)^+$. Lovász et al and Evans et al obtained the dimension of a cycle of length n for most n. In the present paper we obtain the dimension of a caterpillar or close bounds for it in various cases.

Keywords: Dimension of a graph, Product dimension, Caterpillar, Graph labeling, Path, Cycle.

1. INTRODUCTION

The graphs considered in this paper are symmetric graphs without loops. The dimension of a graph G is defined as the minimal number of complete graphs whose product contains G as an induced subgraph. An equivalent way of defining dimension is as follows:

Represent vertices of the graph by vectors of length n with nonnegative integer coordinates in such a way that two vertices are adjacent if and only if all their corresponding coordinates are different. The least such n is called the dimension of the graph. More information regarding this concept is found in [3], [4]. The dimension of G is also called the product dimension of G denoted by $\mathrm{pdim}(G)$. For a related concept viz. representation number of a graph, see [1], [2], [5].

For a graph G, $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set of G.

Definition 1.1. *A spanned (or induced) subgraph of G is a graph H with $V(H) \subset V(G)$ and $E(H) = E(G) \cap (V(H) \times V(H))$. For every $M \subset V(G)$ there is exactly one spanned subgraph H of G with $V(H) = M$. It will be referred to as the subgraph of G spanned by M.*

$\mathbb{N}$ denotes the set of all nonnegative integers. $\mathbb{N}$ also denotes the complete graph on vertices represented by nonnegative integers. $\mathbb{N}^n$ denotes the cartesian product of the complete graph $\mathbb{N}$ taken n times, so $\mathbb{N}^n$ is a graph with vertices as n-tuples of nonnegative integers and two such n-tuples are

2012 AMS Subject Classification: 05C78 (primary)

joined by an edge if and only if all the corresponding coordinates in the two vertices are different. An embedding of a graph G_1 into a graph G_2 is a one to one map ϕ from $V(G_1)$ to $V(G_2)$ such that $\{x, y\} \in E(G_1)$ if and only if $\{\phi(x), \phi(y)\} \in E(G_2)$.

An embedding of a graph G into the graph $\mathbb{N}^n$ is called an n-labeling of G. Thus by an n-labeling of G we mean associating the vertices $x \in V(G)$ with distinct vectors $v(x) = (v_1(x), \ldots, v_n(x))$ of nonnegative integers in such a way that $\{x, y\} \in E(G)$ if and only if the vectors $v(x)$ and $v(y)$ differ in all the corresponding coordinates. For finite graphs G, the dimension of G is the least natural number n such that G can be embedded into $\mathbb{N}^n$. Encoding of a graph means an n-labeling for some n. The labeling vectors will be written simply as words in the coordinates (e.g., 0102 stands for (0, 1, 0, 2)). A particular choice of the vectors above will be referred to as an *encoding*.

Dimension of G is 1 if and only if G is a complete graph. Let P_n (resp. C_n) denote a path (resp. cycle) with n edges. The encoding in the following Figure 1 yields a proof that $\dim(P_3) = 2$ and $\dim(C_4) = 2$.

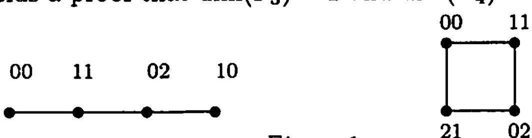

Figure 1

We have $\dim(P_1) = 1$, $\dim(P_2) = 2$. Let $(r)^+$ denote the upper integral approximation of the real number r. In [4], Lovász et al have shown that for $n \geq 3$, dimension of P_n is $(\log_2 n)^+$. Also $\dim(C_3) = 1$. In [4], Lovász et al obtained the following results for the dimension of a cycle:

For $n \geq 3$, $\dim(C_{2n}) = (\log_2(n-1))^+ + 1$.

For $n \geq 2$, $(\log_2 n)^+ + 1 \leq \dim(C_{2n+1}) \leq (\log_2 n)^+ + 2$.

If n is a power of 2 of the form 2^{2t+1}, then $\dim(C_{2n+1}) = (\log_2 n)^+ + 2$.

It was proved by Evans et al [1] that if n is not a power of 2, $\dim(C_{2n+1}) = (\log_2 n)^+ + 1$. If n is a power of 2 of the form 2^{2t}, the dimension of C_{2n+1} is still unknown.

In this paper we shall obtain the dimension or close upper and lower bounds for the dimension for some classes of caterpillars. In Sections 2 and 3 (resp.), we get the main results regarding lower and upper bounds (resp.) for the dimension of a caterpillar. In Section 4, we present results regarding the dimension of certain classes of caterpillar.

2. A LOWER BOUND FOR THE DIMENSION OF A CATERPILLAR

Definition 2.1. *A caterpillar is a tree in which there is a path that contains at least one end-point of every edge. Such a path in a caterpillar is called a spine.*

A point and a path are trivial caterpillars. In what follows we shall assume that the caterpillar is nontrivial.

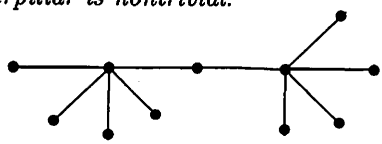

Figure 2

The caterpillar considered in Figure 2 has many spines. The minimum spine called *m-spine* which contains all non-pendent vertices is uniquely determined. There are many maximal spines called *M-spines* which are spines of maximum length. These have all the non-pendent vertices and also two pendent vertices of the caterpillar as extreme vertices of the spine. For any caterpillar, any M-spine contains the m-spine and is obtained by attaching one pendent vertex on each extremity of the m-spine. The m-spine is the intersection of all M-spines. The number of vertices in any M-spine is equal to the number of vertices in m-spine +2. The number of edges in any M-spine of a caterpillar is called as the length of the caterpillar and it is equal to the diameter of the caterpillar.

In this paper we consider families R_n of caterpillars of length n and vertex degrees ≤ 3 and for all these families we get

$$(\log_2 n)^+ \leq \dim(R_n) \leq (\log_2(n+2))^+ + 1.$$

For particular families we get better bounds in which the upper and lower bounds differ by at most 1. We get cases when the bounds become equal and in those cases we are able to find $\dim(R_n)$.

To prove our results we mainly use the ideas of Lovász et al in [4].

Notation 2.2. Let x^0-x^1-...-x^n be an M-spine of a caterpillar R_n and $\deg(x^i) \leq 3$, $1 \leq i \leq n-1$. Let $B = \{i|\ 1 \leq i \leq n-1,\ \deg(x^i) = 3\}$. For $i \in B$, x^i is called a leg vertex. For $i \in B$, let y^i be the pendent vertex of R_n adjacent to x^i. Thus the vertex set of R_n is $V = V(R_n) = \{x^i|\ 0 \leq i \leq n\} \cup \{y^i|\ i \in B\}$ and the edge set of R_n is $E(R_n) = \{(x^i, x^{i+1})|\ 0 \leq i \leq n-1\} \cup \{(x^i, y^i)|\ i \in B\}$. x^0, x^n and y^s for $s \in B$ are the pendent vertices of R_n. If $i \notin B$, x^i is called a gap vertex or a non-leg vertex. Thus a leg vertex is of degree 3 and a gap vertex is of degree ≤ 2.

Let $x^{r+1}, x^{r+2}, \ldots, x^{r+t}$ be consecutive leg vertices of R_n and suppose that x^r and x^{r+t+1} are gap vertices, i.e. $i \in B$ for $r+1 \leq i \leq r+t$ but $r, r+t+1 \notin B$. We call the induced subgraph on x^s and y^s, $r+1 \leq s \leq r+t$, a bunch of legs. The induced subgraph (path) on all gap vertices between consecutive bunches of legs is called a bunch of middle gap vertices. Initial and final bunch of gap vertices or set of gap vertices can be defined in an obvious manner. (See Figure 3.)

We shall get a lower bound for $\dim(R_n)$ in our Main Theorem I using the ideas from [4] directly or indirectly:

Remark 2.3. A Criterion for Adjacent Vertices in Terms of Inner Product : *Put* $S(n) = \{A : A \subset \{1, 2 \ldots, n\}\}$.
Then $|S(n)| = 2^n$. *Let* $\mathbb{N}$ *be the set of all non-negative integers. For a vector* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{N}^n$ *and* $A \in S(n)$ *define vectors* $\bar{x}, \tilde{x} \in \mathbb{N}^{S(n)}$ *by putting*

$$\bar{x}(A) = \prod_{i \in A} x_i, \tilde{x}(A) = \prod_{i \notin A} (-x_i). \tag{2.1}$$

Then $\bar{x}$ *and* $\tilde{x}$ *have* 2^n *coordinates. We see immediately that*

$$\prod_{i=1}^{n} (x_i - y_i) = \bar{x}.\tilde{y} \tag{2.2}$$

where the notation $\bar{x}.\tilde{y}$ *designates the inner product of* $\bar{x}$ *and* $\tilde{y}$. *Thus* $\mathbf{x}$ *and* $\mathbf{y}$ *are labelings of adjacent vertices if and only if* $\bar{x}.\tilde{y} \neq 0$.

Proposition 2.4. (L. Lovász et al)([4], Proposition 5.3) *Let* $x^1, \ldots, x^k$ *be distinct elements of* $V(G)$ *such that for some* $y^1, \ldots, y^k \in V(G)$, $\{x^i, y^i\} \in E(G)$ *and* $\{x^i, y^j\} \notin E(G)$ *for* $i < j$, $1 \leq i, j \leq k$. *Then*

$$\dim(G) \geq \log_2 k.$$

Theorem 2.5. **(Main Theorem I)** *Let* R_n, $n \geq 4$, *be a caterpillar of length* n *with* $x^0, \ldots, x^n$ *as vertices of an M-spine and* $\deg(x^i)$, $1 \leq i \leq n-1$, *be at most* 3. *Let the caterpillar* R_n *contain* t_0 *bunches of gap (non-leg) vertices consisting of odd number of vertices. Let* $t_1 = 1$ *provided at least one of the initial and final bunches of non-leg vertices consists of exactly* 1 *vertex, and* $t_1 = 0$ *otherwise. Then* $\dim(R_n)$ *satisfies the inequality,*

$$(\log_2(|V| - t_0 + t_1))^+ \leq \dim(R_n).$$

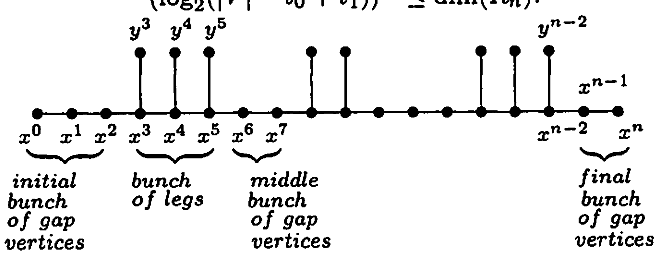

Figure 3

Proof. Let W be the subset of V obtained from V by removing certain vertices as follows:

1) If at least one of the initial or final sets of gap vertices has exactly one element then renaming the vertices of the M-spine if necessary we arrange that the final set of gap vertices has exactly one element.

2) If after Step 1, the initial set of gap vertices has odd number of vertices we remove x^0.
3) If any bunch of middle gap vertices contains odd number of vertices, then remove the pendent vertex of the preceding leg.
4) If the final bunch of gap vertices contains 3 or more odd number of vertices, then remove the pendent vertex of last leg.
5) Call the remaining set of vertices as W. Let $|W| = t$. Let the elements of W be denoted by w^i, $1 \leq i \leq t$.

Let k be the dimension of the caterpillar R_n. Let f be an embedding (encoding) of R_n in $\mathbb{N}^k$. Let $f(w^i) = \mathbf{w}^i$, $1 \leq i \leq t$. We assume by adding 1, if necessary, to all coordinates of $\mathbf{w}^i$ for all i, $1 \leq i \leq t$, that all coordinates in all $\mathbf{w}^i$ are positive integers. These new $\mathbf{w}^i$ satisfy the requirements of an embedding.

Let $\bar{w}^i$ and $\tilde{w}^i \in \mathbb{N}^{2^k}$ be 2^k-tuples of non-negative integers, as defined in Section 2, corresponding to the k-tuples $\mathbf{w}^i$. We show that $\bar{w}^i$ are $\mathbb{R}$-linearly independent. Let

$$(*) \qquad \sum_{i=1}^{t} a_i \bar{w}^i = 0, a_i \in \mathbb{R}.$$

w^i can be one of the x^j, i.e. a vertex of the M-spine, or one of the y^j, i.e. a pendent vertex other than x^0 and x^n. If y^j is any pendent vertex (of a leg) in V, then taking dot product of the last equation with $\tilde{y}^j$ we get

$$a\tilde{y}^j . \bar{x}^j = 0,$$

where a is the coefficient of x^j in (*). Since $\tilde{y}^j . \bar{x}^j \neq 0$, we get $a = 0$. Let $W_1 = W \backslash \{x^j | x^j \text{ is a leg vertex }\}$.

- Suppose we have an odd number of initial gap vertices $x^0, x^1, \ldots, x^s$. Then $x^0 \notin W$, so $x^0 \notin W_1$. Also W and W_1 contain $x^1, \ldots, x^s$. From (*), consider the resulting equation omitting $\bar{x}^j$ corresponding to the leg vertices x^j. **Call it (**)**. Taking dot product successively with $x^0, x^1, \ldots, x^{s-1}$, we get that the coefficients of $\bar{x}^j$, $1 \leq j \leq s$, in (**) are 0.
- If we have even number of initial gap vertices $x^0, x^1, \ldots, x^s$, then W and W_1 contain all of them. Taking dot product successively with $\tilde{x}^0, \tilde{x}^2, \ldots, \tilde{x}^{s-1}$ (even superscripts) we get that the coefficients of $\bar{x}^1, \bar{x}^3, \ldots, \bar{x}^s$ (odd superscripts) are 0. Then taking dot product with $\tilde{x}^s, \tilde{x}^{s-2}, \ldots, \tilde{x}^1$ (odd superscripts in reverse order) successively, we get that the coefficients of $\bar{x}^{s-1}, \bar{x}^{s-3}, \ldots, \bar{x}^2, \bar{x}^0$ (even superscripts in reverse order) are 0. Thus coefficients of $\bar{x}^0, \bar{x}^1, \ldots, \bar{x}^{s-1}$ are 0.

Consider the equation (**) and remove the terms containing $\bar{x}^i$ where $1 \leq i \leq s$ if s is even, (i.e. number of initial leg vertices is odd), and $0 \leq i \leq s$ if s

is odd. We deal with middle bunches of gap vertices one by one successively from the left. If $x^{j+1}, x^{j+2}, \ldots, x^{j+s}$ form a middle bunch of gap vertices, then if s is odd, we have $y^j \notin W$. Then taking dot product successively with $\tilde{x}^j, \tilde{x}^{j+1}, \ldots, \tilde{x}^{j+s-1}$, we get that the coefficients of $\bar{x}^{j+1}, \ldots, \bar{x}^{j+s}$ in the equation are 0. If s is even, then $y^j \in W$. We take dot product successively with $\tilde{x}^{j+1}, \tilde{x}^{j+3}, \ldots, \tilde{x}^{j+s-1}$ to get the coefficients of $\bar{x}^{j+2}, \bar{x}^{j+4}, \ldots, \bar{x}^{j+s}$ as 0. Then take successively dot product with $\tilde{x}^{j+s}, \tilde{x}^{j+s-2}, \ldots, \tilde{x}^{j+2}$. We get that the coefficients of $\bar{x}^{j+s-1}, \bar{x}^{j+s-3}, \ldots, \bar{x}^{j+1}$ are 0. In this way we deal with the middle bunches of gap vertices one by one.

Next we look at the final bunch of gap vertices. If this bunch has exactly one vertex viz. x^n, then W contains y^{n-1}. We are left with an equation of type

$$a\bar{y}^{n-1} + b\bar{x}^n = 0.$$

Since x^n and y^{n-1} have at least one coordinate same but not all, and since the coordinates are positive, we get $a = 0$ and $b = 0$.

If the final bunch has 3 or more odd number of gap vertices, say $x^{j+1}, \ldots,$ $x^{j+s} = x^n$, then the pendent vertex of the last leg i.e. y^j is not in W. Hence successively taking dot product with $\tilde{x}^j, \tilde{x}^{j+1}, \ldots, \tilde{x}^{j+s-1}$, we get that the coefficients of $\bar{x}^{j+1}, \ldots, \bar{x}^{j+s} = \bar{x}^n$ are zero.

If the final bunch of gap vertices has even number of vertices, then x^j is in W. In this case take successively dot product with $\tilde{x}^{j+1}, \tilde{x}^{j+3}, \ldots, \tilde{x}^{j+s-1} = \tilde{x}^{n-1}$, and then with $\tilde{x}^n, \tilde{x}^{n-2}, \ldots, \tilde{x}^{j+2}$, we get that the coefficients of $\bar{x}^{j+1}, \bar{x}^{j+2}, \ldots, \bar{x}^n$ are zero.

This shows that the vectors $\mathbf{w}^i$, $1 \leq i \leq t$, are $\mathbb{R}$-linearly independent. Hence $t \leq 2^n$, so $(\log_2 t)^+ \leq n$. Now $t = |V| - t_0 + t_1$, so

$$(\log_2(|V| - t_0 + t_1))^+ \leq \dim(R_n).$$

□

3. An upper bound for the dimension of a caterpillar

Theorem 3.1. (**Main Theorem II**) *Let R_n, $n \geq 4$, be a caterpillar of length n and let $x^0, x^1, \ldots, x^n$ be the vertices of an M-spine of R_n and let $deg(x^i) = 3$ for $2 \leq i \leq n-2$ and* $\deg(x^i) = 2$ *for $i = 1, n-1$. For $2 \leq i \leq n-2$, let y^i be the pendent vertex adjacent to x^i. Then*

$$\dim(R_n) \leq (\log_2 n)^+ + 1.$$

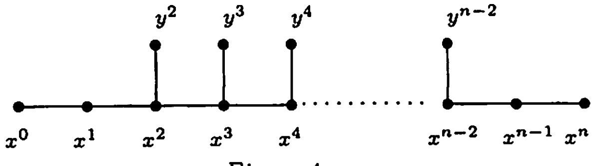

Figure 4

Proof. To prove the upper bound for the dimension of the caterpillar R_n, we first show that the caterpillar R_{2^k} can be embedded in $\mathbb{N}^{k+1}$. We consider the M-spine of the caterpillar R_n given by x^0-x^1-...-x^n. In analogy with a theorem of Lovász et al [[4], Theorem 5.6], we define

$$v_k(i) \in K_3^{k+1}, \; 0 \le i \le 2^k$$

(i refers to vertex number in the M-spine), and

$$v_k'(i) \in K_3^{k+1}, \; 2 \le i \le 2^k - 2$$

(i refers to vertex number of the pendent vertex adjacent to the i^{th} vertex in the M-spine), and define them inductively as follows:
For $k = 2$, define $v_2(i)$, $0 \le i \le 4$, as

$$v_2(0) = 000, v_2(1) = 111, v_2(2) = 022, v_2(3) = 110, v_2(4) = 001.$$

Again for $k = 2$, define $v_2'(i)$, for $i = 2$, as

$$v_2'(2) = 101.$$

For $k \ge 2$, we shall now define $v_{k+1}(i)$ for $0 \le i \le 2^{k+1}$. We first define

$$v'(i) = \begin{cases} 0 & \text{if } i \text{ is even,} \\ 1 & \text{if } i \text{ is odd,} \end{cases} \qquad \textit{and } v''(i) = \begin{cases} 1 & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

Define

$$v_{k+1}(i) = \begin{cases} v_k(i)v'(i) & \text{if } 0 \le i \le 2^k - 2, \\ v_k(2^k - 1)1 = 11\ldots1\ldots101 & \text{if } i = 2^k - 1, \\ v_k(2^k)2 = 00\ldots0\ldots012 & \text{if } i = 2^k, \\ v_k(2^k + 1)0 = 11\ldots1\ldots100 & \text{if } i = 2^k + 1, \\ v_k(2^{k+1} - i)v''(i) & \text{if } 2^k + 2 \le i \le 2^{k+1}. \end{cases}$$

Again for $k \ge 2$, we define $v_{k+1}'(i)$, $2 \le i \le 2^{k+1} - 2$, as

$$v_{k+1}'(i) = \begin{cases} v_k'(i)v''(i) & \text{if } 2 \le i \le 2^k - 2, \\ 0200\ldots0\ldots010 & \text{if } i = 2^k - 1, \\ 1211\ldots1\ldots101 & \text{if } i = 2^k, \\ 0200\ldots0\ldots011 & \text{if } i = 2^k + 1, \\ v_k'(2^k - i)v'(i) & \text{if } 2^k + 2 \le i < 2^{k+1}. \end{cases}$$

We claim that the correspondence sending x^i to $v_k(i)$ and y^i to $v_k'(i)$ is an encoding of R_{2^k}. We shall see that neighbors agree in no coordinate and non-neighbors agree in at least one coordinate. We will prove by induction that, for $1 \le i, j \le 2^k$,

- **A:** if $|i-j| > 1$, $v_k(i)$ and $v_k(j)$ agree in a coordinate and for $|i-j| = 1$, they agree in no coordinate.
- **B:** if $|i-j| > 1$, $v_k'(i)$ and $v_k'(j)$ agree in a coordinate and for $|i-j| = 1$, they agree in no coordinate.

C: for $i \neq j$, $v_k(i)$ and $v'_k(j)$ agree in a coordinate and for $i = j$, they agree in no coordinate.

This is true for $k = 2$. Let it hold until k and let us consider $v_{k+1}(i)$ and $v'_{k+1}(j)$.

A: Let $v_{k+1}(i)$ and $v_{k+1}(j)$, $0 \leq i, j \leq 2^{k+1}$, $i \neq j$, be vertices of the M-spine of the caterpillar $R_{2^{k+1}}$. The first $k+1$ coordinates of $v_{k+1}(i)$ are $v_k(i')$ where $i' = i$ if $0 \leq i \leq 2^k$, and $i' = 2^{k+1} - i$ if $i > 2^k$.

Case	i	j	i'	j'	$\lvert i' - j' \rvert$
1	$\leq 2^k$	$\leq 2^k$	i	j	$\lvert i - j \rvert$
2	$> 2^k$	$> 2^k$	$2^{k+1} - i$	$2^{k+1} - j$	$\lvert i - j \rvert$
3	$\leq 2^k$	$> 2^k$	i	$2^{k+1} - j$	$\lvert 2^{k+1} - i - j \rvert$

Let $v(i)$ and $w(j)$ be vectors of the same length for certain indices i and j. We shall say that $A(i,j)$ (resp. $B(i,j)$) holds for $v(i)$ and $w(j)$ if $v(i)$ and $w(j)$ agree in at least one of the non-final coordinates (resp. in the final (i.e. last) coordinate). We shall say that $A'(i,j)$ (resp. $B'(i,j)$) holds for $v(i)$ and $w(j)$ if $v(i)$ and $w(j)$ do not agree in any of non-final coordinate (resp. in the last coordinate).

- In Case (1) and Case (2), $|i' - j'| = |i - j|$, so if $|i - j| > 1$, then $|i' - j'| > 1$, so $A(i,j)$ holds by induction. If $|i - j| = 1$, then $|i' - j'| = 1$, so $A'(i,j)$ holds by induction and further, i, j being of opposite parity, $B'(i,j)$ holds.
- In Case (3), consider the subcase $i < 2^k$ and $j > 2^k$. Then $|i - j| > 1$. Now $|i' - j'| = |2^{k+1} - i - j|$.
  - If $i' = j'$, i.e. $2^{k+1} = i + j$, all the first $k+1$ corresponding coordinate of $v_{k+1}(i)$ and $v_{k+1}(j)$ are the same, so clearly $A(i,j)$ holds.
  - If $|i' - j'| > 1$, then by induction $A(i,j)$ holds.
  - If $|i' - j'| = 1$, then $|2^{k+1} - i - j| = 1$. Hence i and j are of opposite parity. As $i < 2^k$ and $j > 2^k$ we see that $B(i,j)$ holds.
- In Case (3), consider the subcase $i = 2^k$ and $j > 2^k$.
  - If $|i - j| = 1$, then $j = 2^k + 1$, $|i' - j'| = 1$, so $A'(i,j)$ holds. Also i, j being of opposite parity, $B'(i,j)$ holds.
  - If $|i - j| > 1$, then $j > 2^k + 1$, so $|i' - j'| > 1$, so $A(i,j)$ holds.

In all the cases we see that if $|i - j| = 1$, then $A'(i,j)$ and $B'(i,j)$ hold and if $|i - j| > 1$, then $A(i,j)$ or $B(i,j)$ holds. Thus i and j correspond to adjacent vertices of the M-spine if and only if all the $k+2$ corresponding coordinates of $v_{k+1}(i)$ and $v_{k+1}(j)$ are different.

B: Let $v'_{k+1}(i)$, $v'_{k+1}(j)$ for $2 \le i, j \le 2^{k+1} - 2$, $i \ne j$, be the pendent vertices of the caterpillar R_n. In this case, by induction we can see that $v'_{k+1}(i)$ always agrees with $v'_{k+1}(j)$ in one of the first $k+1$ coordinates, for $k \ge 2$.

C: Let $v_{k+1}(i)$ be a vertex of the M-spine and $v'_{k+1}(j)$ be a pendent vertex of the caterpillar R_n, $0 \le i \le 2^{k+1}$, $2 \le j \le 2^{k+1} - 2$.

- If $i > 2^k$ and $j < 2^k$, we see from definition that $v_{k+1}(i)$ agrees with $v'_{k+1}(j)$ for $2 \le i \le n-2$ in the first coordinate, if i, j are of opposite parity, and in the last coordinate, if i, j are of the same parity. The same holds for $j > 2^k$ and $i < 2^k$.
- If $i = j$, $2 \le i \le 2^{k+1} - 2$, then $v_{k+1}(i)$ and $v'_{k+1}(j)$ are adjacent. Here $v_{k+1}(i)$ defers in all corresponding coordinates of $v'_{k+1}(j)$. This can be clearly seen by induction, in the cases $i < 2^k$, $i = 2^k$, $i > 2^k$ separately.
- If both $i, j < 2^k$ and $i \ne j$, $v_{k+1}(i)$ and $v'_{k+1}(j)$ are non-adjacent vertices. By definition, $v_{k+1}(i)$ does agree with $v'_{k+1}(j)$ in one of the first $k+1$ coordinates by induction. The same holds if both $i, j > 2^k$. This argument holds for $i = 2^k$ and $j < 2^k$ or $j > 2^k$ also.
- If $j = 2^k$ and $0 \le i \le 2^{k+1}$, $i \ne 2^k$, for $i = 2m+1$, i and $2^{k+1}-i$ are odd, so by induction we see that the first 4 coordinates of $v_{k+1}(i)$ are given by $v_2(1) = 111$ or $v_2(3) = 110$. In this case, $v'_{k+1}(j)$ agrees with $v_{k+1}(i)$ in the first coordinate. If $i = 2m$, $0 \le m \le 2^k$, $i \ne 2^{k-1}$, we have the following cases:
  - If $m = 0$ or 2^k, $v_{k+1}(i)$ starts with $v_2(0) = 000$. In this case, by induction, the first $(k+1)$ coordinates of $v_{k+1}(i)$ are zero. Therefore, by the definition of $v'_{k+1}(j)$, it agrees with $v_{k+1}(i)$ in the $(k+1)^{th}$ coordinate, which is 0.
  - If m is odd, $1 \le m \le 2^k - 1$, by induction, $v_{k+1}(i)$ starts with $v_2(2) = 022$. In this case, $v'_{k+1}(j)$ agrees with $v_{k+1}(i)$ in the second coordinate.
  - If m is even, $1 < m < 2^k - 1$, by induction, $v_{k+1}(i)$ starts with $v_2(4) = 001$. In this case, $v'_{k+1}(j)$ agrees with $v_{k+1}(i)$ in the 3^{th} coordinate.

This shows that the caterpillar R_{2^k} can be embedded in $\mathbb{N}^{k+1}$. Thus $\dim(R_{2^k}) \le k+1$. Now if $2^{k-1} < n \le 2^k$, then R_n is an induced subgraph of R_{2^k} and so $\dim(R_n) \le \dim(R_{2^k}) \le k+1 = (\log_2 n)^+ + 1$. Therefore for any value of n,

$$\dim(R_n) \le (\log_2 n)^+ + 1.$$

□

4. Dimension of a caterpillar

In this section we shall get results about dimensions of certain types of caterpillars using results of Sections 2 and 3.
First we get close bounds for the dimension of a general caterpillar considered in Theorem 2.5. Then we consider special types of caterpillars for which we get dimension for most n.

Theorem 4.1. *Let R_n be a caterpillar of diameter n as considered in Theorem 2.5. Then*

$$(\log_2 n)^+ \le \dim(R_n) \le (\log_2(n+2))^+ + 1.$$

If one of the initial and final sets of gap vertices has 2 or more vertices, then $\dim(R_n) \le (\log_2(n+1))^+ + 1$. *If both the initial and final sets of gap vertices have 2 or more vertices then* $\dim(R_n) \le (\log_2 n)^+ + 1$.

Proof. In the notation of Theorem 2.5, there are $n+1$ x^i's and at least $t_0 - 1$ y^i's, so $|V| \ge n+1+t_0-1$. Hence $|V| - t_0 + t_1 \ge (n+t_0) - t_0 + 0 = n$. Hence $(\log_2 n)^+ \le \dim(R_n)$ (or use that $\dim(M\text{-}spine) = (\log_2 n)^+ \le \dim(R_n)$).

Now R_n is an induced subgraph of the caterpillar considered in Theorem 3.1, but having length $n+2$. Hence $\dim(R_n) \le (\log_2(n+2))^+ + 1$.

If any one or both (resp.) of the initial and final sets of gap vertices have 2 or more vertices, then the length $n+2$ can be replaced by $n+1$ or n (resp.) by considering suitable caterpillar in Theorem 3.1. □

Theorem 4.2. *Let R_n, $n \ge 4$, be the caterpillar considered in Section 3. Then* $\dim(R_n) = (\log_2 n)^+ + 1$ *if n is not of the form 2^k+1. For $n = 2^k+1$,* $k+1 \le \dim(R_n) \le k+2$.

Proof. By Theorem 2.5, for R_n, $|V| = 2(n-1)$, $t_0 = t_1 = 0$, so $(\log_2(n-1))^+ + 1 \le \dim(R_n)$. From Section 4, $\dim(R_n) \le (\log_2 n)^+ + 1$. For $n \ne 2^k+1$, both the bounds are equal and so $\dim(R_n) = (\log_2 n)^+ + 1$. For $n = 2^k+1$, $k+1 \le \dim(R_n) \le k+2$. □

Theorem 4.3. *Let R_n, $n \ge 2$, be a caterpillar of length n and let $x^1, x^2, \ldots, x^{n-1}$ be the vertices of the m-spine of R_n such that $deg(x^i) = 3$ for $1 \le i \le n-1$. Then* $\dim(R_n)$ *satisfies the inequality,*

$$(\log_2 n)^+ + 1 \le \dim(R_n) \le (\log_2(n+2))^+ + 1.$$

In particular, $\dim(R_n) = (\log_2 n)^+ + 1$ *if n is not of the form $2^k - 1$ or 2^k, and* $k+1 \le \dim(R_n) \le k+2$ *if $n = 2^k - 1$ or 2^k.*

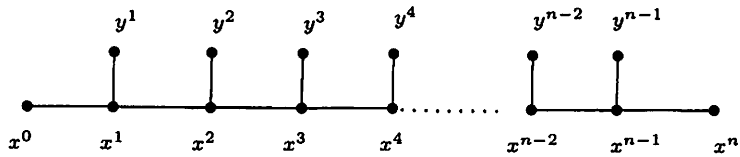

Figure 5

Proof. Here $|V| = 2n$, $t_0 = 2$ and $t_1 = 1$, so $(\log_2(2n-1))^+ \leq \dim(R_n)$. Now $(\log_2(2n-1))^+ = (\log_2(2n))^+ = (\log_2 n)^+ + 1$, so

$$(\log_2 n)^+ + 1 \leq \dim(R_n).$$

Now joining x^0 to a new vertex x^{-1} and x^n to a new vertex x^{n+1}, we get a new caterpillar say R'_{n+2} which is of the same type as Theorem 3.1. R_n being an induced subgraph of R'_{n+2}, we get $\dim(R_n) \leq \dim(R'_{n+2})$. By Theorem 3.1, $\dim(R'_{n+2}) \leq (\log_2(n+2))^+ + 1$. Thus

$$(\log_2 n)^+ + 1 \leq \dim(R_n) \leq (\log_2(n+2))^+ + 1.$$

Hence for n not of the form $2^k - 1$ and 2^k, $\dim(R_n) = (\log_2 n)^+ + 1$ and for $n = 2^k - 1$ or 2^k, $k+1 \leq \dim(R_n) \leq k+2$. □

Now we shall consider a caterpillar (a train-compartment graph) with sets of bunches with $p-1$ leg vertices followed by a gap vertex.

Theorem 4.4. *Let R_n, $n \geq p$, be a caterpillar of length n and let $x^0, x^1, \ldots, x^{n-1}$ be the vertices of the M-spine of R_n and for $1 \leq i \leq n-1$, let $\deg(x^i) = 3$ or 2 according as $p \nmid i$ or $p | i$. For $p \nmid i$, $1 \leq i \leq n$, the pendent vertex adjacent to x^i is denoted by y^i. Let $n \equiv r \pmod{p}$, $0 \leq r \leq p-1$. Let $h = 2$ if $r = 1$, and $h = 1$ if $r = 0, 2, 3, \ldots, p-1$. Then $\dim(R_n)$ satisfies the inequality,*

$$(\log_2(n - \lceil \tfrac{n}{p} \rceil + h))^+ + 1 \leq \dim(R_n) \leq (\log_2(n+2))^+ + 1.$$

For $r = 1$, $(\log_2(n - \lceil \frac{n}{p} \rceil + h))^+ + 1 \leq \dim(R_n) \leq (\log_2(n+1))^+ + 1$.
In particular, for $2^{k-1} + \frac{2^{k-1}-2}{p-1} < n \leq 2^k - 1$, $\dim(R_n) = k+1$.
If $n = 2^k - 1$, where $n \equiv 1 \pmod{p}$, then $\dim(R_n) = k+1$.
Further for $n = 2^{k-1}, 2^k$, $k+1 \leq \dim(R_n) \leq k+2$.
If $2^{k-1} + 1 \leq n \leq 2^{k-1} + \frac{2^{k-1}-2}{p-1}$, $k \leq \dim(R_n) \leq k+1$.

Figure for the case $n = pr$:

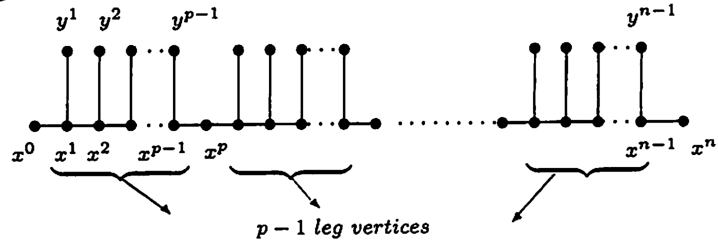

Figure 6

Proof. Here $|V| = 2n - \lceil \frac{n}{p} \rceil + 1$, $t_0 = \lceil \frac{n}{p} \rceil + l$ where $l = -1$ if $r = 1$ and $l = 1$ if $r = 0, 2, 3, \ldots, p-1$, $t_1 = 1$. Then

$$|V| - t_0 - t_1 = \begin{cases} 2n - \lceil \frac{n}{p} \rceil + 1 - \lceil \frac{n}{p} \rceil + 1 + 1 = 2(n - \lceil \frac{n}{p} \rceil + \frac{3}{2}) \text{ if } r = 1, \\ 2n - \lceil \frac{n}{p} \rceil + 1 - \lceil \frac{n}{p} \rceil - 1 + 1 = 2(n - \lceil \frac{n}{p} \rceil + \frac{1}{2}) \text{ if } r \neq 1. \end{cases}$$

Therefore by Theorem 2.5,

$$(\log_2(n - \lceil \tfrac{n}{p} \rceil + h))^+ + 1 \leq \dim(R_n),$$

where $h = 2$ if $r = 1$ and $h = 1$ if $r = 0, 2, 3, \ldots, p-1$.

Now as R_n is an induced subgraph of the caterpillar of Theorem 4.3, say R'_n, we get $\dim(R_n) \leq \dim(R'_n)$. By Theorem 4.3,

$$\dim(R'_n) \leq (\log_2(n+2))^+ + 1.$$

Thus

$$(\log_2(n - \lceil \tfrac{n}{p} \rceil + h))^+ + 1 \leq \dim(R_n) \leq (\log_2(n+2))^+ + 1.$$

Take k so that $2^{k-1} < n + 1 \leq 2^k$. Then the lower bounds will be equal if $2^{k-1} < n - \lceil \frac{n}{p} \rceil + h$, i.e. $2^{k-1} < n - \frac{n+r'}{p} + h$, where $r' = p.\lceil \frac{n}{p} \rceil - n$ $(0 \leq r' \leq p-1)$, i.e. $2^{k-1} < n\frac{p-1}{p} + \frac{ph-r'}{p}$, i.e. $2^{k-1} + \frac{2^{k-1}}{p-1} - \frac{ph-r'}{p-1} < n$.
Now

$$-\frac{ph-r'}{p-1} \begin{cases} = -\frac{2p-(p-1)}{p-1} = -1 - \frac{2}{p-1} & \text{if } r = 1, h = 2, r' = p-1, \\ \leq -\frac{p-(p-2)}{p-1} = -\frac{2}{p-1} & \text{if } r \neq 1, h = 1, 0 \leq r' \leq p-2. \end{cases}$$

Hence in any case we get equality for lower and upper bounds if $2^{k-1} + \frac{2^{k-1}-2}{p-1} < n \leq 2^k - 2$, and then $\dim(R_n) = k + 1$.
When $n \equiv 1 \pmod p$, the final set of non-leg vertices has two vertices, so by Theorem 4.1,

$$(\log_2(n - \lceil \tfrac{n}{p} \rceil + h))^+ + 1 \leq \dim(R_n) \leq (\log_2(n+1))^+ + 1.$$

If $n = 2^k - 1$ and $r = 1$, i.e. $2^k \equiv 2 \pmod p$ then $\dim(R_n) \leq k+1$. Thus $\dim(R_n) = k+1$.

Also $k+1 \leq \dim(R_n) \leq k+2$ if $n = 2^k$, and $k \leq \dim(R_n) \leq k+1$ if $2^{k-1}+1 \leq n \leq 2^{k-1} + \frac{2^{k-1}-2}{p-1}$. □

Example 4.5. If $p = 2$, in Theorem 4.4, then for $n = 2^k - 1$ we have, $2^k - 1 \equiv 1 \pmod 2$, so $\dim(R_n) = k+1$.
If $p = 3$, then for k *odd*, $2^k - 1 \equiv 1 \pmod 3$. Hence $\dim(R_n) = k+1$.

Example 4.6. *Let R_n, $n \geq 3$, be a caterpillar of length n and let $x^0, x^1, \ldots,$ x^n be the vertices of the M-spine and let $\deg(x^i) = 3$ or 2 according as i is even or odd for $1 \leq i \leq n-1$. For even i, y^i is the pendent vertex adjacent to x^i. Then $\dim(R_n)$ satisfies the inequality,*

$$(\log_2(n+2))^+ \leq \dim(R_n) \leq (\log_2 n)^+ + 1.$$

In particular,
if $n = 2^k$ or $2^k - 1$, $\dim(R_n) = k+1$.
If $2^{k-1}+1 \leq n \leq 2^k - 2$, then $k \leq \dim(R_n) \leq k+1$.

Proof. Use Theorem 2.5 and Theorem 4.1. □

ACKNOWLEDGEMENT

The first author thanks S. P. Pune University for support under Research and Development Grant.

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