Decomposition of a $\lambda K_{m,n}$ into Graphs of Four Vertices and Five Edges

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ABSTRACT. Recently the authors proposed a fundamental theorem for the decomposing of a complete bipartite graph. They applied the theorem to obtain complete results on the decomposition of a complete bipartite graph into connected subgraphs on four vertices and up to four edges. In this paper, we decompose a complete multi-bipartite graph into its subgraphs of four vertices and five edges. We show that necessary conditions are sufficient for the decompositions, with some exceptions where decompositions do not exist.

1. Introduction

The decomposition problem of a graph into subgraphs all of which belong to a specific class of graphs has been well studied where the subgraphs are simple (see [1], [2], and references therein). We consider connected graphs G with vertex set V of size/order n and edge set E of size e, and we allow the edges to occur with a frequency greater or equal to 1. By λ copies of a simple graph G, denoted by λG, we mean the graph with the same vertex set of G with each edge of G having multiplicity λ. For example, a λK_n is a λ-fold complete multigraph of order n and a $\lambda K_{m,n}$ is a λ-fold complete bipartite graph with V partitioned into two subsets V_1 and V_2 such that the size of V_1 equals m and the size of V_2 equals n. The decomposition of copies of a complete graph into proper multigraphs has received some attention, see [**3**, **4**, **5**, **6**, **8**, **9**], but the decomposition problem of a complete bipartite graph was completely open till [**7**].

D. G. Sarvate wishes to thank the College of Charleston for granting sabbatical and Mbarara University of Science and Technology, Uganda for their hospitality. P. A. Winter wishes to thank the Howard College for its support. L. Zhang wishes to thank The Citadel Foundation for its support.

In [7], Sarvate, Winter and Zhang proposed a fundamental theorem (Theorem 1) on the decomposition of a $\lambda K_{m,n}$ into isomorphic subgraphs, and applied the theorem to settle graph decomposition problem for several subgraphs with number of vertices less than or equal to 4 and the number of edges less than or equal to 4. In this paper, we extend their study to the decomposition problem for subgraphs with number of vertices equaling four and number of edges equaling five. All graphs considered in this paper are connected graphs. We show that necessary conditions are sufficient for the decompositions, with some exceptions where decompositions do not exist.

THEOREM 1. *(The fundamental theorem for the decomposition of a $\lambda K_{m,n}$ [7]): If a decomposition of a $\lambda K_{t,s}$ into a subgraph H exists, then a decomposition of a $\lambda\mu K_{pt,qs}$ into subgraphs H exists for any positive integers μ, p and q.*

COROLLARY 1. [7] *If a decomposition of a $\lambda K_{a,n}$ and a decomposition of a $\lambda K_{b,n}$ are known where $a, b = 1, 2$ or $2, 3$, then we know a decomposition of a $\lambda K_{m,n}$ for any positive integer m.*

COROLLARY 2. [7] *If a decomposition of a $\lambda K_{a,c}$ and a decomposition of a $\lambda K_{b,d}$ are known where $a, b = 1, 2$ or $2, 3$, same for c and d, then we know a decomposition of a $\lambda K_{m,n}$ for any positive integers m and n.*

REMARK 1. [7] *Essentially a $\lambda K_{mt,ns}$ is $t \times s$ copies of disjoint $\lambda K_{m,n}$. Therefore, once a decomposition of a $\lambda K_{m,n}$ is known, we know a decomposition of a $\lambda K_{mt,ns}$. Similarly, if a decompositions of a $\lambda K_{m,n}$ and a $\mu K_{m,n}$ are known, then we know a decomposition of a $(a\lambda + b\mu)K_{m,n}$.*

We will refer to the fundamental theorem and its corollaries and remarks as the FT in the rest of the paper.

2. Decompositions of a $\lambda K_{m,n}$ into graphs with 4 vertices and 5 edges

Figure 1 includes all connected graphs with 4 vertices and 5 edges.

Each of the last four graphs on the second row in Figure 1 has an odd cycle, the decomposition of a $\lambda K_{m,n}$ into any of these graphs does not exist (since bipartite graphs do not have odd cycles).

2.1. OLO graph decompositions.

DEFINITION 1. *Let $V = \{a, b, c, d\}$. An OLO graph $< a, b, c, d >$ on V is a graph where the frequency of edges $\{a, b\}$ and $\{b, c\}$ and $\{c, d\}$ are 2, 1 and 2, respectively (see the first graph in the first row of Figure 1 for an example).*

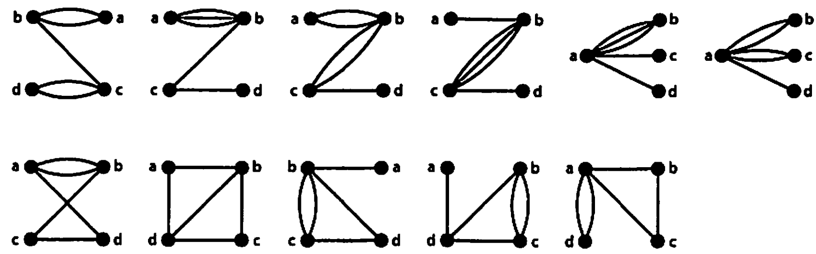

FIGURE 1. All connected graphs on 4 vertices and 5 edges

The necessary conditions for a decomposition of a $\lambda K_{m,n}$ into OLO graphs are $m \geq 2$ and $n \geq 2$ and λmn is 0(mod 5).

If $\lambda = 2$, the necessary conditions are $m \geq 2$ and $n \geq 2$ and mn is 0(mod 5). A $2K_{2,5t}$ over $V_1 = \{a, b\}$ and $V_2 = \{1, 2, \ldots, 5t\}$ can not be decomposed into OLO graphs. Suppose one of the OLO graphs in the decomposition is $< a, 1, b, 2 >$, then it is impossible to have another OLO graph in the decomposition with $\{b, 1\}$ as a single edge because the edge $\{a, 1\}$ has already occurred twice in the OLO graph $< a, 1, b, 2 >$.

The decomposition of a $2K_{3,5}$ over $V_1 = \{1, 2, 3\}$ and $V_2 = \{4, 5, 6, 7, 8\}$ is given by $\{2416, 3415, 5371, 6372, 5283, 6281\}$ where $abcd$ represents the OLO graph $< a, b, c, d >$. The decomposition of a $2K_{4,5}$ over $V_1 = \{1, 2, 3, 4\}$ and $V_2 = \{5, 6, 7, 8, 9\}$ is given by $\{6152, 7153, 7263, 8264, 9384, 7381, 5492, 7491\}$.

The decomposition of a $2K_{5,5}$ is given as follows using the difference sets concept. Assume $V_1 = \{0\} \times Z_5$ (i.e. $\{(0,0), (0,1), (0,2), (0,3), (0,4)\}$) and $V_2 = \{1\} \times Z_5$ (0 and 1 just help us know which point is in which part). Note differences between V_1 and V_2 are $0, 1, 2, 3$ and 4. Consider the difference sets $< (0,0), (1,0), (0,1), (1,3) >$ and $< (0,4), (1,0), (0,1), (1,4) >$. The first difference set gives these five graphs for the decomposition: $< (0,0), (1,0), (0,1), (1,3) >$, $< (0,1), (1,1), (0,2), (1,4) >$, $< (0,2), (1,2), (0,3), (1,0) >$, $< (0,3), (1,3), (0,4), (1,1) >$, $< (0,4), (1,4), (0,0), (1,2) >$. Note this takes care of pairs with difference 0 twice and also pairs with difference 2, but pairs with difference 4 occur only once, next difference set takes care of the remaining pairs with differences 1 and 3 twice and 4 once. These decompositions can be used with the FT to obtain decomposition for any $2K_{n,5t}$ for any $n > 2$.

If $\lambda = 3$, the necessary conditions are $m \geq 2$ and $n \geq 2$ and mn is $0 \pmod 5$. A $3K_{m,5t}$ cannot be decomposed into OLO graphs. If a decomposition exists, there should be $3mt$ OLO graphs and $6mt$ double edges in the decomposition (since each OLO graph has two double edges). An edge can be a double edge in at most one of the OLO graph since $\lambda = 3$. In a $3K_{m,5t}$, there are only $5mt$ distinct edges but we need $6mt$ distinct edges as double edges in a decomposition.

If $\lambda = 5$, the necessary conditions are $m \geq 2$ and $n \geq 2$. Note that a $5K_{2,2}$ (on $V_1 = \{1,2\}$ and $V_2 = \{3,4\}$) can be decomposed into OLO graphs $\{1324, 3142, 4132, 1423\}$. A $5K_{2,3}$ (on $V_1 = \{1,2\}$ and $V_2 = \{3,4,5\}$) can be decomposed into six OLO graphs $\{4132, 5142, 3152, 4231, 5241, 3251\}$. Using the FT, we can decompose any $5K_{a,b}$ for a and b greater than 1.

We showed earlier that a decomposition of a $2K_{2,5t}$ into OLO graphs does not exist. For a $4K_{2,5}$ over $V_1 = \{a,b\}$ and $V_2 = \{1,2,3,4,5\}$, it can be decomposed into eight OLO graphs $\{5a1b, 5a2b, 2a1b, 1a2b, a3b5, a4b5, a3b4, a4b3\}$. A $6K_{2,5}$ over $V_1 = \{a,b\}$ and $V_2 = \{1,2,3,4,5\}$ can be decomposed into 12 OLO graphs $\{a1b4, a1b4, a2b4, a2b5, a3b5, a3b5, 4a1b, 4a1b, 4a2b, 5a2b, 5a3b, 5a3b\}$. A $7K_{2,5}$ over $V_1 = \{a,b\}$ and $V_2 = \{1,2,3,4,5\}$ can be decomposed into 14 OLO graphs $\{a1b2, a2b3, a3b4, a4b5, a5b3, a2b4, a2b5, 3a1b, 1a2b, 4a3b, 5a4b, 3a5b, 4a1b, 5a1b\}$. By the FT, a $\lambda K_{2,5t}$ can be decomposed into OLO graphs if $\lambda \geq 4$. Applying the FT, we conclude the results obtained in this section in the following theorem.

THEOREM 2. *The necessary conditions of decomposing a $\lambda K_{m,n}$ into OLO graphs are sufficient except for $\lambda = 3$ and a $2K_{2,5t}$ where a decomposition does not exist .*

2.2. Lamp graphs decompositions.

DEFINITION 2. *The multigraph $< a,b,c,d >$ where $\{a,b\}$ is an edge with multiplicity 3 and remaining two edges $\{b,c\}$ and $\{c,d\}$ with multiplicity 1 is called a Lamp graph (see the second graph in the first row of Figure 1 for an example). We write abcd to denote an Lamp graph $< a,b,c,d >$ when there is no confusion.*

The necessary conditions for a decomposition of a $\lambda K_{m,n}$ into Lamp graphs are $\lambda \geq 3$ and $m \geq 2$ and $n \geq 2$ and λmn is $0 \pmod 5$.

For $\lambda = 3$, a $3K_{m,n}$ can be decomposed into Lamp graphs only if either m or n is divisible by 5. A $3K_{5,2}$ on $V_1 = \{1,2,3,4,5\}$ and $V_2 = \{6,7\}$ can be decomposed into six Lamp graphs $\{1647, 264, 3647, 1756, 2756, 3756\}$. A $3K_{5,3}$ on $V_1 = \{1,2,3,4,5\}$ and $V_2 = \{a,b,c\}$ can be decomposed into nine Lamp graphs $\{1a4b, 2a4b, 3a4b, 5b1c, 2b1c, 3b1c, 2c5a, 3c5a, 4c5a\}$. Using the FT, a decomposition of a $3K_{5t,n}$ into Lamp graphs exists.

A $4K_{5,2}$ on $V_1 = \{2,3,4,5,6\}$ and $V_2 = \{0,1\}$ can be decomposed into eight Lamp graphs $\{0216, 0316, 0416, 0516, 1206, 1306, 1406, 1506\}$. A $4K_{5,3}$ on $V_1 = \{3,4,5,6,7\}$ and $V_2 = \{0,1,2\}$ can be decomposed into 12 Lamp graphs $\{0317, 0417, 0517, 0617, 1327, 1427, 1527, 1627, 2307, 2407, 2507, 2607\}$. Using the FT, a decomposition of a $3K_{5t,n}$ into Lamp graphs exists.

A $5K_{2,2}$ on $V_1 = \{2,4\}$ and $V_2 = \{1,3\}$ can be decomposed into four Lamp graphs $\{1234, 3214, 1432, 3412\}$. A $5K_{3,2}$ on $V_1 = \{1,2,3\}$ and $V_2 = \{4,5\}$ can be decomposed six Lamp graphs $\{5142, 5243, 5341, 4152, 4253, 4351\}$. A $5K_{3,3}$ on $V_1 = \{0,1,2\}$ and $V_2 = \{3,4,5\}$ can decomposed into nine Lamp graphs $\{0315, 1423, 2504, 0413, 1524, 2305, 0514, 1325, 2403\}$. Using the FT, a decomposition of a $5K_{m,n}$ into Lamp graphs exists. Using the FT, we have the following result.

THEOREM 3. *Necessary conditions are sufficient for a decomposition of a $\lambda K_{m,n}$ into Lamp graphs.*

2.3. OOL graph decompositions.

DEFINITION 3. *Let $V = \{a,b,c,d\}$. An OOL graph $< a,b,c,d >$ on V is a graph where the frequency of edges $\{a,b\}$ and $\{b,c\}$ and $\{c,d\}$ are 2, 2 and 1, respectively (see the third graph in the first row of Figure 1 for an example). We write abcd to denote an OOL graph $< a,b,c,d >$ when there is no confusion.*

The necessary conditions for the decomposition of a $\lambda K_{m,n}$ into *OOL* graphs are $m \geq 2$ and $n \geq 2$ and λmn is 0(mod 5).

If $\lambda = 2$, the necessary conditions are $m \geq 2$ and $n \geq 2$ and m or n is 0(mod 5). A $2K_{2,5}$ on $V_1 = \{a,b\}$ and $V_2 = \{1,2,3,4,5\}$ can be decomposed into four *OOL* graphs $\{a1b2, a3b2, b4a2, b5a2\}$. A $2K_{3,5}$ on $V_1 = \{a,b,c\}$ and $V_2 = \{1,2,3,4,5\}$ can be decomposed into six *OOL* graphs $\{1a2b, 4a3c, c4b2, b1c3, 3b5a, 2c5a\}$. By the FT, a decomposition of a $2K_{m,5t}$ into *OOL* graphs exists.

If $\lambda = 3$, the necessary conditions are $m \geq 2$ and $n \geq 2$ and m or n is 0(mod 5). A $3K_{m,5t}$ cannot be decomposed into *OOL* graphs. If a decomposition exists, there should be $3mt$ *OOL* graphs and $6mt$ double edges in the decomposition (since each *OOL* graph has two double edges). An edge can be a double edge in at most one of the *OOL* graph since $\lambda = 3$. In a $3K_{m,5t}$, there are only $5mt$ distinct edges but we need $6m$ distinct edges as double edges in a decomposition.

If $\lambda = 5$, the necessary conditions are $m \geq 2$ and $n \geq 2$. A $5K_{2,3}$ on $V_1 = \{a, b\}$ and $V_2 = \{1, 2, 3\}$ can be decomposed into six OOL graphs $\{b2a1, b3a2, b1a3, a2b1, a3b2, a1b3\}$. A $5K_{3,3}$ on $V_1 = \{a, b, c\}$ and $V_2 = \{1, 2, 3\}$ can be decomposed into nine OOL graphs $\{b2a1, c3a2, b1a3, c2b1, a3b2, c1b3, a2c1, b3c2, a1c3\}$. By the FT, a decomposition of a $5K_{m,n}$ into OOL graphs exists. Also, any $\lambda > 3$ can be written as a linear combination of 2 and 5, by the FT, a decomposition of a $\lambda K_{m,n}$ into OOL graphs exists. We conclude the results obtained in this section in the following theorem.

THEOREM 4. *The necessary conditions of decomposing a $\lambda K_{m,n}$ into OOL graphs are sufficient except for $\lambda = 3$ where a decomposition does not exist.*

2.4. LEL graph decompositions.

DEFINITION 4. *Let $V = \{a, b, c, d\}$. An LEL (phi) graph $< a, b, c, d >$ on V is a graph where the frequency of edges $\{a, b\}$ and $\{b, c\}$ and $\{c, d\}$ are $1, 3$ and 1, respectively (see the fourth graph in the first row of Figure 1 for an example). We write abcd to denote an LEL graph $< a, b, c, d >$ when there is no confusion.*

The necessary conditions for the decomposition of a $\lambda K_{m,n}$ into LEL graphs are $m \geq 2$, $n \geq 2$, $\lambda \geq 3$ and λmn is 0(mod 5).

For $\lambda = 4$, the necessary condition $4mn \equiv 0 \pmod 5$ implies that either m or $n \equiv 0 \pmod 5$. Both a $4K_{2,5}$ (on $V_1 = \{a, b\}$ and $V_2 = \{1, 2, 3, 4, 5\}$) and a $4K_{3,5}$ (on $V_1 = \{a, b, c\}$ and $V_2 = \{1, 2, 3, 4, 5\}$) can be decomposed into eight LEL graphs and twelve LEL graphs, respectively, as follows: $\{b1a5, b2a5, b3a5, b4a5, a1b5, a2b5, a3b5, a4b5\}$ and $\{b1a5, b2a5, b3a5, b4a5, c1b5, c2b5, c3b5, c4b5, a1c5, a2c5, a3c5, a4c5\}$. By the FT, a decomposition of a $4K_{m,5s}$ into LEL graphs exists.

For $\lambda = 5$, the necessary condition $5mn \equiv 0 \pmod 5$ implies that there is no additional condition for m and n except for both of them need to be greater than or equal to 2. A $5K_{2,2}$ (on $V_1 = \{a, b\}$ and $V_2 = \{1, 2\}$), a $5K_{2,3}$ (on $V_1 = \{a, b\}$ and $V_2 = \{1, 2, 3\}$) and a $5K_{3,3}$ (on $V_1 = \{a, b, c\}$ and $V_2 = \{1, 2, 3\}$) can be decomposed into four, six and nine LEL graphs, respectively, as follows: $\{b1a2, 1a2b, 1b2a, a1b2\}$, $\{a1b2, a2b3, a3b1, b1a2, b2a3, b3a1\}$ and $\{b1a2, b2a3, b3a1, c1b2, c2b3, c3b1, a1c2, a2c3, a3c1\}$. By the FT, a decomposition of a $5K_{m,n}$ into LEL graphs exists.

For $\lambda = 3$, the necessary condition $3mn \equiv 0 \pmod 5$ implies that either m or $n \equiv 0 \pmod 5$. A $3K_{2,5t}$ on $V_1 = \{a, b\}$ and $V_2 = \{1, \ldots, 5t\}$ cannot be decomposed into LEL graphs. If a decomposition exists, since $\lambda = 3$, an edge will only appear as a triple edge in an LEL graph or a single edge in three LEL graphs in the decomposition. Also, there should be $6t$ LEL graphs in the decomposition. Notice that each LEL graph should have one single edge with a and one single edge with b and one triple edge with a or with b. By the pigeonhole principle, without loss of generality, we can assume that at least $3t$ LEL graphs in the decomposition have a triple edge with a, i.e., there are $3t$ distinct edges $\{a, i\}$ occurring triply and hence $3t$ distinct edges $\{i, b\}$ occurring singly. On the other hand, there are $6t$ LEL graphs and therefore $6t$ single edges with b in the decomposition, that is, there can only be $\frac{6t}{3} = 2t$ distinct single edges with b. This is a contradiction. Thus, a decomposition of a $3K_{2,5t}$ into LEL graphs does not exist.

A $3K_{3,5}$ on $V_1 = \{a, b, c\}$ and $V_2 = \{1, 2, 3, 4, 5\}$ also cannot be decomposed into LEL graphs. Notice that each vertex in V_1 has degree 15 and each vertex in V_2 has degree 9. If an LEL decomposition exists, there should be 9 LEL graphs. Thus, each vertex in V_1 must appear in exactly three LEL graphs as degree 4. Also, same vertex from V_2 can not occur with all three vertices a, b and c to form triple edges. Hence one vertex in V_2, say 5 occurs with only one vertex in V_1, say a. Now b and c must occur with 5 to form single edges. Without loss of generality, suppose an LEL graph has $\{a, 5\}$ as the triple edge, i.e. $< x, a, 5, b >$ where x is any vertex from V_2 which has not occurred with a to form triple edges. Hence c must occur with 5 singly in the three graphs where c has degree 4 with three other vertices from V_2. Then it is impossible to let c occur with the fifth element from V_2 three more times as that element occurs at most twice: once with a and once with b triply to let c occur with it singly.

A $3K_{3,10}$ on $V_1 = \{a, b, c\}$ and $V_2 = \{1, \ldots, 9, x\}$ can be decomposed into 18 LEL graphs $\{xa1b, xa2b, xa3b, xc1b, xc2b, xc3b, xb7a, xb8a, xb9a, 4c7a, 5c8a, 6c9a, 7a4c, 8a5c, 9a6c, 1b4c, 2b5c, 3b6c\}$. A $3K_{3,15}$ on $V_1 = \{a, b, c\}$ and $V_2 = \{1, \ldots, 9, u, v, w, x, y, z\}$ can be decomposed into 27 LEL graphs $\{xa1b, xa2b, ya3b, za4b, wa5c, va6c, ua7c, ua8c, ua9c, 1b6c, 2b7c, 3b8c, 4b9c, ubxa, ubya, 5bza, 5bwa, 5bva, 5cya, 6cza, 7cwa, 8cva, 9cub, xc1b, xc2b, xc3b, 5c4b\}$. By the FT, a $3K_{3,5t}$ can be decomposed into LEL graphs except for a $3K_{3,5}$ where a decomposition does not exist.

A $3K_{4,5}$ on $V_1 = \{a, b, c, d\}$ and $V_2 = \{1, 2, 3, 4, 5\}$ can be decomposed into twelve LEL graphs $\{c1a5, c2a5, d3a5, c2b1, d3b1, a4b5, d3c2, a4c1, b5c1, a4d2, b5d2, b1d2\}$. A $3K_{6,5}$ on $V_1 = \{a, b, c, d, e, f\}$ and $V_2 = \{1, 2, 3, 4, 5\}$ can be decomposed into 18 LEL graphs $\{e1a4, f3a4, d2a5, f3b4, d5b4, e2b1, d5c4, e1c4, f2c3, b1d2, c3d2, a4d5, c3e2, a5e2, b4e1, a5f2, b1f2, c4f3\}$. Since any even number $2m > 4$ can be written as a linear combination of 4 and 6, by the FT, a $3K_{2m,5}$ and hence a $3K_{2m,5t}$ can be decomposed into LEL

graphs.

A $3K_{5,5}$ on $V_1 = \{a,b,c,d,e\}$ and $V_2 = \{1,2,3,4,5\}$ can be decomposed into 15 LEL graphs $\{b1a4, c2a4, d3a4, c2b5, d3b5, e4b5, d3c1, e4c1, a5c1, e4d2, a5d2, b1d2, a5e3, b1e3, c2e3\}$. A $3K_{7,5}$ on $V_1 = \{1,\ldots,7\}$ and $V_2 = \{a,\ldots,e\}$ can be decomposed into 21 LEL graphs $\{c1a5, b2a6, b3a5, d4a7, a5b2, c6b4, c7b4, c1b4, b2c6, b3c6, d4c7, d5c7, a6d3, a7d3, c1d3, e2d5, b3e1, d4e1, d5e1, a6e2, a7e2\}$. By the FT, a $3K_{2m+1,5}$ (for $2m+1 \geq 5$) and hence a $3K_{2m+1,5t}$ can be decomposed into LEL graphs (since any odd number greater than 7 can be written as a linear combination of 5 and an even number). Combining with the case of a $3K_{2m,5t}$, a $3K_{n,5t}$ for $n \geq 4$ can be decomposed into LEL graphs. Furthermore, combining the result obtained for a decomposition of a $3K_{3,5t}$ into LEL graphs, a $3K_{n,5t}$ for $n \geq 3$ can be decomposed into LEL graphs except for a $3K_{3,5}$ where a decomposition does not exist.

A $6K_{2,5t}$ on $V_1 = \{a,b\}$ and $V_2 = \{1,2,3,4,5,\ldots,5t\}$ can not be decomposed into LEL graphs. Note that degree of any vertex in V_2 is 12. An LEL graph in a decomposition must have two vertices from V_2, one of degree one and the other of degree four. No vertex in V_2 can appear more than three times as degree four as degree is 12. Also, we show below that no vertex can appear as degree four in three LEL graphs in a decomposition. Assume a decomposition into LEL graphs exists, then there are $6 \times 2 \times t = 12t$ LEL graphs in the decomposition. Without loss of generality, suppose vertex 1 appears as degree four in three LEL graphs, and the first two LEL graphs are $< *, a, 1, b >$ and $< *, a, 1, b >$ where $*$ is some vertex in V_2. Now we can not complete the third LEL graph $< *, b, 1, x >$ as $\lambda = 6$ and 1 has already occurred with a six times. Thus, a vertex from V_2 can appear as degree four only in at most two LEL graphs, so the total number of LEL graphs in the decomposition is at most $5t \times 2 = 10t$, a contradiction to $12t$ LEL graphs in a decomposition if it exists.

A $6K_{3,5}$ on $V_1 = \{a,b,c\}$ and $V_2 = \{1,2,3,4,5\}$ can be decomposed into 18 LEL graphs $\{b1a2, b1a2, c2a4, b3a5, b4a5, b5a4, c2b3, c2b3, a3b4, c1b5, c4b5, c5b4, a3c4, a3c4, b1c5, a2c5, a4c1, a5c1\}$. Both a $6K_{4,5}$ and a $6K_{5,5}$ can be decomposed into LEL graphs by the FT since an LEL decomposition exists for a $3K_{4,5}$ and for a $3K_{5,5}$, respectively. By the FT, a $6K_{n,5t}$ can be decomposed into LEL graphs except for a $6K_{2,5t}$ where a decomposition does not exist.

A $7K_{2,5}$ on $V_1 = \{a,b\}$ and $V_2 = \{1,2,3,4,5\}$ can be decomposed into 14 LEL graphs $\{b3a5, b1a4, b1a5, b2a4, b2a3, b4a3, b5a3, a1b3, a2b3, a3b2, a4b2, a4b3, a5b1, a5b1\}$. A $7K_{3,5}$ on $V_1 = \{a,b,c\}$ and $V_2 = \{1,2,3,4,5\}$ can be decomposed into 21 LEL graphs $\{c1a2, c1a4, b2a4, b2a5, b3a5, b4a5, b5a4,$

$c1b4, c1b4, a3b5, a3b5, c2b1, c4b5, c5b4, b2c3, b2c4, a3c5, a3c5, a1c4, a4c5, a5c4\}$. By the FT, an LEL decomposition exists for a $7K_{n,5}$ and hence a $7K_{n,5t}$.

Also, by the FT, an LEL decomposition of a $\lambda K_{2,5t}$ exists except for a $3K_{2,5t}$ and a $6K_{2,5t}$ (notice that an LEL decomposition of a $\lambda K_{2,5t}$ exists for $\lambda = 4, 5$ and 7, and any $\lambda > 7$ can be written as a linear combination of 4, 5 and 7). An LEL decomposition of a $\lambda K_{3,5t}$ exists except for a $3K_{3,5}$ (notice that an LEL decomposition of a $\lambda K_{3,5t}$ exists for $\lambda = 4, 5, 6$ and 7, and any $\lambda > 7$ can be written as a linear combination of 4, 5 and 6). Recall that an LEL decomposition of a $\lambda K_{n,5t}$ for $n \geq 4$ exists and an LEL decomposition of a $5tK_{m,n}$ exists by the FT. We summarize the results obtained in this section in the following theorem.

THEOREM 5. *The necessary conditions of decomposing a $\lambda K_{m,n}$ into LEL graphs are sufficient except for a $3K_{2,5t}$, a $6K_{2,5t}$ and a $3K_{3,5}$ where a decomposition does not exist.*

2.5. $ELL_{1,3}$ graph decompositions.

DEFINITION 5. *Let $V = \{a, b, c, d\}$. An $ELL_{1,3}$ graph $< a, b, c, d >$ on V is a graph where the frequency of edges $\{a, b\}$ and $\{a, c\}$ and $\{a, d\}$ are $3, 1$ and 1, respectively (see the fifth graph in the first row of Figure 1 for an example). We write abcd to denote an $ELL_{1,3}$ graph $< a, b, c, d >$ when there is no confusion.*

The necessary conditions for the decomposition of a $\lambda K_{m,n}$ into $ELL_{1,3}$ graphs are $m \geq 3$ or $n \geq 3$ and $\lambda \geq 3$ and λmn is 0(mod 5).

A $3K_{1,5}$ on $V_1 = \{a\}$ and $V_2 = \{1, 2, 3, 4, 5\}$ can be decomposed into three $ELL_{1,3}$ graphs $\{a145, a245, a345\}$. By the FT, a decomposition of a $3K_{m,5t}$ into $ELL_{1,3}$ graphs exists.

A $4K_{1,5}$ on $V_1 = \{a\}$ and $V_2 = \{1, 2, 3, 4, 5\}$ can be decomposed into four $ELL_{1,3}$ graphs $\{a312, a413, a514, a215\}$. By the FT, a decomposition of a $4K_{m,5t}$ into $ELL_{1,3}$ graphs exists.

For $\lambda = 5$, the necessary condition $5mn \equiv 0(\text{mod } 5)$ implies that there is no additional condition for m and n except for one of them needs to be greater than or equal to 3. A $5K_{1,3}$ (on $V_1 = \{a\}$ and $V_2 = \{1, 2, 3\}$), a $5K_{1,4}$ (on $V_1 = \{a\}$ and $V_2 = \{1, 2, 3, 4\}$) and a $5K_{1,5}$ (on $V_1 = \{a\}$ and $V_2 = \{1, 2, 3, 4, 5\}$) can be decomposed into three, four and five $OOL_{1,3}$ graphs, respectively, as follows: $\{a312, a213, a123\}$, $\{a312, a412, a134, a234\}$ and $\{a312, a423, a534, a145, a215\}$. By the FT, a decomposition of a $5K_{m,n}$ into $OOL_{1,3}$ graphs exists. Also, by the FT, a $\lambda K_{m,5t}$ ($\lambda \geq 3$) can be decomposed into $ELL_{1,3}$ graphs (since λ can be written as a linear combination of 3, 4 and 5). We conclude the results obtained in this section in the following theorem.

THEOREM 6. *The necessary conditions of decomposing a $\lambda K_{m,n}$ into $ELL_{1,3}$ graphs are sufficient.*

2.6. $OOL_{1,3}$ graph decompositions.

DEFINITION 6. *Let $V = \{a, b, c, d\}$. An $OOL_{1,3}$ graph $< a, b, c, d >$ on V is a graph where the frequency of edges $\{a, b\}$ and $\{a, c\}$ and $\{a, d\}$ are $2, 2$ and 1, respectively (see the last graph in the first row of Figure 1 for an example). We write abcd to denote an $OOL_{1,3}$ graph $< a, b, c, d >$ when there is no confusion.*

The necessary conditions for the decomposition of a $\lambda K_{m,n}$ into $OOL_{1,3}$ graphs are $m \geq 3$ or $n \geq 3$ and λmn is 0(mod 5).

A $2K_{1,5}$ on $V_1 = \{a\}$ and $V_2 = \{1, 2, 3, 4, 5\}$ can be decomposed into two $OOL_{1,3}$ graphs $\{a123, a453\}$. By the FT, a decomposition of a $2K_{m,5t}$ into $OOL_{1,3}$ graphs exists.

A $3K_{m,5t}$ cannot be decomposed into $OOL_{1,3}$ graphs. If a decomposition exists, there should be $3mt$ $OOL_{1,3}$ graphs and $6mt$ double edges in the decomposition (since each $OOL_{1,3}$ graph has two double edges). An edge can be a double edge in at most one of the $OOL_{1,3}$ graph since $\lambda = 3$. In a $3K_{m,5t}$, there are only $5mt$ distinct edges but we need $6m$ distinct edges as double edges in a decomposition.

For $\lambda = 5$, the necessary condition $5mn \equiv 0(\text{mod } 5)$ implies that there is no additional condition for m and n except for one of them needs to be greater than or equal to 3. A $5K_{1,3}$ (on $V_1 = \{a\}$ and $V_2 = \{1, 2, 3\}$), a $5K_{1,4}$ (on $V_1 = \{a\}$ and $V_2 = \{1, 2, 3, 4\}$) and a $5K_{1,5}$ (on $V_1 = \{a\}$ and $V_2 = \{1, 2, 3, 4, 5\}$) can be decomposed into three, four and five $OOL_{1,3}$ graphs, respectively, as follows: $\{a123, a231, a132\}$, $\{a123, a234, a341, a142\}$ and $\{a123, a234, a345, a451, a152\}$. By the FT, a decomposition of a $5K_{m,n}$ into $OOL_{1,3}$ graphs exists (since any number $n > 2$ can be written as a linear combination of $3, 4$ and 5).

By the FT, a $\lambda K_{m,n}$ ($\lambda \neq 3$) can be decomposed into $OOL_{1,3}$ graphs (since λ can be written as a linear combination of 2 and 5). We conclude the results obtained in this section in the following theorem.

THEOREM 7. *The necessary conditions of decomposing a $\lambda K_{m,n}$ into $OOL_{1,3}$ graphs are sufficient except for $\lambda = 3$ where a decomposition does not exist.*

2.7. OX graph decompositions.

DEFINITION 7. *Let* $V = \{a, b, c, d\}$. *An* OX *graph* $< a, b, c, d >$ *on* V *is a graph where the frequency of edges* $\{a, b\}$ *and* $\{b, c\}$ *and* $\{c, d\}$ *and* $\{d, a\}$ *are* $2, 1, 1$ *and* 1, *respectively (see the first graph in the first row of Figure 1 for an example). We write abcd to denote an* OX *graph* $< a, b, c, d >$ *when there is no confusion.*

Notice that an OX graph has a cycle of length four. The necessary conditions for the decomposition of a $\lambda K_{m,n}$ into OX graphs are $m \geq 2$ and $n \geq 2$ and λmn is $0(\text{mod } 5)$.

A $2K_{2,5t}$ (on $V_1 = \{a, b\}$ and $V_2 = \{1, \ldots, 5t\}$) cannot be decomposed into OX graphs. Notice that the degree of a vertex in an OX graph is either two or three. The degree of each vertex in V_2 is four, but the degree of some vertex in V_2 has to be three in an OX graph in the decomposition, so it is impossible for that vertex to have a total degree of four in the decomposition.

A $2K_{3,5}$ (on $V_1 = \{a, b, c\}$ and $V_2 = \{1, 2, 3, 4, 5\}$) and a $2K_{4,5}$ (on $V_1 = \{a, b, c, d\}$ and $V_2 = \{1, 2, 3, 4, 5\}$) can be decomposed into six and eight OX graphs, respectively: $\{a1c4, a2b5, b1c5, b3a4, c2b4, c3a5\}$ and $\{a1b3, a2b3, b4d1, b5c2, c1d3, c4a5, d5a4, d2c3\}$. A $2K_{5,5}$ on $V_1 = \{a, b, c, d, e\}$ and $V_2 = \{1, 2, 3, 4, 5\}$ can be decomposed into 10 OX graphs $\{a1c5, a2d3, b2d1, b3e4, c3e2, c4a5, d4a3, d5k1, e5b4, e1c2\}$. By the FT, a $2K_{n,5t}$ for $n \geq 3$ can be decomposed into OX graphs.

A $3K_{2,5}$ (on $V_1 = \{a, b\}$ and $V_2 = \{1, 2, 3, 4, 5\}$) can be decomposed into six OX graphs $\{a1b4, a2b4, a3b4, b1a5, b2a5, b3a5\}$. A $3K_{3,5}$ (on $V_1 = \{a, b, c\}$ and $V_2 = \{1, 2, 3, 4, 5\}$) can be decomposed into nine OX graphs $\{c1b2, a2c3, b3a1, a4b1, b4c2, c4a3, a5b2, b5c3, c5a1\}$. By the FT, a $3K_{n,5t}$ can be decomposed into OX graphs.

For $\lambda = 5$, the necessary condition $5mn \equiv 0(\text{mod } 5)$ implies that there is no condition on m and n except for $m \geq 2$ and $n \geq 2$. A $5K_{2,2}$ on $V_1 = \{a, b\}$ and $V_2 = \{1, 2\}$ can be decomposed into four OX graphs $\{a1b2, a2b1, b1a2, b2a1\}$. A $5K_{2,3}$ on $V_1 = \{a, b\}$ and $V_2 = \{1, 2, 3\}$ can be decomposed into six OX graphs $\{a1b2, a2b3, a3b1, b1a2, b2a3, b3a1\}$. A $5K_{3,3}$ on $V_1 = \{a, b, c\}$ and $V_2 = \{1, 2, 3\}$ can be decomposed into nine OX graphs $\{a1b2, a2c3, a3b1, b1c2, b2a3, b3c1, c1a2, c2b3, c3a1\}$. By the FT, a $5K_{m,n}$ can be decomposed into OX graphs.

A $4K_{2,5}$ (on $V_1 = \{a, b\}$ and $V_2 = \{1, 2, 3, 4, 5\}$) can be decomposed into eight OX graphs $\{b1a2, b2a3, b3a4, b4a1, a1b5, a2b5, a3b5, a4b5\}$. Since a decomposition of a $3K_{2,5}$ into OX graphs and a decomposition of a $5K_{2,5}$

into OX graphs exist, by the FT, a decomposition of a $\lambda K_{2,5}$ for $\lambda \geq 3$ into OX graphs exists and hence a $\lambda K_{2,5t}$ for $\lambda \geq 3$. We conclude the results obtained in the section in the following theorem.

THEOREM 8. *The necessary conditions of decomposing a $\lambda K_{m,n}$ into OX graphs are sufficient except for a $2K_{2,5t}$ where a decomposition does not exist.*

3. Summary

In this paper we applied the fundamental theorem to the decomposition of a $\lambda K_{m,n}$ into subgraphs having four vertices and five edges. Applying the fundamental theorem to prove that certain necessary conditions are sufficient reduces the proofs to find examples of decompositions for certain small bipartite graphs. We also showed the non-existence of the decompositions for the following cases: a $3K_{m,n}$ and a $2K_{2,5t}$ can not be decomposed into OLO graphs, a $3K_{m,n}$ can not be decomposed into OOL graphs, a $3K_{2,5t}$, a $6K_{2,5t}$ and a $3K_{3,5}$ can not be decomposed into LEL graphs, a $3K_{m,n}$ can not be decomposed into $OOL_{1,3}$ graphs, and a $2K_{2,5t}$ can not be decomposed into OX graphs.

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