Further Results On SD-prime Labeling

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Abstract

Let $G = (V(G), E(G))$ be a simple, finite and undirected graph with n vertices. Given a bijection $f : V(G) \to \{1, \dots, n\}$, one can associate two integers $S = f(u) + f(v)$ and $D = |f(u) - f(v)|$ with every edge $uv \in E(G)$. The labeling f induces an edge labeling $f' : E(G) \to \{0, 1\}$ such that for any edge uv in $E(G)$, $f'(uv) = 1$ if $gcd(S, D) = 1$, and $f'(uv) = 0$ otherwise. Such a labeling is called an SD-prime labeling if $f'(uv) = 1$ for all $uv \in E(G)$. We say that G is SD-prime

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if it admits an SD-prime labeling. A graph G is said to be a *strongly SD-prime graph* if for every vertex v of G there exists an SD-prime labeling f satisfying $f(v) = 1$. In this paper, we first give some sufficient conditions for a theta graph to be strongly SD-prime. We then give constructions of new SD-prime graphs from known SD-prime graphs and investigate the SD-primality of some general graphs.

Keywords: Prime labeling, SD-prime labeling, Strongly SD-prime labeling.

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1 Introduction

Let $G = (V(G), E(G))$ (or $G = (V, E)$ for short if not ambiguous) be a simple, finite and undirected graph of order $|V| = n$. All notations not defined in this paper can be found in [1].

The first paper on graph labeling was introduced by Rosa in 1967. Since then, there have been more than 1500 research papers published on graph labeling (see the dynamic survey by Gallian [5]).

The concept of prime graphs was introduced in [16, 17].

Definition 1.1. A bijection $f : V \to \{1, \ldots, n\}$ induces an edge labeling $f' : E \to \{0, 1\}$ such that for an edge uv in G $f'(uv) = 1$ if $gcd(f(u), f(v)) = 1$ and $f'(uv) = 0$ otherwise. Such a labeling is called a *prime labeling* if $f'(uv) = 1$ for all $uv \in E$. We say G is a *prime graph* if it admits a prime labeling.

For an edge labeling $f' : E \to \{0, 1\}$ of a graph G, we let $e_{f'}(i)$ be the number of edges labeled with $i \in \{0, 1\}$.

Several results on prime graphs can be found in [2–4, 8, 10, 11]. In [7], Lau and Shiu introduced a variant of prime graph labeling.

Given a bijection $f : V \rightarrow \{1, \ldots, n\}$, and every edge uv in E, one can associate two integers $S = f(u)+f(v)$ and $D = |f(u)-f(v)|$.

Definition 1.2. A bijection $f : V \rightarrow \{1, \ldots, n\}$ induces an edge labeling $f' : E \rightarrow \{0, 1\}$ such that for any edge uv in G, $f'(uv) = 1$ if $gcd(S, D) = 1$, and $f'(uv) = 0$ otherwise. Such a labeling is called an *SD-prime labeling* if $f'(uv) = 1$ for all $uv \in E$. We say G is *SD-prime* if it admits an SD-prime labeling.

Definition 1.3. A graph G is said to be a *strongly SD-prime graph* (or simply SSD-prime) if for every vertex v of G there exists an SD-prime labeling f satisfying $f(v) = 1$.

The following results are obtained in [7].

Theorem 1.1. *A graph G of order n is SD-prime if and only if G is bipartite and that there exists a bijection $f : V \rightarrow \{1, 2, \ldots, n\}$ such that for each edge uv of G, $f(u)$ and $f(v)$ are of different parity and $gcd(f(u), f(v)) = 1$.*

Theorem 1.2. *A connected graph G is SSD-prime only if it is SD-prime of even order. Moreover, G is a spanning subgraph of $K_{m,m}$ for some $m \geq 1$.*

All the graphs we consider in this paper are necessarily bipartite unless stated otherwise. In the next sections, we first give some sufficient conditions for a theta graph to be strongly SD-prime. We then give two constructions of new SD-prime graphs from known SD-prime graphs and investigate the SD-primality of some general graphs.

2 SSD-prime Theta Graphs

Definition 2.1. A *cycle with a long chord* (or *theta graph*) [12] is a graph obtained from a cycle C_m, $m \geq 4$, by adding a chord of length l where $l \geq 1$. Namely, let $C_m = u_0u_1 \cdots u_{m-1}u_0$. Without loss of generality, we may assume that the long chord joins u_0 with u_a, where $2 \leq a \leq m-2$. That is, $u_0u_mu_{m+1} \cdots u_{m+l-2}u_a$ is the chord. We denote this graph by $C_m(a;l)$. Note that when $l = 1$, the chord is u_0u_a.

In [7], we have

Theorem 2.1. *For $a \geq 2$, $l \geq 1$, $C_m(a;l)$ is SD-prime if and only if both m and $a+l$ are even.*

Theorem 2.2. *For odd a, l,*

(a) $C_m(a;1)$ is SSD-prime if and only if m is even.

(b) $C_{a+l}(a;l)$ is SSD-prime if

(1) $a = 3$; or

(2) $a = 2^k + 5, k \geq 1$; or

(3) $a - 5 \not\equiv 0 \pmod p$ where p is any prime factor of $l+2$.

In determining the SSD-primality of $C_m(a;l), a, l \geq 5$, Theorem 1.2 implies that m must be even and both a and l must be odd. Without loss of generality, we assume that $m - a \geq l \geq a$. By a suitable symmetry reflection, we only need to show that there exists SD-prime labeling such that $f(u_i) = 1$ for $i \in \{1, 3, \ldots, a-2, a, a+2, a+4, \ldots, m-1, m, m+2, m+4, \ldots, m+l-3\}$. Note that the labeling we used to prove Theorem 2.1 gives $f(u_a) = 1$. We first consider $i \in \{1, 3, 5, \dots, a-2\}$. Now, define a new labeling f as follow:

$f(u_i) = 1$, $f(u_0) = 2$, $f(u_j) = j - m + 3$ for $m \le j \le m + l - 2$;

$f(u_j) = a + l + 2 - j$ for $i + 1 \le j \le a$;

$f(u_j) = a + l + 1 - i + j$ for $1 \le j \le i - 1$;

$f(u_j) = m + a + l - j$ for $a + 1 \le j \le m - 1$.

Note that the labels of all pairs of adjacent vertices $uv \ne u_a u_{a+1}$ are relatively prime. By Theorem 1.1, f is an SD-prime labeling if $gcd(f(u_a), f(u_{a+1})) = gcd(l + 2, m + l - 1) = 1$. Similarly, we can also define another labeling g as above with $g(u_i) = 1$ for $i \in \{a+2, a+4, \dots, m-1\}$ such that $g(u_a) = l+2, g(u_{a-1}) = m+l-1$ (respectively, with $g(u_i) = 1$ for $i \in \{m, m+2, \dots, m+l-3\}$ such that $g(u_a) = a+2, g(u_{a+1}) = m+l-1$). As above, g is an SD-prime labeling if $gcd(l+2, m+l-1) = 1$ (respectively, if $gcd(a+2, m+l-1) = 1$). This approach is shown in Figure 1.

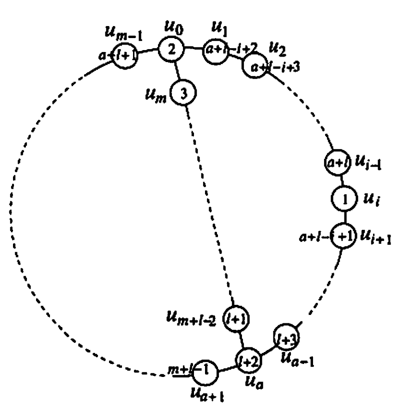


Figure 1: A labeling of $C_m(a; l)$ that could be SD-prime

Suppose $gcd(l+2, m+l-1) \ne 1$ (or $gcd(a+2, m+l-1) \ne 1$). Since

$l \geq a \geq 5$, we look for a prime p (or q) $\in \{(m+l+1)/2, \ldots, m+l-4\}$ such that $gcd(l+2, p+1) = 1$ (or $gcd(l+2, q+1) = 1$). Note that p (or q) $\geq l+4$ and that p and q may not be distinct. If this p (or q) exists, we relabel the vertex labels $p+1$ (or $q+1$) to $m+l-1$ in reverse order to get a required SD-prime labeling since $gcd(p, m+l-1) = 1$ (or since $gcd(q, m+l-1) = 1$).

In Figure 2, we have the labeling of $C_{18}(7;7)$ under the original function f. We can see that $gcd(f(u_5), f(u_6)) = gcd(9, 24) = 3$. So we choose a prime $p = 19 \in \{13, \ldots, 22\}$ such that $gcd(l+2, p+1) = gcd(9, 20) = 1$. After the relabeling from 20 to 24 in reverse order we get a required SD-prime labeling as in Figure 3.

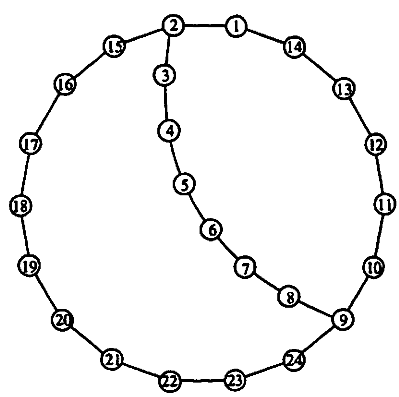


Figure 2: $C_{18}(7;7)$ under the original labeling function f

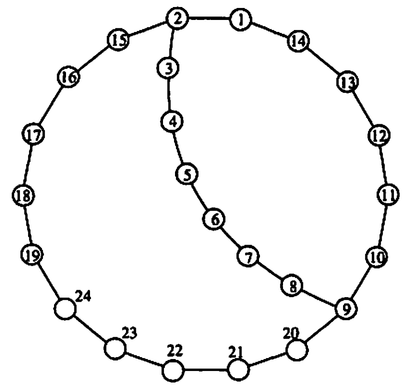


Figure 3: $C_{18}(7;7)$ after the relabeling from 20 to 24

Corollary 2.3. *For even $m \geq 6$ and odd $a, l \geq 5$ such that $m - a \geq l \geq a$, the graph $C_m(a;l)$ is SSD-prime if one of the following holds:*

(a) $gcd(l+2, m+l-1) = gcd(a+2, m+l-1) = 1$;

(b) whenever $gcd(l+2, m+l-1) \neq 1$ (or $gcd(a+2, m+l-1) \neq 1$), there exists a prime p (or q) $\in \{(m+l+1)/2, \ldots, m+l-3\}$ such that $gcd(l+2, p+1) = 1$ (or $gcd(a+2, q+1) = 1$).

Observe that the new label of the vertex u_{m-1} is always odd after relabeling. Note that it may be possible to choose a prime $p < (m+l+1)/2$. For example, taking $p = 7$ and relabeling the vertex labels 8 to 24 in reverse order gives us an SD-prime labeling.

Since $gcd(l+2, m+l-1) = gcd(a+2, m+l-1) = 1$ if $m+l-1$ has no odd factors, we have

Corollary 2.4. *Suppose $m+l-1$ has no odd factors, then the graph $C_m(a;l)$ is SSD-prime if and only if m is even and a, l are odd.*

Question 2.1. *Is there a sufficient condition for the existence of a prime p (or q) in Corollary 2.3(b)?*

We now look at another possible relabeling whenever the original labeling as in Figure 1 is not SD-prime. Note that if there exists a suitable exponent of 2, say t, such that $gcd(t-1, m+l-1) = 1$, we can then relabel the vertex labels from t to $m+l-1$ in reverse order. The resulting labeling is SD-prime since any two adjacent labels that are not consecutive integers must be one of the types $(1,x), (2,y), (t-1, m+l-1), (t,z)$ where x is even, and y, z are odd. Hence, if $m+l-1$ has no prime factor of 3, we choose $t = 4$. Otherwise, we consider the following cases:

(a) 3 is the only odd prime factor, take $t = 8$;

(b) $m+l-1$ contains only 2 distinct odd prime factors: if 3 and 5, take $t = 8$; if 3 and 7, take $t = 16$; if 3 and 11 (or larger), take $t = 8$;

(c) contains only 3 distinct odd prime factors:

   (i) 3 and 5 and 7, take $t = 32$;

   (ii) 3 and 5 and 11 or larger, take $t = 8$;

   (iii) 3 and 7 and 11 (or 13, 17, 23, 29, 37 or larger), take $t = 32$;

   (iv) 3 and 7 and 31, take $t = 128$;

   (v) 3 and 2 more distinct prime factors ≥ 11, take $t = 8$.

Corollary 2.5. *The graph $C_m(a;l)$ is SSD-prime if m is even, a, l are odd and 3 is not a prime factor of $m+l-1$. Moreover, if 3 is a prime factor of $m+l-1$ that contains at most three distinct prime factors, then $C_m(a;l)$ is SSD-prime.*

Conjecture 2.1. *The graph $C_m(a;l)$ is SSD-prime if and only if a, l are odd.*

3 Constructions of SD-prime graph

Lemma 3.1. *Suppose G is of order n and admits an SD-prime labeling f. The following graphs are also SD-prime:*

(1) Add new vertices u_i $(i \geq 1)$ to G such that $f(u_i) = n + i$.

(2) Join a new vertex v to at least one vertex u of G such that $f(v) = n + 1$ and the corresponding $gcd(S, D) = 1$.

(3) Delete any edge of G or add any edge such that the corresponding $gcd(S, D) = 1$.

Theorem 3.2. *Let G_1 and G_2 be SD-prime. Suppose that G_1 has m vertices, and $p_1, p_2, \ldots, p_k$ are all the odd primes $\leq m - 1$. Suppose that G_2 has n vertices such that $n = \pi(p_1)(p_2)\cdots(p_k)$, and π is any integer, then $G_1 \cup G_2$ is SD-prime.*

Proof. Keep the vertex labels in G_2 unchanged. So, all the edge labels in G_2 are still 1. Re-label the vertices in G_1 by adding n to each of them. Then the vertex labels in $G_1 \cup G_2$ are from 1 to $m + n$ inclusive. It remains to show that all edge labels in G_1 are still 1.

Consider an edge uv in G_1. The original vertex labels are $f(u)$ and $f(v)$ where f is an SD-prime labeling of G_1. Since $f(u)$ and $f(v)$ have opposite parity, so do $f(u) + n$ and $f(v) + n$. Assume that $gcd(f(u) + n, f(v) + n) = d > 1$. Let p (necessarily odd) be a prime factor of d.

Case 1: p divides n. This implies that p divides both $f(u)$ and $f(v)$, contradicting $gcd(f(u), f(v)) = 1$.

Case 2: p does not divide n. This implies that p is not any of the primes $p_1, p_2, \ldots, p_k$. Since p divides both $f(u) + n$ and $f(v) + n$, it divides $f(u) - f(v)$. Note that $-(m-1) \leq f(u) - f(v) \leq m-1$, and so the prime factors of $f(u) - f(v)$ are all $\leq m-1$. Then p must be one of $p_1, p_2, \ldots, p_k$, a contradiction. □

Furthermore, we can add new edges joining G_1 and G_2 and keep the resulting graph SD-prime. This can be done in more than one way. For example, join the vertex labeled 1 in G_2 to any vertex in G_1 with even label. As another example, join the vertex labeled n in G_2 and the vertex labeled $n+1$ in G_1.

In Figure 4, we give an example by taking G_1 to be the SD-prime graph of order $m = 11$ so that $p_1 = 3, p_2 = 5, p_3 = 7$. Take $\pi = 2$ and let $G_2 = P_{210}$ with vertex labels 1 to 210 consecutively. Clearly, the labeling of $G_1 \cup G_2$ is also SD-prime.

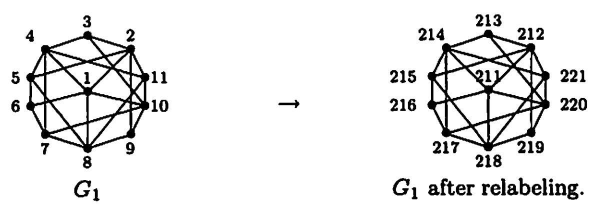


Figure 4: Union of two SD-prime graphs

Theorem 3.3. *Let f and g be SD-prime labelings for the graphs G_1 and G_2 respectively. Suppose that G_1 has m vertices, and $p_1, p_2, \ldots, p_k$ are all the odd primes $\leq m-2$. Suppose that G_2 has n vertices such that $n = 1 + \pi(p_1)(p_2)\cdots(p_k)$, where π is any even integer. Let a and b be vertices in G_1 and G_2 respectively with $f(a) = g(b) = 1$. Then the one-point-union of G_1 and G_2 formed by identifying a and b is SD-prime.*

Proof. Note that n is odd. Label the vertices in the one-point-union as follows.

Keep the vertex labels in G_2 unchanged. So, all the edge labels in G_2 are still 1. Re-label the vertices, other than a, in G_1 by adding $n-1$ to each of them. The vertex a in G_1, which is now the same as the vertex b in G_2, keeps its label 1. Therefore, the vertex labels in the one-point-union of G_1 and G_2 are from 1 to $m+n-1$ inclusive. It remains to show that all edge labels in G_1 are still 1.

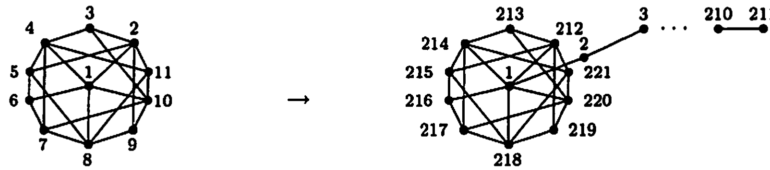
First consider an edge uv in G_1, where neither u nor v is a. The original vertex labels are $f(u)$ and $f(v)$. Since $f(u)$ and $f(v)$ have opposite parity, so do $f(u)+n-1$ and $f(v)+n-1$. Assume that $gcd(f(u)+n-1, f(v)+n-1) = d > 1$. Let p (necessarily odd) be a prime factor of d.

Case 1: p divides $n-1$. This implies that p divides both $f(u)$ and $f(v)$, contradicting $gcd(f(u), f(v)) = 1$.

Case 2: p does not divide $n-1$. This implies that p is not any of the primes $p_1, p_2, \ldots, p_k$. Since p divides both $f(u)+n-1$ and $f(v)+n-1$, it divides $f(u)-f(v)$. As neither $f(u)$ or $f(v)$ is 1, we have $-(m-2) \leq f(u)-f(v) \leq m-2$, and so the prime factors of $f(u)-f(v)$ are all $\leq m-2$. Therefore, p must be one of $p_1, p_2, \ldots, p_k$, a contradiction.

Now consider an edge au in G_1. Recall that a retains its vertex label 1. Since the original vertex labeling f is SD-prime, $f(u)$ must be even, and so $f(u)+n-1$ is also even. In addition, since $gcd(1, f(u)+n-1) = 1$, the edge au has label 1. □

By taking the same G_1 as in Figure 4 and $G_2 = P_{211}$ with vertex labels from 1 to 211 consecutively, we see that the one-point-union given in Figure 5 also has an SD-prime labeling.



G_1 $\quad$ G_1 after relabeling.

Figure 5: One-point-union of two SD-prime graphs

Let G be a connected bipartite graph of order n that has a 4-independent partition $\{A', B', A, B\}$ such that (A', B') is connected and every vertex of A (respectively, B) is adjacent to a vertex of A' (respectively, B'), whereas $(A', B), (A, B')$ and (A, B) are null graphs. Without loss of generality, assume $|A'| \geq |B'| \geq 1$. Let $I_1 = \{1,$ all odd primes p such that $n/2 < p \leq n\}$ and $I_2 = \{2^a : 2 \leq 2^a \leq n\}$.

Theorem 3.4. *Suppose G is a graph as defined above. Let $|A'| - |B'| = d \geq 0$. A necessary condition for G to be SD-prime is that $|A| - |B| = d-1, d$ or $d+1$. If I_1 and I_2 have enough elements for the constructions in the sufficiency part below, then $|A| - |B| = d-1, d$ or $d+1$ is also sufficient for SD-primality of G.*

Proof. (Necessity) G has an SD-prime labeling.

Case 1. Since (A', B') is connected, for any two vertices x, y in A', there is a path, say $x, u_1, u_2, \ldots, u_k, y$ (k is odd) such that u_i is in B' for odd i and in A' for even i. Let A' contain an odd-labeled vertex. By SD-primality, all vertices in A' must be assigned odd labels, and all vertices in B' must be assigned even labels. Consequently, A (and B) must contain even-labeled (and odd-labeled) vertices. Since the number of odd labels is the same as or one more than the number of even labels, we have $|A'| + |B| = |A| + |B'|$ or $|A| + |B'| + 1$. Thus

$|A| - |B| = |A'| - |B'|$ or $|A'| - |B'| - 1$.

Case 2. Let A' contains an even-labeled vertex. As in Case 1 above, we have $|A| + |B'| = |A'| + |B|$ or $|A'| + |B| + 1$. Thus $|A| - |B| = |A'| - |B'|$ or $|A'| - |B'| + 1$.

(Sufficiency) Let $|A'|+|B'| = m$. Since $|A'|-|B'| = d$, we have $|A'| = (m+d)/2$ and $|B'| = (m-d)/2$. Thus m and d must have the same parity. In addition, $n = |A'|+|B'|+|A|+|B| = m+(|A|-|B|)+2|B|$.

Case 1. $|A| - |B| = d - 1$. This means that n is odd. Use the labels in I_1 for the vertices in A', and the labels in I_2 for the vertices in B'. There are $(n+1)/2 - |A'| = (n+1)/2 - |B'| - d = (n+1)/2 - (m+d)/2 = |B|$ odd labels and $(n-1)/2 - |B'| = (n-1)/2 - (m-d)/2 = |A|$ even labels left. They are exactly enough for B and A respectively.

Case 2. $|A| - |B| = d$. This means that n is even. Similar to the argument for Case 1, we can obtain an SD-prime labeling.

Case 3. $|A| - |B| = d + 1$. This means that n is odd. Use the labels in I_2 for the vertices in A', and the labels in I_1 for the vertices in B'. There are $(n+1)/2 - |B'|$ odd labels and $(n-1)/2 - |A'| = (n-1)/2 - |B'| - d$ even labels left. They are exactly enough for A and B respectively. □

Let $\pi(x)$ denotes the number of primes not exceeding x. In [9], Ramanujan gave a method of finding the minimum number of x such that $\pi(x) - \pi(x/2) \geq n \geq 1$. This leads to the following definition.

Definition 3.1. [13, 14] For $n \geq 1$, the n-th Ramanujan prime is the smallest positive integer R_n with the property that if $x \geq R_n$, then $\pi(x) - \pi(x/2) \geq n$.

In [14], Sondow obtained an upper bound for R_n as follows:

Theorem 3.5. *The n-th Ramanujan prime satisfies the inequalities* $2n\ln(2n) < R_n < 4n\ln(4n)$ *for* $n \geq 1$.

Remark 1. If $n \geq \max\{2^{(m+d)/2}, 4((m+d)/2-1)\ln(4((m+d)/2-1))\}$, then there are sufficient elements in I_1 and I_2.
Proof. In each of the cases in the sufficiency proof, I_1 and I_2 have enough elements if each of them has at least $(m+d)/2$ elements. Clearly, $n \geq 2^{|I_2|}$. The number of odd primes greater than $n/2$ and not exceeding n is $\pi(n) - \pi(n/2) = |I_1| - 1$. Thus we want n to be at least the $(((m+d)/2)-1)$-st Ramanujan prime. The result follows from Theorem 3.5. □

A caterpillar is a tree that becomes a path when all its leaves are removed. We denote by $CT(m; n_1, n_2, \ldots, n_m)$ the caterpillar that becomes P_m when all its leaves are removed, and for each a_k in $V(P_m) = \{a_1, a_2, \ldots, a_m\}$, the vertex a_k has n_k pendant edges. The path P_m is known as the spine of the caterpillar. Note that such a caterpillar has $n = m + n_1 + n_2 + \cdots + n_m$ vertices. By Theorem 3.4, the following result follows directly.

Corollary 3.6. *Suppose that* $|I_1|, |I_2| \geq \lceil \frac{m}{2} \rceil$. *The caterpillar* $CT(m; n_1, n_2, \ldots, n_m)$ *is SD-prime if and only if*

(1) $n_1 + n_3 + \cdots = n_2 + n_4 + \cdots$, *or* $1 + n_1 + n_3 + \cdots = n_2 + n_4 + \cdots$, *or* $n_1 + n_3 + \cdots = 1 + n_2 + n_4 + \cdots$ *for even* m.

(2) $n_1 + n_3 + \cdots = n_2 + n_4 + \cdots$, *or* $n_1 + n_3 + \cdots = 1 + n_2 + n_4 + \cdots$, *or* $n_1 + n_3 + \cdots = 2 + n_2 + n_4 + \cdots$ *for odd* m.

Proof. Case 1. m is even. Let $A' = \{a_1, a_3, \ldots, a_{m-1}\}$ and $B' = \{a_2, a_4, \ldots, a_m\}$, or $A' = \{a_2, a_4, \ldots, a_m\}$ and $B' =$

$\{a_1, a_3, \ldots, a_{m-1}\}$. Then $d = 0$. By Theorem 3.4, we have SD-primality if and only if $n_1+n_3+\cdots = n_2+n_4+\cdots$, or $1+n_1+n_3+\cdots = n_2+n_4+\cdots$, or $n_1+n_3+\cdots = 1+n_2+n_4+\cdots$.

Case 2. m is odd. Let $A' = \{a_1, a_3, \ldots, a_m\}$ and $B' = \{a_2, a_4, \ldots, a_{m-1}\}$. Then $d = 1$. By Theorem 3.4, we have SD-primality if and only if $n_1+n_3+\cdots = n_2+n_4+\cdots$, or $n_1+n_3+\cdots = 1+n_2+n_4+\cdots$, or $n_1+n_3+\cdots = 2+n_2+n_4+\cdots$. □

In Figure 6, we give an SD-prime labeling for $CT(8; 2, 3, 0, 0, 4, 3, 0, 1)$.

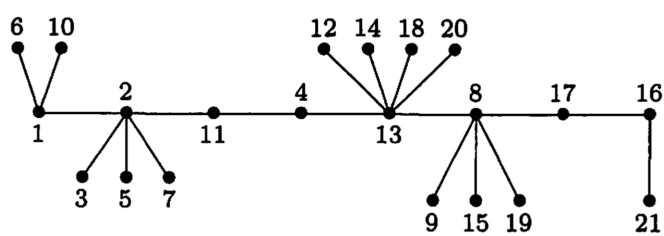


Figure 6: An SD-prime labeling for $CT(8; 2, 3, 0, 0, 4, 3, 0, 1)$

Note that by Corollary 3.6, Theorem 2.8 in [7] (on double star $DS(a,b) = CT(2,a,b)$, $a \geq b \geq 2$) follows directly since the condition $|I_1|, |I_2| \geq \lceil \frac{m}{2} \rceil = 1$ always holds.

The ring worm $RW(m; n_1, n_2, \ldots, n_m)$ is obtained from $CT(m+1; n_1, n_2, \ldots, n_m, 0)$ by identifying the vertices a_1 and a_{m+1}. The following results are obvious.

Corollary 3.7. *For even m and $|I_1|, |I_2| \geq m/2$, the graph $RW(m; n_1, n_2, \ldots, n_m)$ is SD-prime if and only if $|(n_1+n_3+\cdots) - (n_2+n_4+\cdots)| \leq 1$.*

Corollary 3.8. *Let C be an even cycle with m vertices, and N be* *a null graph of any order. The corona of C with N is SD-prime if $|I_1|, |I_2| \geq m/2$.*

Theorem 3.9. *Let $m > 1$. A full m-ary tree T is SD-prime if and only if (i) T is a single vertex or (ii) $m = 2$ and T has height 1.*

Proof. Sufficiency is obvious. For necessity, since adjacent vertex labels must have opposite parity, the number of odd and even vertices must be $1+m^2+m^4+\cdots$ and $m+m^3+m^5+\cdots$ respectively or vice versa. First assume that T has even height $h > 0$. The first sum has at least two terms, and has exactly one more term than the second sum, rendering their difference to be more than 1. Thus T cannot be SD-prime. Now assume that T has odd height h. The two sums have the same number of terms, and the first sum is obviously less than the second by more than 1, unless $h = 1$ and $m = 2$. □

Theorem 3.10. *If $2kn + 1$ is prime for $k = 1, 2, \ldots, 2m - 1$, then the grid $P_n \times P_{2m}$ is SD-prime.*

Proof. View the $2m$ copies of P_n as m pairs of horizontal paths, with the paths stacked one above another. For the uppermost pair, label the vertices of the top P_n from left to right by $1, 2, \ldots, n$, and the vertices of the bottom P_n from right to left by $n+1, n+2, \ldots, 2n$. In general, for the i-th pair, label the j-th vertex of the top P_n by $2(i-1)n + j$, and the j-th vertex of the bottom P_n by $2in + 1 - j$. Clearly, adjacent labels have opposite parity.

Two horizontally adjacent labels are obviously relatively prime, since they differ by 1. Two vertically adjacent labels have the form $2(i-1)n + j$ and $2in + 1 - j$ if they belong to the same pair of P_n's, or $2in + 1 - j$ and $2in + j$ if they belong to different pairs of P_n's.

Assume that these two labels are not relatively prime, and let p be a prime dividing both. In the first case, p also divides the sum $2(2i-1)n+1$, where $1 \le i \le m$. In the second case, p also divides the sum $2(2i)n+1$, where $1 \le i \le m-1$. In other words, p divides $2kn+1$, where $1 \le k \le 2m-1$. Since $2kn+1$ is prime, p is equal to this $2kn+1$. However, $2kn+1$ is the sum of two smaller positive integers that are multiples of p, a contradiction. Hence, any two vertically adjacent labels are relatively prime. □

Remark 2. In [6], it was proved that "the prime numbers contain infinitely many arithmetic progressions of length k for all k". Since such progressions must have even common difference, our assumption for Theorem 3.10 can be satisfied infinitely often.

Corollary 3.11. *If $2n+1$ is prime, then the ladder $P_n \times P_2$ is SD-prime.*

Corollary 3.12. *If $2kn+1$ is prime for $k = 1, 2, \ldots, 2m-1$, then the grid $P_n \times P_{2m-1}$ is SD-prime.*

Proof. Use exactly the same labeling as in the above proof, and delete the bottom copy of P_n. □

Let the parity function, P, be such that $P(\text{odd integer}) = 1$ and $P(\text{even integer}) = 0$.

Corollary 3.13. *If $2kn+1$ is prime for $k = 1, 2, \ldots, m-1+P(m)$, then the grid $P_n \times P_m$ is SD-prime.*

Proof. This follows from Theorem 3.10 and Corollary 3.12, with a slight change in notation. □

Using symmetry, we have the following.

Corollary 3.14. *Suppose that* $2kn+1$ *is prime, for* $k = 1, 2, \ldots, m-1+P(m)$*, or that* $2km+1$ *is prime, for* $k = 1, 2, \ldots, n-1+P(n)$*. Then the grid* $P_n \times P_m$ *is SD-prime.*

In [7], it was shown that a complete bipartite graph $K_{m,n}$, $1 \leq m \leq n$ is SD-prime if and only if (i) $m = 1$, $n = 1, 2$; or (ii) $m = 2$, $n = 2, 3$. However, we have

Theorem 3.15. *Any bipartite graph* G *is an induced subgraph of an SD-prime graph* H.

Proof. Let $G = (A, B)$ with $|A| = r$ and $|B| = s$. Let $v = \max\{2r-1, 2^s\}$. Label the vertices in A by $1, 3, \ldots, 2r-1$, and the vertices in B by $2^1, 2^2, \ldots, 2^s$. Add $(v - r - s)$ isolated vertices and label them using the unused integers in $\{1, 2, \ldots, v\}$ to get H. □

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