Fully Cordial Trees

Ebrahim Salehi and Daniel Corral

Department of Mathematical Sciences
University of Nevada, Las Vegas
Las Vegas, NV 89154-4020

ebrahim.salehi@unlv.edu dacorral@unlv.nevada.edu

Abstract

For a graph $G = (V, E)$ and a coloring $f : V(G) \to \mathbb{Z}_2$ let $v_f(i) = |f^{-1}(i)|$. f is said to be friendly if $|v_f(1) - v_f(0)| \leq 1$. The coloring $f : V(G) \to \mathbb{Z}_2$ induces an edge labeling $f_+ : E(G) \to \mathbb{Z}_2$ defined by $f_+(xy) = |f(x) - f(y)|$, $\forall xy \in E(G)$. Let $e_f(i) = |{f_+}^{-1}(i)|$. The friendly index set of the graph G, denoted by $FI(G)$, is defined by

$$FI(G) = \{|e_f(1) - e_f(0)| : f \text{ is a friendly vertex labeling of } G\}.$$

In this paper we determine the friendly index set of certain classes of trees and introduce a few classes of fully cordial trees.

Key Words: Friendly coloring, friendly index set, near perfect matching, Fibonacci and Lucas trees.
AMS Subject Classification: 05C15, 05C25, 05C78

1 Introduction

In this paper all graphs $G = (V, E)$ are connected, finite, simple, and undirected. For graph theory notations and terminology not described in this paper, we refer the readers to [7]. Let $G = (V, E)$ be a graph and $f : V(G) \to \mathbb{Z}_2$ a binary vertex labeling (coloring) of G. For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$. The coloring f is said to be *friendly* if $|v_f(1) - v_f(0)| \leq 1$. That is, the number of vertices labeled 1 is almost the same as the number of vertices labeled 0. Any friendly labeling $f : V(G) \to \mathbb{Z}_2$ induces an edge labeling $f_+ : E(G) \to \mathbb{Z}_2$ defined by $f_+(xy) = |f(x) - f(y)|$, $\forall xy \in E(G)$. For $i \in \mathbb{Z}_2$, let $e_f(i) = |{f_+}^{-1}(i)|$ be the number of edges of G that are labeled i. The number $N(f) = |e_f(1) - e_f(0)|$ is

called the *friendly index* (or *cordial index*) of f. A graph G is said to be *cordial* if it admits a friendly labeling with index 0 or 1.
To illustrate the above concepts, consider the graph G of Figure 1, which has ten vertices. The condition $|v_f(1) - v_f(0)| \leq 1$ implies that five vertices be labeled 0 and the other five 1.

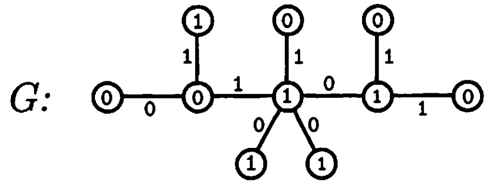

Figure 1: A typical friendly labeling of G.

Figure 1 also shows the associated edge labeling of G, where five edges have label 1 while the other four edges have label 0. Therefore, the friendly index provided by this labeling is $5 - 4 = 1$ and G is cordial.
I. Cahit [2, 3, 4] introduced the concept of cordial labeling as a weakened version of the less tractable graceful and harmonious labeling. A graph G is said to be cordial if it admits a friendly labeling with index 0 or 1. Hovay [10], later generalized the concept of cordial graphs and introduced A-cordial labelings, where A is an abelian group. A graph G is said to be A-cordial if it admits a labeling $f : V(G) \to A$ such that for every $i, j \in A$,

$$|v_f(i) - v_f(j)| \leq 1 \text{ and } |e_f(i) - e_f(j)| \leq 1.$$

Cairnie-Edwards [5] proved that the problem of deciding whether or not a graph G is cordial is NP-complete, as conjectured by Kirchherr [12]. Cordial graphs have been studied extensively. Interested readers are referred to a number of relevant literature that are mentioned in the bibliography section, including [1, 8, 9, 11, 13, 14, 16, 17, 18].
Chartrand-Lee-Zhang [6] introduced the concept of *friendly index set* of a graph G defined by

$$FI(G) = \{N(f) : \ f \text{ is a friendly labeling of } G \ \}.$$

For the graph G in Figure 1, it is easy to verify that $FI(G) = \{1, 3, 5, 9\}$. The friendly colorings of G that provide the other friendly indices are presented in Figure 2.
In this paper, we will focus on the group $A = \mathbb{Z}_2$ and determine the friendly index sets of certain classes of trees. Note that if 0 or 1 is in $FI(G)$, then G is cordial. Thus the concept of friendly index sets could be viewed as a generalization of cordiality. A friendly labeling $f : V(G) \to \mathbb{Z}_2$ is called a *maximum friendly labeling* of G if its friendly index is the maximal, that is,

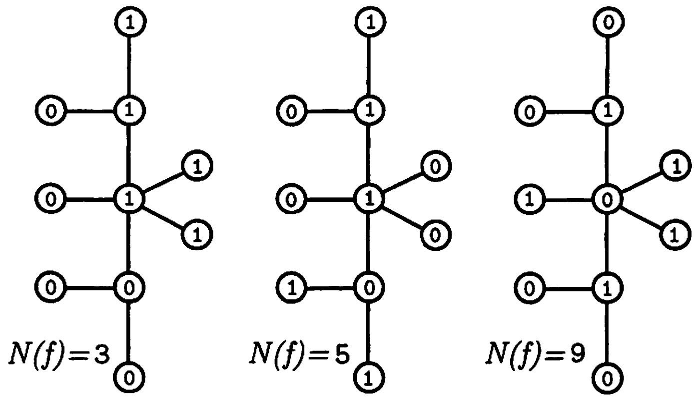

Figure 2: Three friendly labelings of G with indices 3, 5 and 9.

$N(f) = |E(G)|$. In this case, we call $N(f)$ the maximum friendly index of G. Also, if $f : V(G) \to \mathbb{Z}_2$ is a friendly labeling, so is its inverse labeling $g : V(G) \to \mathbb{Z}_2$ defined by $g(v) = 1 - f(v)\ \forall v \in V(G)$. Moreover, $N(g) = N(f)$. First we state a few known results from [15] and [18] to be used in the following sections.

Theorem 1.1. *For any graph G with q edges,*
$$FI(G) \subseteq \{q - 2i : i = 0, 1, 2, \ldots, \lfloor q/2 \rfloor\}.$$

Theorem 1.2. *Let $1 \le m \le n$. For the complete bipartite graph $K_{m,n}$ we have*

$$FI(K_{m,n}) = \begin{cases} \{(m-2i)^2 : 0 \le i \le \lfloor m/2 \rfloor\} & \text{if } m+n \text{ is even;} \\ \{i(i+1) : 0 \le i \le m\} & \text{if } m+n \text{ is odd.} \end{cases}$$

For any $n \ge 2$, the complete bipartite graph $K(1, n)$ is called a *star* and is denoted by $ST(n)$. Stars are the trees of diameter 2, for which we have:

Corollary 1.3. $FI(ST(n)) = \begin{cases} \{0, 2\} & \text{if } n \text{ is even;} \\ \{1\} & \text{if } n \text{ is odd.} \end{cases}$

Theorem 1.4. *The friendly index set of a full binary tree with depth $d > 1$ is* $\{0, 2, 4, \ldots, 2^{d+1} - 4\}$.

2 Fully Cordial Trees

In what follows, whenever there is no ambiguity, we suppress the index f and denote $e_f(i)$ by simply $e(i)$. For a graph $G = (p, q)$ of size q, and a friendly labeling $f : V(G) \to \mathbb{Z}_2$ of G, we have

$$N(f) = |e_f(0) - e_f(1)| = |q - 2e_f(1)| = |q - 2e_f(0)|. \qquad (2.1)$$

Therefore, to find the index of f it is enough to find $e_f(1)$ (or $e_f(0)$). Moreover, to determine the friendly index set of G it is enough to compute $e_f(1)$, or $e_f(0)$, for all different friendly colorings f of G. Another immediate consequence of (2.1) is the following useful fact:

Observation 2.1. *For a graph G of size q,* $FI(G) \subseteq \{q-2k : 0 \leq k \leq \lfloor q/2 \rfloor\}$.

Definition 2.2. A graph G is said to be *fully cordial* if
$$FI(G) = \{q - 2i : i = 0, 1, 2, \ldots, \lfloor q/2 \rfloor\}.$$

The Observation 2.1 indicates that the friendly index set of a graph G is a subset of $\{q-2i : i = 0, 1, 2, \ldots, \lfloor q/2 \rfloor\}$. As illustrated by the example provided in Figure 2, we may not have equality. However, Salehi-Lee [18] proved the following theorem concerning the fully cordial graphs.

Theorem 2.3. *If $T = (p, q)$ is a tree with perfect matching, then $FI(T) = \{1, 3, 5, \ldots, q\}$. That is, T is fully cordial.*

For any graph $G = (p, q)$, the maximum possible element of its friendly index set is q, the number of its edges. By equation (2.1), this maximum can be achieved if $e_f(1) = 0$ or $e_f(0) = 0$. The following observation indicates that $e_f(1) \neq 0$.

Observation 2.4. *Let G be a non trivial connected graph and $f : V(G) \to \mathbb{Z}_2$ any friendly coloring of G. Then $e_f(1) \geq 1$.*

Proof. The two sets $A = \{u \in V(G) : f(u) = 0\}$ and $B = \{v \in V(G) : f(v) = 1\}$ partition $V(G)$. Since G is connected, there are vertices $u \in A$ and $v \in B$ that are adjacent. The label of edge uv is 1. therefore, $e_f(1) \geq 1$. □

Corollary 2.5. *For any graph $G = (p, q)$, $q \in FI(G)$ if and only if $e_f(0) = 0$ for some friendly coloring $f : V(G) \to \mathbb{Z}_2$.*

3 Near Perfect Matching Trees

In [18], Salehi-Lee showed that any tree with perfect matching is fully cordial. However, there are many other fully cordial trees that do not have perfect matchings. Paths of odd orders P_{2n+1} are the most obvious examples. In this section we introduce another class of fully cordial trees. Namely, near perfect matching trees.

Definition 3.1. A matching of a graph G is called *near perfect matching* if it covers all the vertices of G but one. G is called a *near perfect matching* graph if any maximal matching of G is near perfect matching.

Observation 3.2. *A tree T with near perfect matching M contains at least a P_3 pendent $u_1 \sim u_2 \sim u_3$ such that* $\deg u_1 = 1$, $\deg u_2 = 2$ *and* $u_1u_2 \in M$.

Proof. Let $P : u_1 \sim u_2 \sim u_3 \sim \cdots \sim u_{k-2} \sim u_{k-1} \sim u_k$ be the longest path in T. Clearly, $\deg u_1 = \deg u_k = 1$. Also, $\deg u_2 = 2$ or $\deg u_{k-1} = 2$. Otherwise, any maximum matching of T would miss at least two vertices. If $\deg u_2 = \deg u_{k-1} = 2$, then $u_1u_2 \in M$ or $u_ku_{k-1} \in M$. Suppose (wlog) $\deg u_2 = 2$ and $\deg u_{k-1} > 2$. Then $u_1u_2 \in M$. Otherwise, any maximum matching of T would miss at least two vertices. □

Theorem 3.3. *Any near perfect matching tree is fully cordial.*

Proof. Note that T is a tree of odd order, $|T| = 2n+1$. We proceed by induction on n. Clearly, the statement of theorem is true for $n = 1$. Suppose the statement is true for any tree of order $2n + 1$ and let T be a tree of order $2n + 3$ with near-perfect matching M. By Observation 3.2, T contains vertices $u \sim v \sim w$ such that $\deg u = 1$, $\deg v = 2$ and the edge uv is in M. Now consider the tree $S = T - \{u, v\}$ which has order $2n + 1$ and has near perfect matching $M' = M - \{uv\}$. Therefore, by the induction hypothesis

$$FI(S) = \{0, 2, 4, \cdots, 2n\}.$$

We need to show that $FI(T) = \{0, 2, \cdots, 2n, 2n+2\}$. Consider a friendly coloring $f : V(S) \to \mathbb{Z}_2$ of S and extend it to $g : V(T) \to \mathbb{Z}_2$ by defining $g(v) = f(w)$, $g(u) = 1 - f(w)$. Then g is a friendly coloring of T with $e_g(1) = e_f(1) + 1$, $e_g(0) = e_f(0) + 1$. Therefore, $N(g) = N(f)$. This implies that

$$FI(S) = \{0, 2, 4, \cdots, 2n\} \subseteq FI(T).$$

It only remains to show that $2n+2 \in FI(T)$. Let $\phi : V(S) \to \mathbb{Z}_2$ be a friendly coloring of S with index $2n$. We may assume that $e(1) = 2n$, $e(0) = 0$, and extend ϕ to $\psi : V(T) \to \mathbb{Z}_2$ by defining $\psi(v) = 1 - \phi(w)$, $\psi(u) = \phi(w)$. Then ψ is a friendly coloring of T with $e_\psi(1) = e_\phi(1) + 2$, $e_\psi(0) = e_\phi(0) = 0$. Therefore, $N(\psi) = N(\phi) + 2 = 2n + 2$. □

Theorem 3.4. *For $n \geq 2$, the path of order n is fully cordial.*

Proof. This is an immediate consequence of theorems 2.3 and 3.3. Because, any path P_n is either near perfect matching or is a perfect matching tree. Therefore, it is fully cordial. □

Definition 3.5. *Fibonacci Trees*, denoted by FT_n, are defined inductively as follows: FT_1 is the trivial tree with one vertex, FT_2 is the path P_2, and for $n \geq 3$, $FT_n = (V_n, E_n)$ is the binary tree of root r_n, whose left and right children are FT_{n-1} and FT_{n-2}, respectively.

In [18] it is shown that any Fibonacci tree is fully cordial. Here, we present a different proof in which we utilize theorems 2.3 and 3.3.

Theorem 3.6. *For $n \geq 1$, every Fibonacci tree FT_n is fully cordial.*

Proof. Note that every Fibonacci tree has either a perfect matching or is a near perfect matching tree. In fact, if $n \equiv 1 \pmod 3$, then FT_n is a near perfect matching tree; otherwise, it has a perfect matching. We prove this statement

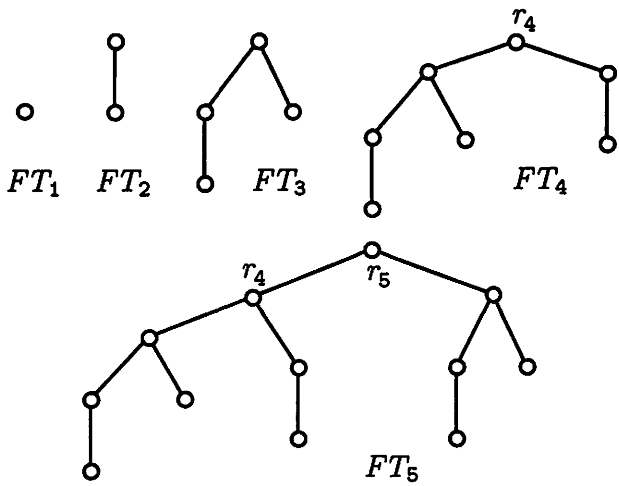

Figure 3: The first five Fibonacci trees.

by induction on n. Clearly the statement is true for $n = 1, 2, 3$. Now suppose the statement is true for all positive integers less than n $(3 < n)$ and let FT_n be the Fibonacci tree of order n. We consider the following cases:

(A) $n \equiv 1 \pmod 3$. In this case, by the induction hypothesis, both the left and right children have perfect matchings. Let M_1 and M_2 be perfect matchings of FT_{n-1} and FT_{n-2}, respectively. Then $M_1 \cup M_2$ is a maximum matching of FT_n that covers all the vertices but its root. Therefore, FT_n is a near perfect matching tree.

(B) $n \equiv 2 \pmod 3$. In this case, by the induction hypothesis, the left child FT_{n-1} is near perfect matching while the right child FT_{n-2} has a perfect matching. Let M_1 be a maximum matching of FT_{n-1} (we may assume that M_1 leaves the root r_{n-1} out) and M_2 be a perfect matching of FT_{n-2}. Then $M_1 \cup M_2 \cup \{r_n r_{n-1}\}$ will form a perfect matching of FT_n.

(C) $n \equiv 0 \pmod 3$. The argument is similar to the previous case.

□

The complete bipartite graphs $K_{1,n}$ are also known as stars, for which we have the following fact:

Theorem 3.7. *[15]* $FI(K_{1,n}) = \begin{cases} \{0,2\} & \text{if } n \text{ is even;} \\ \{1\} & \text{if } n \text{ is odd.} \end{cases}$

One can view the star $ST(n) = K_{1,n}$ as a graph formed by n copies of P_2 all of them sharing an endpoint. From this perspective, one can introduce the star-like trees, where P_2 is replaced with P_k $(k \geq 2)$.

Definition 3.8. *Star-like tree,* denoted by $ST(n,k)$, is a graph formed by n copies of P_k when all of them share exactly an end-vertex. This common end-vertex is clearly the center of the graph.

We observe that $ST(1,k) \simeq P_k$, $ST(2,k) \simeq P_{2k-1}$, $ST(n,2) \simeq K_{1,n}$ and $ST(n,1)$ the trivial graph having just one vertex. The friendly index sets of these graphs have been determined. Therefore, from now on we assume that $n,k \geq 3$.

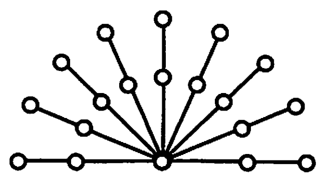

Figure 4: Star-like $ST(9,3)$ and with the FI-set $\{0,2,4,\cdots,16,18\}$.

Theorem 3.9. *For any $n,k \geq 3$, the star-like tree $ST(n,k)$ is fully cordial if and only if k is odd.*

Proof. If k is odd, then $ST(n,k)$ is a near perfect matching tree and by Theorem 3.3 it is fully cordial. When k is even, then $e(0) \neq 0$ holds for any friendly coloring of the graph. Hence, by Corollary 2.5, the maximum possible friendly index cannot be achieved. In fact, when k is even, then $e(0) \geq \lfloor (n-1)/2 \rfloor$. □

4 Caterpillars of Diameter 4

A *double star* is a tree of diameter 3. Double stars have two central vertices u and v and are denoted by $DS(a,b)$, where $\deg u = a$ and $\deg v = b$, as illustrated in Figure 5.
Double star $DS(a,b)$ has $a+b$ vertices and its friendly index set is known [17]:

Theorem 4.1. *Let $a \leq b$. Then*

$$FI(DS(a,b)) = \begin{cases} \{1,3,\ldots,2a-1\} & \text{if } a+b \text{ is even;} \\ \{0,2,\ldots,2a\} & \text{if } a+b \text{ is odd.} \end{cases}$$

Corollary 4.2. *Double star $DS(a,b)$ is fully cordial if and only if $|a-b| \leq 1$.*

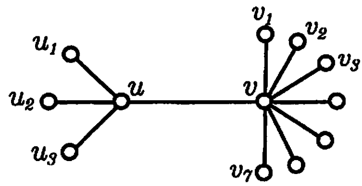

Figure 5: Double Star, $DS(4, 8)$, with central vertices u and v.

Double stars are also caterpillars of diameter of 3. A *caterpillar* is a tree having the property that the removal of its end-vertices results in a path (the spine). We use $CR(a_1, a_2, \ldots, a_n)$ to denote the caterpillar with a P_n-spine, where the ith vertex of P_n has degree a_i. Since $CR(1, a_1, \ldots, a_n, 1) = CR(a_1, \ldots, a_n)$ and $a_i \neq 1$ $(1 \leq i \leq n-1)$, we will assume that $a_i \geq 2$.

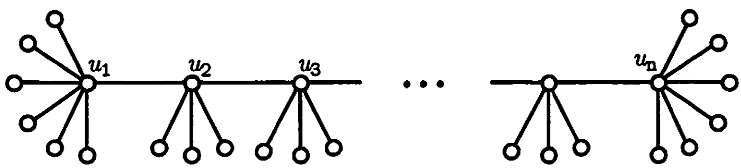

Figure 6: A Caterpillar of diameter $n+1$ (P_n-spine).

In this section we concentrate on caterpillars of diameter four, whose spines are P_3, and will use the notation $G = CR(a, b, c)$, where $\deg u = a$, $\deg v = b$, and $\deg w = c$, as illustrated in Figure 7. This caterpillar has $a+b+c-1$ vertices and $a+b+c-2$ edges.

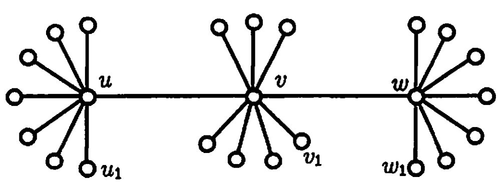

Figure 7: A Caterpillar of diameter 4, $CR(8, 9, 8)$.

The friendly index set of $G = CR(a, b, c)$, when $a+b+c$ is odd, is determined in [17]:

Theorem 4.3. *Let* $a, b, c \geq 2$ *and* $a+b+c$ *be odd. Then* $FI(CR(a, b, c)) = A \cup B \cup C$, *where*

$$\begin{aligned}A &= \{|\,2a-4i-1\,| : m_A \le i \le M_A\};\\ B &= \{|\,2b-4j-1\,| : m_B \le j \le M_B\};\\ C &= \{|\,2c-4k-1\,| : m_C \le k \le M_C\};\end{aligned}$$

and

$$\begin{aligned}m_A &= \max\{0, \tfrac{a-b-c+3}{2}\}; & M_A &= \min\{a-1, \tfrac{a+b+c-3}{2}\};\\ m_B &= \max\{0, \tfrac{-a+b-c+1}{2}\}; & M_B &= \min\{b-2, \tfrac{a+b+c-3}{2}\};\\ m_C &= \max\{0, \tfrac{-a-b+c+3}{2}\}; & M_C &= \min\{c-1, \tfrac{a+b+c-3}{2}\}.\end{aligned}$$

Lemma 4.4. *The caterpillar $G = CR(a,b,c)$ has the maximum possible friendly index if and only if $|b-a-c+1| \le 1$.*

Proof. By the Corollary 2.5, such a friendly labeling f exists if and only if all edges are labeled 1. Let $f(u) = f(w) = 1$ and $f(v) = 0$. Then all the end-vertices adjacent to u and w are labeled 0, and all end-vertices adjacent to v are labeled 1. That is, $a+c-1$ vertices are labeled 0 and b vertices are labeled 1. But for this labeling to be friendly we require $|b-a-c+1| \le 1$. □

Theorem 4.5. *Let $a+b+c$ be odd. Then $G = CR(a,b,c)$ is fully cordial if and only if $b = a+c-1$ and $a = 2$ or $c = 2$.*

Proof. Suppose G is fully cordial. Then by Lemma 4.4, $b = a+c-1$. Also, $a = 2$ or $c = 2$. Otherwise, using the notation of Theorem 4.3, the sets A and C are subsets of B and $FI(G) = B = \{|\,2b-4j-1\,| : 0 \le j \le b-2\}$. However, this set has $b-1$ odd numbers; the smallest is 1 and the largest element is $2b-1$. Therefore, one odd number between 1 and $2b-1$ is missing. In fact, $2b-3$ is not in $FI(G)$.
Conversely, let $b = a+c-1$ and $a = 2$ or $c = 2$. Without loss of generality, we may assume $a = 2$. In this case $G = CR(2, c+1, c)$. Using 4.3, one can easily see that $FI(G) = \{1, 3, \ldots, 2c+1\}$ which shows that G is fully cordial. □

In what follows, we consider the caterpillar $G = CR(a,b,c)$, when $a+b+c$ is even. First we determine its friendly index set, then we completely identify those that are fully cordial. As mentioned before, G has $a+b+c-1$ vertices and $a+b+c-2$ edges.
We observe that any friendly coloring $f : G \to \mathbb{Z}_2$ that labels the central vertices the same will result in either index $N(f) = 0$ or $N(f) = 2$, which are not very interesting. Therefore, we consider the cases in which the central vertices are labeled differently.
Case 1. Suppose we label the central vertices by $f(u) = 0$, and $f(v) = f(w) = 1$ and label all other vertices by 1 except

$$\begin{aligned}f(u_1) &= f(u_2) &= \cdots &= f(u_i) = 0;\\ f(v_1) &= f(v_2) &= \cdots &= f(v_j) = 0;\\ f(w_1) &= f(w_2) &= \cdots &= f(w_k) = 0.\end{aligned} \tag{4.1}$$

Then $v(0) = i + j + k + 1$ and $e(1) = a - i + j + k$. For this labeling to be friendly we require either

$$i + j + k + 1 = \frac{a+b+c}{2}, \tag{4.2}$$

or

$$i + j + k + 1 = \frac{a+b+c-2}{2}, \tag{4.3}$$

Equation (4.2) yields $N(f) = |e(1) - e(0)| = |2a - 4i|$. In this situation, $i + 1 \leq (a + b + c)/2$ and $a - i + 1 \leq (a + b + c - 2)/2$, which provide the inequalities $(a-b-c+4)/2 \leq i \leq (a+b+c-2)/2$. Therefore, the friendly indices obtained in this case would be

$$A = \{|2a - 4i| : m_A \leq i \leq M_A\},$$

where $m_A = \max\{0, (a-b-c+4)/2\}$ and $M_A = \min\{a-1, (a+b+c-2)/2\}$. Equation (4.3) gives us $|e(1) - e(0)| = |2a - 4i - 2|$. In this situation, $i + 1 \leq (a + b + c - 2)/2$ and $a - i + 1 \leq (a + b + c)/2$, which provide the inequalities $(a-b-c+2)/2 \leq i \leq (a+b+c-4)/2$. Therefore, the friendly indices obtained in this situation would be

$$D = \{|2a - 4i - 2| : m_D \leq i \leq M_D\},$$

where $m_D = \max\{0, (a-b-c+2)/2\}$ and $M_D = \min\{a-1, (a+b+c-4)/2\}$.
Case 2. Let $f(v) = 0$, and $f(u) = f(w) = 1$ be the labeling of the central vertices and all other vertices be labeled 1 except for those specified in (4.1). In this case, $v(0) = i+j+k+1$ and $e(1) = b-j+i+k$. Again, for this labeling to be friendly we require either (4.2) or (4.3).
The equation (4.2) gives us $N(f) = |e(1) - e(0)| = |2b - 4j|$. In this instance, $j+1 \leq (a+b+c)/2$ and $b-j \leq (a+b+c-2)/2$, which provide the inequalities $(b-a-c+2)/2 \leq j \leq (a+b+c-2)/2$. Therefore, the friendly indices obtained in this case would be

$$B = \{|2b - 4j| : m_B \leq j \leq M_B\},$$

where $m_B = \max\{0, (b-a-c+2)/2\}$ and $M_B = \min\{b-2, (a+b+c-2)/2\}$. The equation (4.3) yields $N(f) = |e(1) - e(0)| = |2b - 4j - 2|$. In this instance, $j+1 \leq (a+b+c-2)/2$ and $b-j \leq (a+b+c)/2$, which provide the inequalities $(b - a - c)/2 \leq j \leq (a + b + c - 4)/2$. Therefore, the friendly indices obtained in this subcase would be

$$E = \{|2b - 4j - 2| : m_E \leq j \leq M_E\},$$

where $m_E = \max\{0, (b - a - c)/2\}$ and $M_E = \min\{b - 2, (a + b + c - 4)/2\}$.
Case 3. Suppose we label the central vertices by $f(w) = 0$, $f(u) = f(v) = 1$ and label all other vertices by 1 except for those specified in (4.1).

Then $v(0) = i + j + k + 1$ and $e(1) = b - k + i + j$. Again, for this labeling to be friendly we require either (4.2) or (4.3).
The equation (4.2) gives us $N(f) = |e(1) - e(0)| = |2c - 4k|$. In this situation, $k + 1 \leq (a + b + c)/2$ and $c - k + 1 \leq (a + b + c - 2)/2$, which provide the inequalities $(c - a - b + 4)/2 \leq k \leq (a + b + c - 2)/2$. Therefore, the friendly indices obtained in this subcase would be

$$C = \{|2c - 4k| : m_C \leq k \leq M_C\},$$

where $m_C = \max\{0, (c-a-b+4)/2\}$ and $M_C = \min\{c-1, (a+b+c-2)/2\}$.
The equation (4.3) gives us $N(f) = |e(1) - e(0)| = |2c - 4k - 2|$. In this situation, $k+1 \leq (a+b+c-2)/2$ and $c-k+1 \leq (a+b+c)/2$, which provide the inequalities $(c-a-b+2)/2 \leq k \leq (a+b+c-4)/2$. Therefore, the friendly indices obtained in this subcase would be

$$F = \{|2c - 4k| : m_F \leq k \leq M_F\},$$

where $m_F = \max\{0, (c-a-b+2)/2\}$ and $M_F = \min\{c-1, (a+b+c-4)/2\}$.
We summarize the above discussion in the following theorem.

Theorem 4.6. *Suppose $a + b + c$ is even and $a, b, c \geq 2$. Then* $FI(CR(a, b, c)) = A \cup B \cup C \cup D \cup E \cup F$, *where*

$$\begin{array}{ll} A = \{|2a - 4i| : m_A \leq i \leq M_A\}; & D = \{|2a - 4i - 2| : m_D \leq i \leq M_D\}; \\ B = \{|2b - 4j| : m_B \leq j \leq M_B\}; & E = \{|2b - 4j - 2| : m_E \leq j \leq M_E\}; \\ C = \{|2c - 4k| : m_C \leq k \leq M_C\}; & F = \{|2c - 4k - 2| : m_F \leq k \leq M_F\}; \end{array}$$

Lemma 4.7. *Let $a + b + c$ be even. Then the caterpillar $G = CR(a, b, c)$ has a maximum friendly index if and only if $|b - a - c + 1| = 1$.*

Proof. By Corollary 2.5, such a friendly labeling f exists if and only if all edges are labeled 1. Without loss of generality we may assume that $f(u) = f(w) = 0$ and $f(v) = 1$. Then all the end-vertices adjacent to u and w are labeled 1, and all end-vertices adjacent to v are labeled 0. That is, $v(0) = b$. However this labeling is friendly if and only if either $2b = a + b + c$ or $2b = a + b + c - 2$ which proves the lemma. □

Lemma 4.8. *Let $|b - a - c + 1| = 1$. Then $FI(CR(a, b, c)) = \Delta \cup \Omega$ where*

$$\Delta = \{|2b - 4j| : 1 \leq j \leq \lfloor \tfrac{b}{2} \rfloor\}; \quad \Omega = \{|2b - 4j - 2| : 0 \leq j \leq \lfloor \tfrac{b-1}{2} \rfloor\}.$$

Proof. We utilize Theorem 4.6 and note that

$$\begin{array}{ll} A = \{|2a - 4i| : 0 \leq i \leq a - 1\}; & D = \{|2a - 4i - 2| : 0 \leq i \leq a - 1\}; \\ B = \{|2b - 4j| : 1 \leq j \leq b - 2\}; & E = \{|2b - 4j - 2| : 0 \leq j \leq b - 2\}; \\ C = \{|2c - 4k| : 0 \leq k \leq c - 1\}; & F = \{|2c - 4k - 2| : 0 \leq k \leq c - 1\}. \end{array}$$

Since $A \cup D$, $C \cup F \subset B \cup E$ then by 4.6, $FI(G) = B \cup E$. We also observe that $|2b - 4j|$ produces the same number for j and $b - j$. Similarly, $|2b - 4k - 2|$ produces the same number for k and $b-k+1$. Therefore, $B = \Delta$ and $E = \Omega$. □

Theorem 4.9. *Let $a+b+c$ be even. Then $G = CR(a,b,c)$ is fully cordial if and only if $|b-a-c+1| = 1$.*

Proof. Suppose G is fully cordial. Then G achieves its maximum friendly index and by Lemma 4.7, $|b-a-c+1| = 1$.
Conversely, let $|b-a-c+1| = 1$. Then by Lemma 4.8, $FI(G) = \Delta \cup \Omega$. Also, we observe that G has either $2b-2$ or $2b$ edges and the set $\Delta \cup \Omega$ generates exactly either $\{0,2,4,\ldots,2b-2\}$ or $\{0,2,4,\ldots,2b\}$. These numbers are the full spectrum of friendly indices of G. □

References

[1] M. Benson and S-M. Lee, On Cordialness of Regular Windmill Graphs, *Congressus Numerantium* **68** (1989), 49-58.

[2] I. Cahit, Cordial Graphs: a weaker version of graceful and harmonious graphs, *Ars Combinatoria* **23** (1987), 201-207.

[3] I. Cahit, On Cordial and 3-equitable Graphs, *Utilitas Mathematica* **37** (1990), 189-198.

[4] I. Cahit, Recent Results and Open Problems on Cordial Graphs, *Contemporary Methods in Graph Theory,* Bibligraphisches Inst. Mannhiem (1990), 209-230.

[5] N. Cairnie and K.Edwards, The Computational Complexity of Cordial and Equitable Labellings, *Discrete Mathematics* **216** (2000), 29-34.

[6] G. Chartrand, S-M Lee and P. Zhang, Uniformly Cordial Graphs, *Discrete Mathematics,* **306** (2006), 726-737.

[7] G. Chartrand and P. Zhang, A First Course in Graph Theory, *Dover Publications, New York* (2012).

[8] Joseph A. Gallian, A dynamic survey of graph labeling, *Electronic Journal of Combinatorics,* (2014).

[9] Y.S. Ho, S-M. Lee, and S.C. Shee, Cordial Labellings of the Cartesian Product and Composition of Graphs, *Ars Combinatoria* **29** (1990), 169-180.

[10] M. Hovay, A-cordial Graphs, *Discrete Mathematics* **93** (1991), 183-194.

[11] W.W. Kirchherr, On the Cordiality of Certain Specific Graphs, *Ars Combinatoria* **31** (1991), 127-138.

[12] W.W. Kirchherr, NEPS operations on cordial graphs, *Discrete Mathematics,* **115** (1993), 201-209.

[13] S. Kuo, G.J. Chang, and Y.H.H. Kwong, Cordial Labeling of mKn, *Discrete Mathematics* **169** (1997)121-131.

[14] S-M. Lee and A. Liu, A Construction of Cordial Graphs from Smaller Cordial Graphs, *Ars Combinatoria* **32** (1991), 209-214.

[15] S-M. Lee and H.K. Ng, On Friendly Index Sets of Bipartite Graphs, *Ars Combinatoria* **86** (2008), 257-271.

[16] E. Salehi and D. Bayot, Friendly Index Sets of Grids, *Utilitas Mathematica* **81** (2010), 121-130.

[17] E. Salehi and Shipra De, On a Conjecture Concerning the Friendly Index Sets of Trees, *Ars Combinatoria* **90** (2009), 371-381.

[18] E. Salehi and S-M. Lee, On Friendly Index Sets of Trees, *Congressus Numerantium* **178** (2006), 173-183.