On 2-steps-Hamiltonian cubic graphs

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Abstract

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A (p, q)-graph $G = (V, E)$ is said to be AL(k)-traversal if there exist a sequence of vertices $(v_1, v_2, \ldots, v_p)$ such for each $i = 1, 2, \ldots, p-1$, the distance for v_i and v_{i+1} is equal to k. We call a graph G a 2-steps Hamiltonian graph if it has a AL(2)-traversal in G and $d(v_p, v_1) = 2$. In this paper we characterize some cubic graphs which are 2-steps Hamiltonian. We show that no forbidden subgraphs characterization for non 2-steps-Hamiltonian cubic graphs is available by showing every cubic graph is a homeomorphe subgraph of a non 2-steps Hamiltonian cubic graph.

1 Introduction

In this paper we consider graphs with no loops. A graph is called Hamiltonian if it contains a cycle passing through all its vertices. Such a cycle is called a Hamiltonian cycle. A graph is cubic if each of its vertex is of degree 3. Cubic graphs have been much studied in graph theory and they seem to be among the most desirable regular graphs.

Historically, cubic Hamiltonian graphs have been associated with the Four-Color Theorem. In 1880 Tait conjecture that every cubic 3-connected planar graph is Hamiltonian. However, Tait's conjecture turned out to be

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false. The first simple 3-polytope with no Hamiltonian circuit was constructed by Tutte [8, 9].

Lau, Lee, et al. [7], extended the general concept of Hamiltonian to k-steps Hamiltonian graphs.

Definition 1. For $k \geq 2$, a (p, q)-graph $G = (V, E)$ is said to have k-step traversal if there exist a sequence $(v_1, v_2, \ldots, v_p)$ such for each $i = 1, 2, \ldots, p-1$, the distance for v_i and v_{i+1} is equal to k. A graph admits a k-step traversal is called the AL(k)-traversal graph.

Example 1. The following graph is AL(2)-travesal.

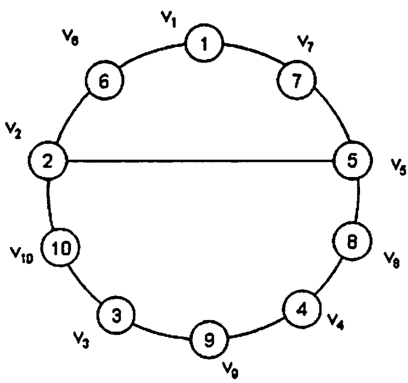


Figure 1:

Example 2. The following graph is AL(2)-traversal but not AL(3)-traversal.

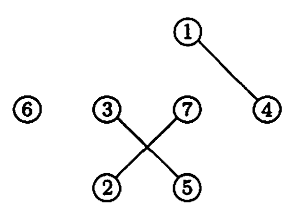


Figure 2: An AL(2)-traversal Graph

Definition 2. We call a graph G a k-step Hamiltonian graph if it has a AL(k)-traversal in G and $d(v_p, v_1) = k$.

A Hamiltonian graph need not be k-step Hamiltonian. The simplest examples are cycles C_n with $n = 0 \pmod k$ which are not AL(k)-traversal, hence cannot be k-step Hamiltonian.

Example 3. A 2-step Hamiltonian cubic graph.

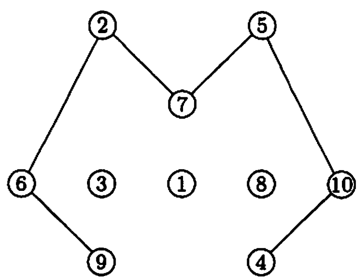


Figure 3: A 2-steps Hamiltonian cubic graph

Example 4. The following graph G is 2-steps Hamiltonian but not Hamiltonian.

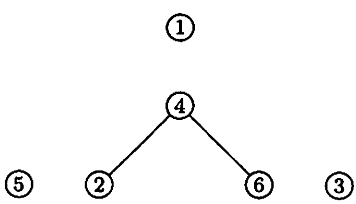


Figure 4: A 2-steps Hamiltonian cubic graph

Example 5. The following Grötzsch graph is Hamiltonian and 2-steps Hamiltonian.

Deciding whether a graph is Hamiltonian, is a notorious difficult problem even for cubic graphs. The same situation is also true for 2-steps Hamiltonian cubic graphs. In section 2, we exhibits infinite classes of cubic graphs which are non 2-steps Hamiltonian. In section 3, we determine generalized Petersen graphs $G(n, k)$ which are 2-steps Hamiltonian and we also find 2-steps Hamiltonian starfish cubic graph SF(n). Finally in section 4, we introduce the bridge-join construction which will produce infinite families of 2-steps Hamiltonian cubic graphs. We also showed that it is impossible

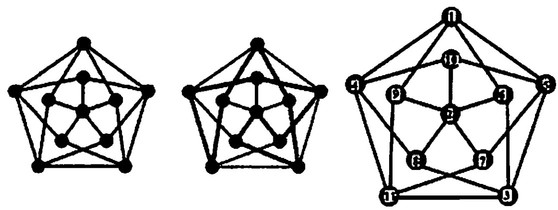


Figure 5:

to have Kuratowski type of characterization of non 2-steps Hamiltonian cubic graphs.

2 Cubic graphs which are not 2-steps Hamiltonian

Barnette's conjecture is an unsolved problem in graph theory, it states that every bipartite cubic polyhedral graph has a Hamiltonian cycle. However, we have the following surprising result

Proposition 2.1. *Every bipartite cubic graph is not 2-steps Hamiltonian.*

Theorem 2.2. *$C_{2n+1} \times K_2$ is 2-steps Hamiltonian for all $n > 1$.*

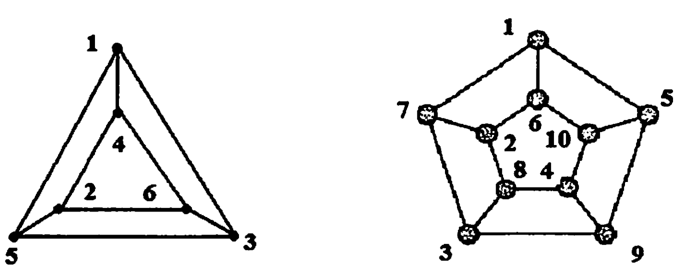
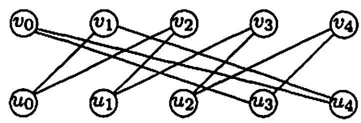


Figure 6: $C_3 \times K_2$ and $C_5 \times K_2$ are 2-steps Hamiltonian.

Corollary 2.3. *For $n \geq 2$, the cylinder graph $C_{2n} \times K_2$ is not 2-steps Hamiltonian.*

We can construct infinitely many bipartite cubic graphs which include the utility graph $K_{3,3}$ as a special case. For integer $n \geq 3$, let $U(n)$ with

$2n$ vertices $\{u_0, u_1, \ldots, u_{n-1}\} \cup \{v_0, v_1, \ldots, v_{n-1}\}$. If we label the vertices $\{u_0, u_1, \ldots, u_{n-1}\} \cup \{v_0, v_1, \ldots, v_{n-1}\}$ and add an edge from each u_i to v_i, v_{i+1} and v_{i+2} (with indices modulo n), then we obtain a cubic bipartite graph. (The vertex v_i is adjacent to u_i, u_{i-1} and u_{i-2}, so it is indeed cubic.) An example when $n = 5$ is given below:



Corollary 2.4. *The graph $U(n)$ is not 2-steps Hamiltonian for any $n \geq 3$.*

Let n be a positive integer. The Möbius ladder (also known as the Möbius wheel) is the cycle C_{2n}, with n additional edges joining diagonally opposite vertices. We will denote this graph by M_{2n}, the vertices by $v_1, v_2, \ldots, v_{2n}$. Then the edges are $(v_1, v_2), (v_2, v_3), \ldots, (v_{2n}, v_1)$ in the cycle, and the n diagonals $(v_1, v_{n+1}), (v_2, v_{n+2}), \ldots, (v_n, v_{2n})$. Figure 7 shows the Möbius ladder M_{2n} for $n = 3, 4$, drawn in both the circulant form and the ladder form. This class of graphs was first named and introduced by F. Harary and R. Guy [4] in 1967.

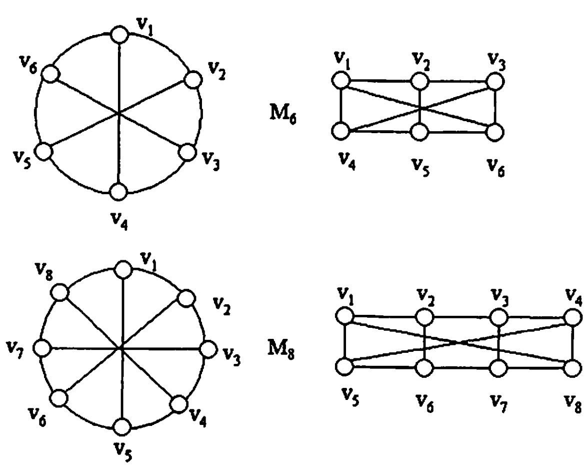


Figure 7:

Lau, Lee et al. showed in [7] that

Theorem 2.5. *Möbius ladder M_n is 2-steps Hamiltonian for $n = 0 \pmod 4$. When $n \equiv 2 \pmod 4$, M_n is not 2-steps Hamiltonian.*

Example 6. The Möbius ladder M_8 is 2-steps Hamiltonian.

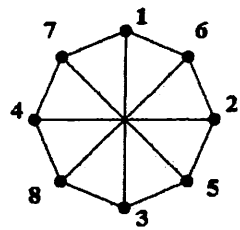


Figure 8: The Möbius ladder M_8 is 2-steps Hamiltonian.

3 Generalized Petersen Graphs and starfish graphs

After Watkins [10], a generalized Petersen graph $G(n,k)$ is defined as

Definition 3. $G(n,k)$ is a graph with vertex set

$$\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$$

and edge set

$$\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} \mid i = 0, 1, \ldots, n-1\}$$

where subscripts are to be read modulo n and $k < \frac{n}{2}$.

Alspach in [2] showed that $G(n,k)$ is Hamiltonian if and only if it is neither $G(n,2) \cong G(n,n-2) \cong G(n,\frac{n-1}{2} \cong G(n,\frac{n+1}{2})$ when $n \equiv 5 \pmod 6$ nor $G(n,\frac{n}{2}$ when $n \equiv 0 \pmod 4$ and $n \geq 8$. We show here

Theorem 3.1. *If n is even and k is odd, $G(n,k)$ is not 2-steps Hamiltonian.*

Proof. For $G(n,k)$ is bipartite if and only if n is even and k is odd. We know that all bipartite graphs are not 2-steps Hamiltonian. □

Theorem 3.2. *For any $n \geq 5$, the generalized Petersen graph $G(n,2)$ is 2-steps Hamiltonian.*

Proof. For $G(n,2)$, label $u_0, u_1, \ldots, u_{n-1}$ by odd integers $1, 3, 5, \ldots, 2n-1$ and $v_2, v_3, \ldots, v_{n-1}, v_0, v_1$ by even integers $2, 4, 6, \ldots, 2n-2, 2n$. Then we see that the 2-steps circuit

$$(u_0, v_2, u_1, v_3, \ldots, v_{n-1}, u_{n-2}, v_0, , u_{n-1}, v_1, u_0)$$

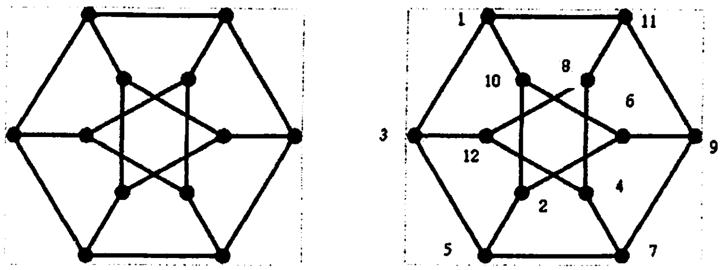


Figure 9: $G(6,2)$ is 2-steps Hamiltonian.

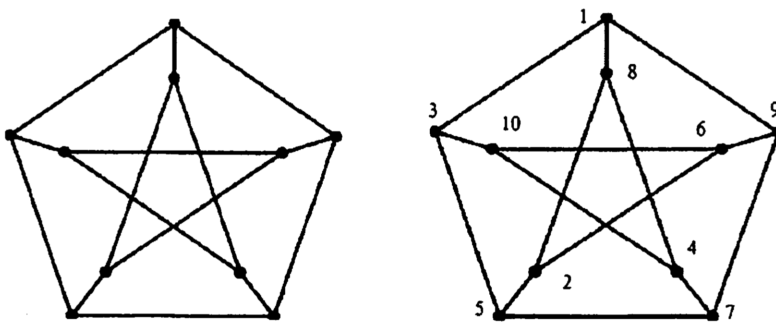
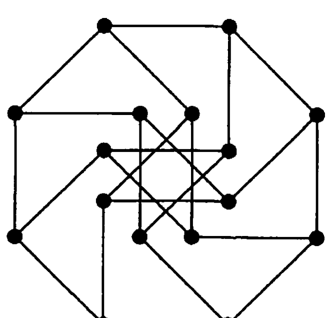


Figure 10: $G(5,2)$ is 2-steps Hamiltonian

is Hamiltonian. Figures 9 and 10 illustrate the cases for $G(5,2)$ and $G(6,2)$. □

Theorem 3.3. *For any $n \geq 6$, the generalized Petersen graph $G(n,3)$ is 2-steps Hamiltonian if and only if n is odd.*

Proof. If n is even, $G(n,3)$ is bipartite. The graph $G(8,3)$ is shown here.



If n is odd, see the general prove in Theorem 3.6. □

Theorem 3.4. *For any $n \geq 9$, the generalized Petersen graph $G(n, 4)$ is 2-steps Hamiltonian.*

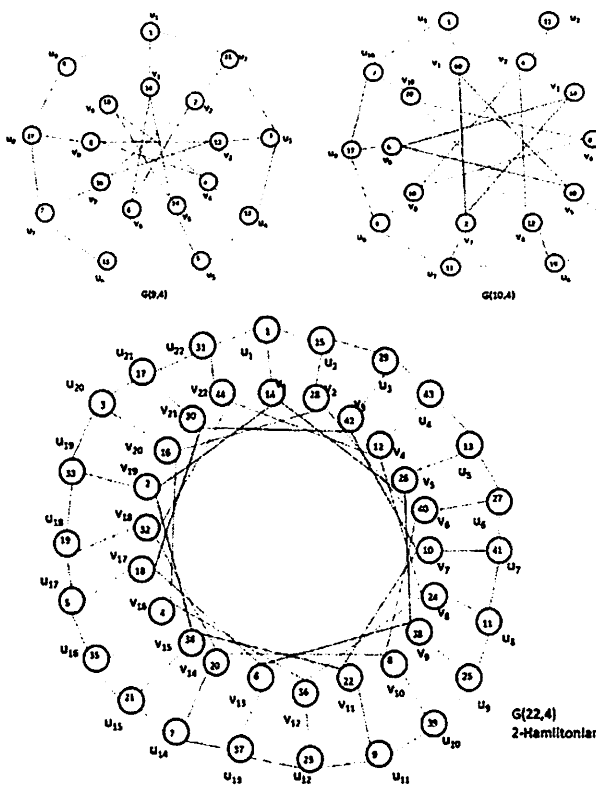


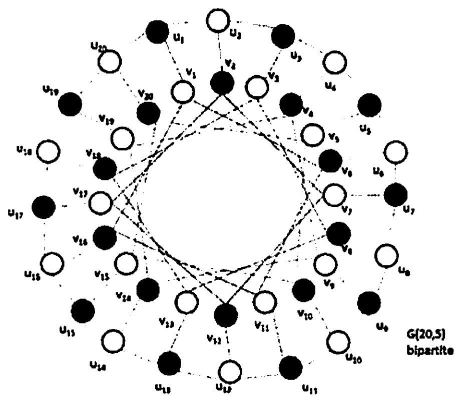
Figure 11:

Theorem 3.5. *For any $n \geq 11$, the generalized Petersen graph $G(n, 5)$ is 2-steps Hamiltonian if and only if n is odd.*

Proof. If n is even, then $G(n, 5)$ must be a bipartite and hence not 2-Hamiltonain. To prove $G(\text{even}, 5)$ is bipartite. Let us color

$$u_1, u_3, u_5, \ldots, u_{2i+1}, \ldots, u_{n-1}$$

black (i.e. u_{odd} black) and u_{even} white. Add color v_{odd} white and v_{even} black. Then $v_j, \ldots, v_{j+10i \text{ (mod } n)}$ for $i = 0, 1, 2, \ldots$ and $j = 1, 3, 5, 7, 9$ are white and $v_{j+5}, \ldots, v_{j+5+10i \pmod n}$ for $i = 0, 1, 2, \ldots$ and $j = 1, 3, 5, 7, 9$ are black. Note: in the v's, v_j is adjacent to v_{j+5}. Now we see the adjacent vertices of the u's have alternate colors and the adjacent vertices of the v's also have alternate colors and also the vertices pairs u_i and v_i have different colors (i.e. if u_i is black then v_i is white, and vise versa). This shows all adjacent vertices have different colors and hence $G(2k, 5)$ is bipartite. An example of $G(20, 5)$ is shown here.



If n is odd, see Theorem 3.6 below.

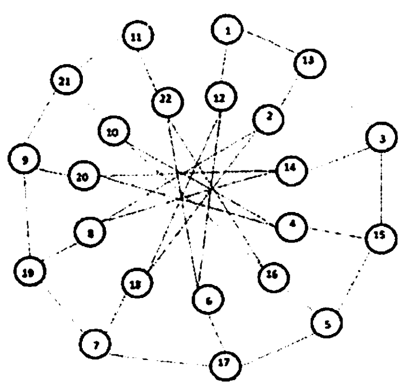


Figure 12: $G(11, 5)$

□

Theorem 3.6. *Any generalized Peterson graph $G(n, k)$ is 2-steps Hamiltonian for n is odd and any $k < \frac{n}{2}$.*

Proof. If n is odd, then we can have the vertex sequence

$$u_1, v_2, u_3, v_4, u_5, \ldots, v_{n-1}, u_n, v_1, u_2, v_3, u_4, v_5, u_6, v_7, \ldots, u_{n-1}, v_n, u_1$$

as the 2-steps Hamiltonian cycle. □

The graph $K_{1,3}$ is called the claw in literature. Assume $V(K_{1,3}) = \{u, x_1, x_2, x_3\}$, we can construct a cubic graph called the starfish SF(k) as follows: take k copies of claw and label the vertices of i copy by $\{u_i, x_{i,1}, x_{i,2}, x_{i,3}\}$. We denote $V(\mathrm{SF}(k)) = \cup\{u_i, x_{i,1}, x_{i,2}, x_{i,3} \mid 1 \le i \le k\}$ and

$$E(\mathrm{SF}(k)) = \cup E(K_{1,3,j} \mid 1 < j < k) \cup \{(x_{i,j}, x_{i,j+1}) \mid 1 \le i \le 3, 1 \le j \le k\}.$$

The graph is not Hamiltonian.

Theorem 3.7. *The starfish SF(n) is 2-steps Hamiltonian if and only if $n \ge 3$ is odd.*

Proof. If n is even, then SF(n) is bipartite and it is not 2-steps Hamiltonian.

If $n \ge 3$ is odd, we see that it is 2-steps Hamiltonian by observe the vertex sequence

$$(u_1, x_{2,1}, x_{2,2}, x_{2,3}, u_3, x_{4,1}, x_{4,2}, x_{4,3}, u_5, x_{6,1}, x_{6,2}, x_{6,3}, \ldots, v_{n-1}, u_n,$$
$$v_1, u_2, v_3, u_4, v_5, u_6, v_7, \ldots, u_{n-2}, x_{n-1,1}, x_{n-1,2}, x_{n-1,3})$$

as the 2-steps Hamiltonian cycle.

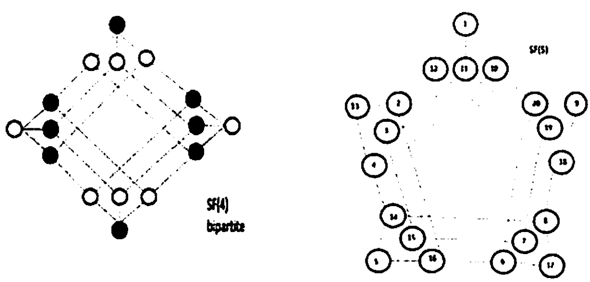


Figure 13:

□

4 Bridge join of cubic graphs

Recall an edge e of a graph G is called a cut edge if $G - e$ is disconnected. It is well known that if G possess a cut edge then it is not Hamitonian. One may propose the following

Conjecture 1. Cubic graph G when $G - e$ is disconnected is not 2-steps Hamiltonian.

However, we have the following infinitely many counterexamples through the following construction:

Bridge join construction: Let (G,e), (H,f) be two graphs in Reg(3), and e,f are edges in G,H, respectively. Insert a vertex u in e and v in f, then join the edge (u,v) between u and v, the resulting graph is a cubic graph, with bridge (u,v). We denote the graph by $B((G,e),(H,f))$.

Denote Reg$(2,3)\{1\}$ the class of all $(2,3)$-regular graphs $(G,\{u\})$ with only one vertex u with degree 2. We give the following sufficient conditions for bridge join cubic graphs are 2-step Hamiltonian.

Theorem 4.1. *If $(G,\{u\})$ and $(H,\{v\})$ are two graphs in Reg$(2,3)\{1\}$, we form $B(G,H)$ by join u and v, the cubic graph $B(G,H)$ is 2-steps Hamiltonian if*

1. *G is* AL(2)*-traversal with 2-steps Hamiltonian path $(u,\ldots,x)$ such that $d_G(u,x)=1$ and H is* AL(2)*-traversal with 2-steps Hamiltonian path $(y,\ldots,v)$ such that $d_H(v,y)=1$ or*

2. *G is* AL(2)*-traversal with 2-steps Hamiltonian path $(x,\ldots,u)$ such that $d_G(u,x)=1$ and H is* AL(2)*-traversal with 2-steps Hamiltonian path $(v,\ldots,y)$ such that $d_H(v,y)=1$.*

Example 7. $B(C_3\times K_2, C_3\times K_2)$ and $B(C_5\times K_2, C_5\times K_2)$ are 2-steps Hamiltonian.

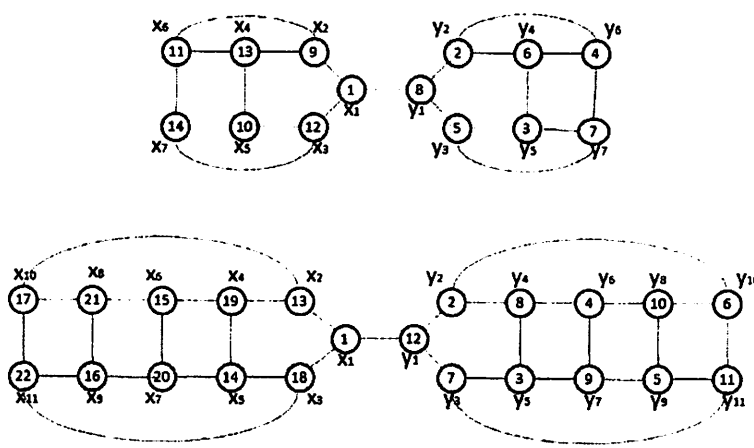


Figure 14: $B(C_3\times K_2, C_3\times K_2)$ and $B(C_5\times K_2, C_5\times K_2)$

Now, we have the general Theorem

Theorem 4.2. *For any odd $m, n \geq 3$, the bridge join of $(C_m \times K_2, e)$ and $(C_n \times K_2, f)$ is 2-steps Hamiltonian with cut edges.*

Examples 8 and 9 illustrate other $(G, \{u\})$ and $(H, \{v\})$ in Reg$(2, 3)\{1\}$, that produce 2-steps Hamiltonian cubic graphs.

Example 8. In the Figure 15.

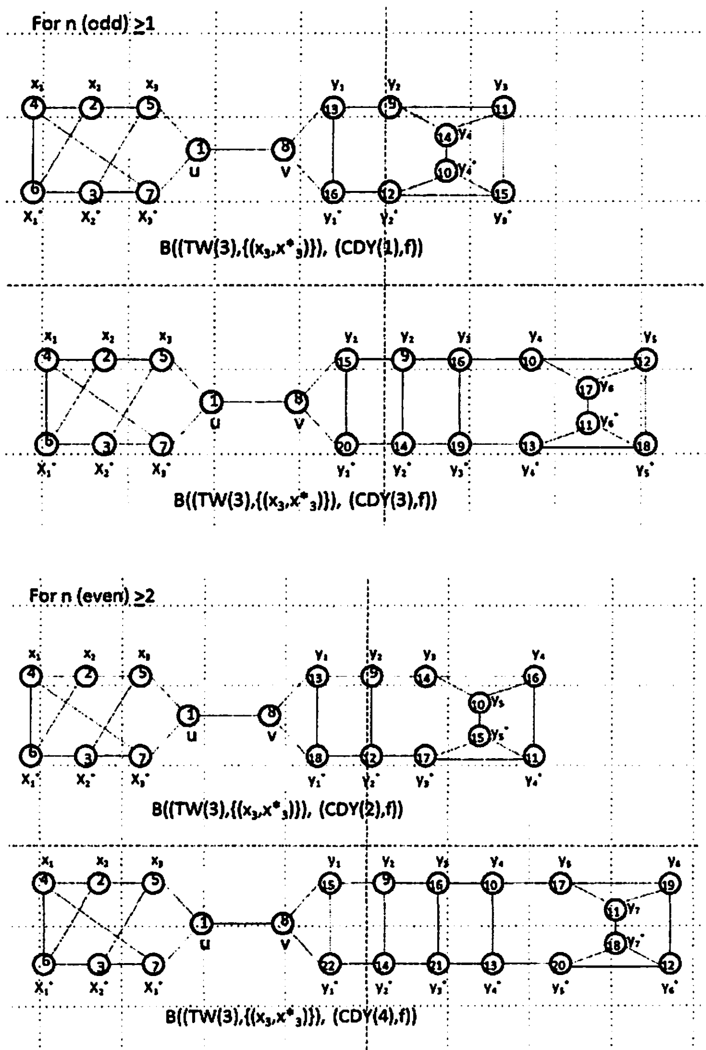


Figure 15:

Example 9.

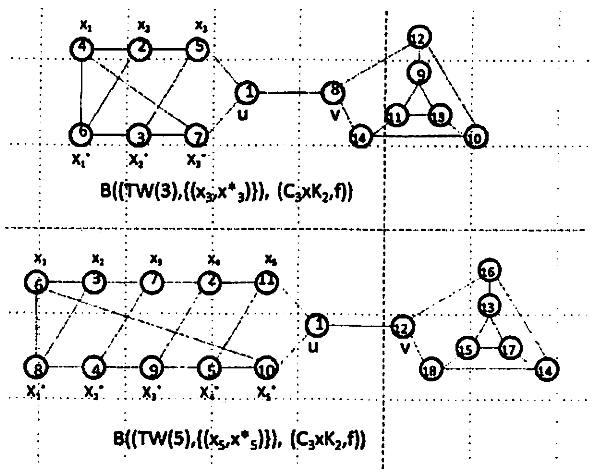
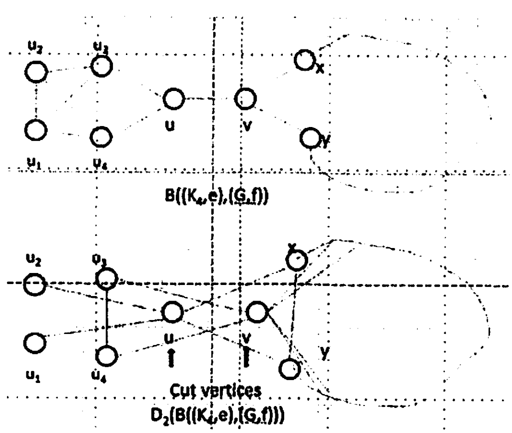


Figure 16:

Theorem 4.3. *The bridge join of (K_4, e) and any cubic graph (G, f) is not 2-steps Hamiltonian.*

Proof. Let $V(K_4) = \{u_1, u_2, u_3, u_4\}$ and $e = (u_3, u_4)$ and let u be the vertex insert on e in K_4. We see that in $D_2(B(K_4, e), (G, f))$, u_1 is connect with u and u_2 is connect with u, u_1 and u_2 are degree 1. Thus the graph $D_2(B(K_4, e), (G, f))$ is non-Hamiltonian.



Thus $B((K_4, e), (G, f))$ is not 2-steps Hamiltonian. □

We also have the following

Theorem 4.4. *The bridge-join graph* $B((K_{3,3}, \{x_3, y_3\}), (H, f))$ *of* $K_{3,3}$, *and any cubic graph* H *and* $f \in E(H)$ *is non* 2*-steps Hamiltonian cubic graph.*

Kuratowski [6] in 1930 showed a graph is non planar if and only if it contains $K_{3,3}$ or K_5 as induced subgraphs. From Theorem 4.4 we have the following

Theorem 4.5. *There does not exist forbidden subgraphs characterization for non* 2*-steps Hamiltonian cubic graphs.*

We propose the following two conjectures.

Conjecture 2. The bridge join of any Möbius ladders $(M_n, e), (M_n, f)$ is not 2-steps Hamiltonian.

Conjecture 3. For any $n \geq 3$, the bridge join of $(U(n), e)$ and any cubic graph (G, f) is not 2-steps Hamiltonian.

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