On ℓ-Path-Hamiltonian and ℓ-Path-Pancyclic Graphs –Results and Problems–

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January 17, 2015

Abstract

Let G be a Hamiltonian graph of order $n \geq 3$. For an integer ℓ with $1 \leq \ell \leq n$, the graph G is ℓ-path-Hamiltonian if every path of order ℓ lies on a Hamiltonian cycle in G. The Hamiltonian cycle extension number of G is the maximum positive integer *ell* for which every path of order ℓ or less lies on a Hamiltonian cycle of G. For an integer ℓ with $2 \leq \ell \leq n-1$, the graph G is ℓ-path-pancyclic if every path of order ℓ in G lies on a cycle of every length from $\ell+1$ to n. (Thus, a 2-path-pancyclic graph is edge-pancyclic.) A graph G of order $n \geq 3$ is path-pancyclic if G is ℓ-path-pancyclic for each integer ℓ with $2 \leq \ell \leq n-1$. In this paper, we present a brief survey of known results on these two parameters and investigate the ℓ-path-Hamiltonian graphs and ℓ-path-pancyclic graphs having small minimum degree and small values of ℓ. Furthermore, highly path-pancyclic graphs are characterized and several well-known classes of ℓ-path-pancyclic graphs are determined. The relationship among these two parameters and other well-known Hamiltonian parameters are investigated along with some open questions in this area of research.

Keywords: Hamiltonian graph, pancyclic graph, ℓ-path-Hamiltonian graph, ℓ-path-pancyclic graph.

AMS subject classification: 05C38, 05C45,

1 Introduction

A *Hamiltonian path* in a graph G is a path containing every vertex of G and a *Hamiltonian cycle* in G is a cycle containing every vertex of G. A graph having a Hamiltonian cycle is a *Hamiltonian graph.* Among the many sufficient conditions for a graph to be Hamiltonian are those concerning the minimum degree $\delta(G)$ of a graph G, the minimum of the degree sums of two nonadjacent vertices in G and the size of G. For a nontrivial graph G that is not complete, let

$$\sigma_2(G) = \min\{\deg u + \deg v : d(u,v) \geq 2\}.$$

The first theoretical result on Hamiltonian graphs occurred in 1952 and is due to Dirac.

Theorem 1.1 [10] *If G is a graph of order $n \geq 3$ such that $\delta(G) \geq n/2$, then G is Hamiltonian.*

In 1960, Ore obtained a result that generalizes Theorem 1.1.

Theorem 1.2 [15] *If G is a graph of order $n \geq 3$ such that $\sigma_2(G) \geq n$, then G is Hamiltonian.*

The following known result gives another sufficient condition for a graph to be Hamiltonian.

Theorem 1.3 [9, p.136] *If G is a graph of order $n \geq 3$ and size $m \geq \binom{n-1}{2} + 2$, then G is Hamiltonian.*

A graph G is *Hamiltonian-connected* if G contains a Hamiltonian $u - v$ path for every pair u, v of distinct vertices of G. In 1963, Ore provided the following similar sufficient conditions for a graph to be Hamiltonian-connected.

Theorem 1.4 [16] *If G is a graph of order $n \geq 4$ such that $\delta(G) \geq (n+1)/2$, then G is Hamiltonian-connected.*

Theorem 1.5 [16] *If G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n+1$, then G is Hamiltonian-connected.*

Theorem 1.6 [16] *If G is a graph of order $n \geq 4$ and size $m \geq \binom{n-1}{2} + 3$, then G is Hamiltonian-connected.*

The concepts of Hamiltonian paths, Hamiltonian cycles and Hamiltonian graphs have been studied extensively in the area of graph theory. The research in this area gave rise to a number of new concepts and properties involving paths and cycles in graphs, such as being panconnected [1, 18] and pancyclic [4]. Recently, two more new concepts involving paths and cycles in graphs were introduced and studied in [2, 5] respectively, namely ℓ-path-Hamiltonian graphs and ℓ-path-pancyclic graphs. In this work, we present a brief survey of known results on these two parameters and several new results on the ℓ-path-Hamiltonian graphs and ℓ-path-pancyclic graphs, as well as some open questions on the relationship among these two parameters and other well-known Hamiltonian parameters. We refer to the book [9] for graph theoretic notation and terminology not described in this paper.

2 On ℓ-Path-Hamiltonian Graphs

A Hamiltonian graph G of order $n \geq 3$ is *ℓ-path-Hamiltonian* for some positive integer ℓ with $1 \leq \ell \leq n$ if every path of order ℓ lies on a Hamiltonian cycle in G. The *Hamiltonian cycle extension number* $\mathrm{hce}(G)$ of G is defined as the largest integer ℓ such that G is i-path-Hamiltonian for each integer i with $1 \leq i \leq k\ell$. These concepts were derived from an 1856 observation of Hamilton when he introduced the *Icosian Game*, which is a two-person game that could be played on the vertices and edges of a dodecahedron (a polyhedron with twenty vertices). Hamilton observed that beginning with any path P of order 5 on the graph G of the dodecahedron, P could be extended to a Hamiltonian cycle of G. That is, for every path P of order 5 in G, there exists a Hamiltonian cycle C of G such that P is a path on C. In fact, this is true for *all paths of order* 5 *or less* on the graph of the dodecahedron. However, Hamilton's observation does not hold for all paths of order 6, that is, there *is* a path of order 6 on the graph G of the dodecahedron that cannot be extended to a Hamiltonian cycle in G. This is illustrated in Figure 1, where the path of order 6 (drawn in bold edges) cannot be extended to a Hamiltonian cycle. This led to a concept of the Hamiltonian cycle extension number defined for every Hamiltonian graph. These concepts were introduced by Gary Chartrand in 2013 and first studied in [5].

2.1 Some Known Results

Among the results obtained on ℓ-path-Hamiltonian graphs are the following.

Theorem 2.1 [5] *If G is a graph of order $n \geq 3$ and $\delta(G) \geq n/2$, then* $\mathrm{hce}(G) \geq 2\delta(G) - n + 1$.

Theorem 2.2 [5] *If G is a graph of order $n \geq 4$ such that $\delta(G) \geq rn$ for some rational number r with $1/2 \leq r < 1$, then* $\mathrm{hce}(G) \geq (2r-1)n + 1$.

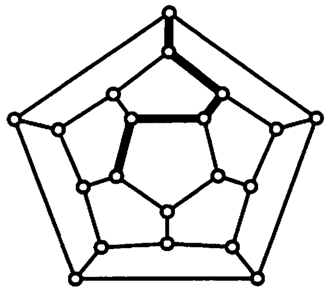

Figure 1: The graph G of the dodecahedron and a 6-path

Theorem 2.3 [5] *Let ℓ and n be positive integers such that $n \geq \ell + 2$. If G is a graph of order n such that $\sigma_2(G) \geq n + \ell - 1$, then* $\text{hce}(G) \geq \ell$.

Theorem 2.4 [5] *Let ℓ and n be positive integers such that $n \geq \ell + 2$. If G is a graph of order n and size $m \geq \binom{n-1}{2} + \ell + 1$, then G is ℓ-path-Hamiltonian.*

Each of the Theorems 2.1–2.4 is best possible. In particular, the lower bound presented in Theorem 2.2 for the Hamiltonian cycle extension number of a graph is sharp for every rational number r. Theorems 2.3 and 2.4 are extensions of Ore's results on Hamiltonian-connected graphs in Theorems 1.5 and 1.6.

All connected graphs G of order n having $\text{hce}(G) = n$ have been characterized in [8]; while all Hamiltonian graphs of order n that are ℓ-path-Hamiltonian are characterized for $\ell \in \{n-3, n-2, n-1\}$ in [13].

Theorem 2.5 [13] *Let G be a graph of order $n \geq 3$.*

(a) *For each integer $\ell \in \{n-2, n-1, n\}$, the graph G is ℓ-path-Hamiltonian if and only if G equals C_n or K_n or $K_{n/2,n/2}$ (when n is even).*

(b) *For $n \geq 4$, the graph G is $(n-3)$-path-Hamiltonian if and only if*

  (i) *G equals C_n or K_n or $K_{n/2,n/2}$ (when n is even) or*

  (ii) *$\overline{G} \in \{P_3 + P_2, C_6, 2P_3, C_4 + C_3\}$ or*

  (iii) $\delta(G) = n - 2$.

2.2 On ℓ-path-Hamiltonian Graphs with Small Minimum Degree

We begin by studying 2-path-Hamiltonian graphs with minimum degree 2. Obviously, the n-cycle C_n of order $n \geq 3$ is ℓ-path-Hamiltonian for $1 \leq \ell \leq$

n and 2-regular. The following result shows that the n-cycle of order $n \geq 4$ is an exception.

Proposition 2.6 *Let ℓ, n be integers satisfying $1 \leq \ell \leq n$ and $n \geq 3$. If G is an ℓ-path-Hamiltonian graph of order n, then* (i) $G = C_n$ *or* (ii) $\ell \leq 2$ *or* (iii) $\delta(G) \geq 3$.

Proof. Since C_3 is the only Hamiltonian graph of order 3, we may assume that $n \geq 4$. Let G be a connected graph of order n with $\delta(G) = 2$. We show that if $G \neq C_n$, then G is not ℓ-path-Hamiltonian for $\ell \geq 3$. Let $(v_1, v_2, \ldots, v_n, v_1)$ be a Hamiltonian cycle in G. Since $\delta(G) = 2$ and $G \neq C_n$, it follows that $\Delta(G) \geq 3$ and so we may assume, without loss of generality, that $\deg v_n = 2$ and $\deg v_1 \geq 3$, say $\{v_2, v_a, v_n\} \subseteq N(v_1)$ for some integer a ($3 \leq a \leq n-1$). Thus, G contains a Hamiltonian path $P = (v_{a-1}, v_{a-2}, \ldots, v_1, v_a, v_{a+1}, \ldots, v_n)$. For $3 \leq \ell \leq n$, therefore, G contains an ℓ-path Q such that $(v_2, v_1, v_a) \subseteq Q \subseteq P$. Since $\deg v_n = 2$ and v_n is adjacent to v_1, there is no n-cycle on which Q lies and we conclude that $\ell \leq 2$. ■

In [5], it was observed that if G is a Hamiltonian graph of order n with minimum degree 2, then

$$\operatorname{hce}(G) = \begin{cases} n & \text{if } G = C_n \\ 1 & \text{if } G \text{ contains an edge not belonging to any} \\ & \text{Hamiltonian cycle in } G \\ 2 & \text{otherwise.} \end{cases}$$

Of course, if $\operatorname{hce}(G) = n$, then G is ℓ-path-Hamiltonian for $1 \leq \ell \leq n$. In a Hamiltonian graph G of order n, some edge may or may not lie on a Hamiltonian cycle in G while every edge must lie on a Hamiltonian path in G. In fact, for each edge e and every integer ℓ with $2 \leq \ell \leq n$, there exists an ℓ-path containing e. Therefore, if G is ℓ-path-Hamiltonian for some $\ell \geq 2$, then there must be a Hamiltonian cycle in G containing e. In other words, if G is not 2-path-Hamiltonian, then G is not ℓ-path-Hamiltonian for each $\ell \in \{2, 3, \ldots, n\}$.

Suppose that G is a 2-path-Hamiltonian graph with $\delta(G) = 2$ and let v be a vertex in G with $N(v) = \{x, y\}$. Since every edge in G lies on a Hamiltonian cycle in G, it follows that $xy \notin E(G)$ unless $G = C_3$. Then by adding the edge xy while deleting the vertex v and the edges vx and vy from G, a 2-path-Hamiltonian graph H such that G is a subdivision of H results. Consequently, we obtain the following.

Proposition 2.7 *Let G be a graph of order at least 4 with $\delta(G) = 2$. If G is 2-path-Hamiltonian, then G is obtained from a 2-path-Hamiltonian graph by subdividing an edge.*

The converse of Proposition 2.7 is not true, however. For example, let G_0 be the complete graph of order 4. Then for G_1 and G_2 in Figure 2, the graph G_i is obtained from G_{i-1} by subdividing an edge for $i = 1, 2$. Although G_0 and G_1 are both 2-path-Hamiltonian, G_2 is not.

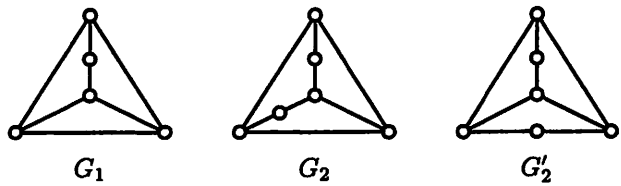

Figure 2: Subdividing edges of K_4

On the other hand, the graph G_2' in Figure 2 is also obtained from G_1 by subdividing an edge and this is 2-path-Hamiltonian. Hence, subdividing an edge of a 2-path-Hamiltonian graph may or may not result in another 2-path-Hamiltonian graph.

In a 2-path-Hamiltonian graph G, suppose that S is a subset of $E(G)$ such that, for each edge e in G, there exists a Hamiltonian cycle C in G with $S \cup \{e\} \subseteq E(C)$. Since G is 2-path-Hamiltonian, the set $S = \emptyset$ certainly has this property. Now let $\mathcal{S}(G)$ be the set of *nonempty* such sets S. If $S \in \mathcal{S}(G)$, then subdividing some edges of S (possibly more than once) results in a 2-path-Hamiltonian graph. Conversely, if $S \neq \emptyset$ and $S \notin \mathcal{S}(G)$, then subdividing every edge of S results in a graph that is *not* 2-path-Hamiltonian.

For example, take $G = K_4$ with $V(G) = \{v_1, v_2, v_3, v_4\}$ and let $S_1 = \{v_1v_2, v_3v_4\}$. Then $S_1 \in \mathcal{S}(G)$ since every edge e in G can be extended to a Hamiltonian cycle C in G that also contains both v_1v_2 and v_3v_4. (In general, if G is complete or regular complete bipartite, then any nonempty set of independent edges belongs to $\mathcal{S}(G)$.) The graph H in Figure 3 as well as the graphs G_1 and G_2' in Figure 2 are 2-path-Hamiltonian graphs obtained from G by subdividing the edges in S_1. On the other hand, the set $S_2 = \{v_1v_2, v_2v_3\}$ does not belong to $\mathcal{S}(G)$ since there is no Hamiltonian cycle C in G with $S_2 \cup \{v_2v_4\} \subseteq E(C)$. Consequently, subdividing both edges in S_2 results in a graph that is not 2-path-Hamiltonian, as the graph G_2 in Figure 2 shows.

As another example, let $G = M_8$ be the Möbius ladder of order 8 obtained from an 8-cycle $C = (v_1, v_2, \ldots, v_8, v_1)$ by adding the four edges v_iv_{i+4} $(1 \leq i \leq 4)$. Note that $\text{hce}(G) = 3$ since G is ℓ-path-Hamiltonian for $1 \leq \ell \leq 3$ while the path (v_1, v_2, v_6, v_7) cannot be extended to a Hamiltonian cycle. In fact, any Hamiltonian cycle in G containing the edge v_2v_6 must contain exactly one of the edges v_1v_2 and v_6v_7. Also, no

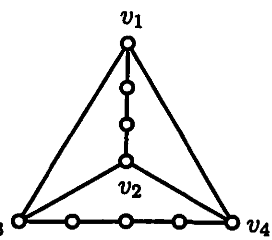

Figure 3: A 2-path-Hamiltonian graph with minimum degree 2

Hamiltonian cycle in G contains both v_1v_5 and v_3v_7. Therefore, a set $S \subseteq E(G)$ belongs to $\mathcal{S}(G)$ if and only if S is a nonempty subset of either $S' = \{v_1v_2, v_3v_4, v_5v_6, v_7v_8\}$ or $E(C)\backslash S'$. Hence, this G is another example of a 2-path-Hamiltonian graph containing an edge e such that a 2-path-Hamiltonian graph results when e is subdivided since $\mathcal{S}(G) \neq \emptyset$. Therefore, we have the following question.

Problem 2.8 *If G is a 2-path-Hamiltonian graph with $\delta(G) \geq 3$, then does G always contain an edge e such that subdividing e results in a 2-path-Hamiltonian graph?*

Next, we determine the Hamiltonian cycle extension number of a well-known class of graphs having minimum degree 2, namely the Cartesian products $P_a \square P_b$ of two paths P_a and P_b of order a and b, respectively. It is straightforward to verify that $P_a \square P_b$ is Hamiltonian if and only if $a, b \geq 2$ and its order ab is even.

Proposition 2.9 *For integers a and b where $2 \leq a \leq b$ and ab is even,*

$$\text{hce}(P_a \square P_b) = \begin{cases} 4 & \text{if } a = b = 2 \\ 1 & \text{if } 2 \leq a \leq 3 \leq b \\ 2 & \text{if } a, b \geq 4. \end{cases}$$

Proof. Let $G = P_a \square P_b$. Since $a, b \geq 2$ and ab is even, G is Hamiltonian and so $\text{hce}(G) \geq 1$. Also, since $\text{hce}(P_2 \square P_2) = \text{hce}(C_4) = 4$, we may assume that at least one of a and b is greater than 2. Let $V(G) = \{v_{i,j} : 1 \leq i \leq a,\ 1 \leq j \leq b\}$ and $v_{i,j}v_{i',j'} \in E(G)$ if and only if either (i) $i = i'$ and $|j - j'| = 1$ or (ii) $j = j'$ and $|i - i'| = 1$.

First, suppose that at least one of a and b, say the former, is less than 4. If $a = 2$, then $b \geq 3$ and the edge $v_{1,2}v_{2,2}$ lies on no Hamiltonian cycle in G. Similarly, if $a = 3$, then no Hamiltonian cycle in G contains the edge $v_{2,1}v_{2,2}$. Consequently, $\text{hce}(G) = 1$ if $2 \leq a \leq 3 \leq b$.

Hence, let us next suppose that $a, b \geq 4$. Then $(v_{1,2}, v_{2,2}, v_{2,1})$ is a 3-path that lies on no Hamiltonian cycle in G. Thus, $\text{hce}(G) \leq 2$ in this case. In order to show that $\text{hce}(G) = 2$, we verify that every edge lies on a Hamiltonian cycle in G. Since ab is even, at least one of a and b is even, that is, either (i) $a \equiv b \equiv 0 \pmod 2$ or (ii) $a \not\equiv b \pmod 2$. In each case, it can be shown that G contains four Hamiltonian cycles such that each edge of G lies on at least one of these four Hamiltonian cycles. This is illustrated in Figure 4 for $P_6 \,\Box\, P_6$ and in $P_6 \,\Box\, P_7$. ■

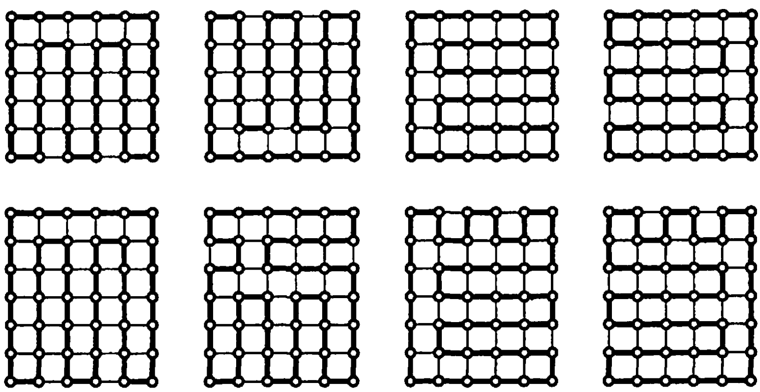

Figure 4: Four Hamiltonian cycles in $P_6 \,\Box\, P_6$ or in $P_6 \,\Box\, P_7$

3 On ℓ-Path-Pancyclic Graphs

A graph G of order $n \geq 3$ is *pancyclic* if G contains a cycle of every possible length, that is, G contains a cycle of length ℓ for each ℓ with $3 \leq \ell \leq n$. The following result is due to Bondy in 1971.

Theorem 3.1 [4] *If G is a graph of order $n \geq 3$ such that $\sigma_2(G) \geq n$, then either G is pancyclic or n is even and G is the complete regular bipartite graph $K_{n/2,n/2}$.*

A graph G of order $n \geq 3$ is *vertex-pancyclic* if each vertex of G lies on a cycle of length from 3 to n. A graph G of order $n \geq 3$ is *edge-pancyclic* if each edge of G lies on a cycle of length from 3 to n. In 1974 Faudree and Schelp established the following result.

Theorem 3.2 [11] *If G is a graph of order $n \geq 5$ such that $\sigma_2(G) \geq n+1$, then for every pair u, v of distinct vertices of G, there is a $u - v$ path of length ℓ for every integer ℓ with $4 \leq \ell \leq n-1$.*

As a consequence of Theorem 3.2, if G is a graph of order $n \geq 5$ such that $\sigma_2(G) \geq n+1$, then every edge of G lies on a cycle of length ℓ for every integer ℓ with $5 \leq \ell \leq n$. In 1993 Zhang and Holton [19] presented a characterization of edge-pancyclic graphs of order $n \geq 5$ such that $\sigma_2(G) \geq n+1$.

Inspired by the concepts of pancyclic graphs, vertex- or edge-pancyclic graphs and ℓ-path-Hamiltonian graphs, the concepts of ℓ-path-pancyclic graphs and path-pancyclic graphs were introduced and studied in [2] and studied further in [3]. For integers ℓ and n with $2 \leq \ell \leq n-1$, a graph G of order n is *ℓ-path-pancyclic* if every path of order ℓ in G lies on a cycle of every length from $\ell+1$ to n. In particular, a 2-path-pancyclic graph is edge-pancyclic. A graph G of order $n \geq 3$ is *path-pancyclic* if G is ℓ-path-pancyclic for each integer ℓ with $2 \leq \ell \leq n-1$. To illustrate these concepts, consider the graph G of order 6 in Figure 5. The graph G is 2-path-pancyclic (or edge-pancyclic). Since the 3-path (u, v, w), for example, does not lie on a cycle of order 4, it is not 3-path-pancyclic. In fact, this graph is also not 3-path-Hamiltonian since the path (u, v, y) does not lie on a Hamiltonian cycle in G.

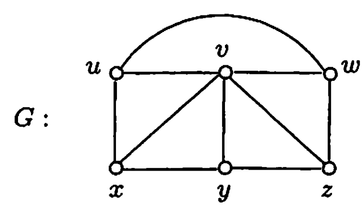

Figure 5: A 2-path-pancyclic graph G that is not 3-path-pancyclic

Certainly, if G is an ℓ-path-pancyclic graph of order $n \geq 4$ where $1 \leq \ell \leq n-1$, then G is also ℓ-path-Hamiltonian. Obviously, the converse is not true. For example, $\text{hce}(C_n) = n$ where $n \geq 4$ and so C_n is ℓ-path-Hamiltonian for each $\ell \in \{1, 2, \ldots, n\}$, but C_n is not ℓ-path-pancyclic for any $\ell \in \{1, 2, \ldots, n-2\}$. In [2, 3], sufficient conditions for a graph G to be ℓ-path-pancyclic were presented in terms of its order, size, minimum degree as well as the value of $\sigma_2(G)$, which we list below.

Theorem 3.3 [2] *Let ℓ and n be integers with $n \geq 4$ and $2 \leq \ell \leq n-1$. If G is a graph of order n with $\delta(G) \geq (n+\ell)/2$, then G is ℓ-path-pancyclic.*

Theorem 3.4 [3] *Let ℓ and n be integers with $n \geq 4$ and $2 \leq \ell \leq n-1$. If G is a graph of order n with $\sigma_2(G) \geq (3n+\ell-5)/2$, then G is ℓ-path-pancyclic.*

Theorem 3.5 [2] *Let ℓ and n be integers with $2 \leq \ell \leq n-2$. If G is a graph of order n and size $m \geq \binom{n-1}{2} + \ell + 1$, then G is ℓ-path-pancyclic.*

It was shown in [2, 3] that each of Theorems 3.3–3.5 is best possible.

3.1 Highly Path-Pancyclic Graphs

We first study connected graphs of order n that are ℓ-path-pancyclic for relatively large values of ℓ, namely, $\ell \in \{n-3, n-2, n-1\}$. Those graphs are referred to as *highly path-pancyclic graphs.* With the aid of Theorem 2.5, we are able characterize all highly path-pancyclic graphs. First, we make some useful observations. If G is a bipartite graph of order $n \geq 4$, then G cannot contain both $(n-1)$-cycles and n-cycles. Therefore, if ℓ and n are integers satisfying $2 \leq \ell \leq n-1$, then G is an ℓ-path-pancyclic bipartite graph of order n if and only if $\ell = n-1$ is odd and G is either C_n or $K_{n/2,n/2}$. Furthermore, each C_n $(n \geq 3)$ is ℓ-path-pancyclic if and only if $\ell = n-1$. We are now prepared to present the following.

Theorem 3.6 *Let G be a graph of order n.*

(a) *For $n \geq 3$, the graph G is $(n-1)$-path-pancyclic if and only if $G \in \{C_n, K_n, K_{n/2,n/2}\}$.*

(b) *For $n \geq 4$, the graph G is $(n-2)$-path pancylic if and only if $G = K_n$.*

(c) *For $n \geq 5$, the graph G is $(n-3)$-path pancylic if and only if $\delta(G) \geq n-2$.*

Proof. Since an ℓ-path-pancyclic graph is ℓ-path-Hamiltonian, (a) and (b) are straightforward by Theorem 2.5(a).

For (c), suppose first that G is an $(n-3)$-path-pancyclic graph of order $n \geq 5$. Consequently, G is $(n-3)$-path-Hamiltonian. As observed above, neither C_n nor $K_{n/2,n/2}$ is $(n-3)$-path-pancyclic. Also, one can verify that G is not $(n-3)$-path-pancyclic if G satisfies (ii) in Theorem 2.5(b). Hence, $G = K_n$ (and so $\delta(G) = n-1$) or $\delta(G) = n-2$ by Theorem 2.5.

Note that K_n is ℓ-path-pancyclic for $2 \leq \ell \leq n-1$. For the converse, therefore, suppose that G is a graph of order $n \geq 5$ and $\delta(G) = n-2$. Let P be an $u-v$ path of order $n-3$ in G and $S = \{x, y, z\} = V(G) \backslash V(P)$. Since G is $(n-3)$-path-Hamiltonian by Theorem 2.5(b), it follows that P belongs to an n-cycle. We claim that P also belongs to an $(n-2)$-cycle and an $(n-1)$-cycle in G. By the fact that $\delta(G) = n-2$, each of u and v is adjacent to at least two vertices in S. Without loss of generality, we may assume that $ux, uy, vx \in E(G)$. Thus, (P, x, u) is an $(n-2)$-cycle. Also, if $xy \in E(G)$, then (P, x, y, u) is an $(n-1)$-cycle. If $xy \notin E(G)$, then $xz \in E(G)$ since $\delta(G) = n-2$. Also, at least one of the $u-v$ paths (u, x, z, v) and (u, z, x, v) exists in G, resulting in an $(n-1)$-cycle containing P. ∎

We saw for each integer $n \geq 3$ that the n-cycle C_n is $(n-1)$-path-pancyclic with minimum degree 2. With the aid of Proposition 2.6, we show that C_n is an exceptional graph in this case as well.

Proposition 3.7 *Let ℓ, n be integers satisfying $2 \leq \ell \leq n-1$. If G is an ℓ-path-pancyclic graph of order n, then* (i) $\ell = n-1$ *and* $G = C_n$ *or* (ii) $\delta(G) \geq 3$.

Proof. Let G be a graph of order n whose minimum degree equals 2. Since C_n is ℓ-path-pancyclic if and only if $\ell = n-1$, let us now assume that $n \geq 4$ and $G \neq C_n$. By Proposition 2.6, the graph G is not ℓ-path-Hamiltonian for $3 \leq \ell \leq n$. Consequently, G is not ℓ-path-pancyclic for $3 \leq \ell \leq n-1$.

Next we show that G is not 2-path-pancyclic. As in Proposition 2.6, supposes that $(v_1, v_2, \ldots, v_n, v_1)$ is a Hamiltonian cycle in G, where $\deg v_1 \geq 3$ and $\deg v_n = 2$. Consider the edge $e = v_1 v_{n-1}$. If $e \in E(G)$, then the edge e belongs to no n-cycle in G. On the other hand, if $e \notin E(G)$, then the edge $v_1 v_n$ belongs to no 3-cycle in G. Thus, G is not 2-path-pancyclic, as claimed. ∎

The graph K_4 is an ℓ-path-pancyclic graph for $\ell = 2, 3$ of order 4 with $\delta(K_4) = 3$. Next, we show that the results presented in Propositions 2.6 and 3.7 are best possible for each pair ℓ, n of integers with $3 \leq \ell \leq n-1$ and $n \geq 6$. The 2-path-pancyclic graph G of Figure 5 has order 6 and $\delta(G) = 3$. We saw that G is not 3-path-pancyclic. In fact, by Theorem 3.6, if G is a graph of order 6 that is 3-path-pancyclic, then $\delta(G) \geq 4$. On the other hand, this example can be extended to show that for each integer $n \geq 7$, there is a graph G_n of order n with $\delta(G_n) = 3$ such that G_n is 2-path-pancyclic but not ℓ-path-pancyclic for any $\ell \in \{3, 4, \ldots, n-1\}$. To see this, we replace the subgraph $K_3 = (u, v, w, u)$ in the graph $G = G_6$ by the complete graph $H = K_{n-3}$ of order $n - 3 \geq 4$ and obtain the graph G_n of order n. Let $u, v, w \in V(H)$ such that the subgraph of G_n induced by $\{u, v, w, x, y, z\}$ is the graph G_6. For each integer $\ell \in \{3, 4, \ldots, n-1\}$, the graph G_n contains an $u - y$ path of order ℓ that does not lie on any Hamiltonian cycle of G_n. Hence, G_n is not ℓ-path-Hamiltonian and so G_n is not ℓ-path-pancyclic.

3.2 Complete Multipartite Graphs

We now determine the values of ℓ for which a complete multipartite graph of order n is ℓ-path-pancyclic. If G is a complete bipartite graph of order $n \geq 4$ that is Hamiltonian, then G is not ℓ-path-pancyclic for each integer $\ell \in \{2, 3, \ldots, n-2\}$ since G contains no odd cycle. Thus, we consider complete multipartite graphs having at least three partite sets.

We represent an arbitrary complete t-partite graph by $G = K_{n_1,n_2,\ldots,n_t}$ where $n_1 \le n_2 \le \cdots \le n_t$ having partite sets $V_1, V_2, \ldots, V_t$ with $|V_i| = n_i$ for $1 \le i \le t$. Thus V_t is a maximum independent set in G and so the independence number of G is $\alpha(G) = |V_t| = n_t$.

The Hamiltonian cycle extension number of a Hamiltonian complete multipartite graph has been determined.

Theorem 3.8 [5] *If G is a Hamiltonian complete multipartite graph of order n, then*

$$\text{hce}(G) = \begin{cases} n & \text{if } G \text{ is complete or } G \text{ is bipartite} \\ n + 1 - 2\alpha(G) & \text{otherwise} \end{cases}$$

where $\alpha(G)$ is the independence number of G.

Again, let G be a complete t-partite graph of order n, where $3 \le t \le n-1$ and V_t is the partite set whose cardinality equals $\alpha(G)$. In the proof of Theorem 3.8 in [5], the fact that the subgraph $G - V_t$ contains an $(n - 2\alpha(G) + 2)$-path P that cannot be extended to a Hamiltonian cycle was used. In fact, one can further observe that this P can be extended to a path of length $n-1$ in G. Thus, for each ℓ with $n - 2\alpha(G) + 2 \le \ell \le n-1$, there is an ℓ-path in G containing P as a subpath and this ℓ-path cannot be extended to a Hamiltonian cycle in G. As a result, we obtain the following.

Observation 3.9 *A complete t-partite graph G of order n, where $t \ge 3$, is ℓ-path-Hamiltonian if and only if $1 \le \ell \le n + 1 - 2\alpha(G)$.*

The next result will be also useful to us.

Proposition 3.10 [3] *Let G be a graph of order n with $\delta(G) \ge (n + \ell - 1)/2$, where n and ℓ are integers satisfying $n \ge 4$ and $2 \le \ell \le n-1$.*

(a) *In G, every path of order ℓ lies on a cycle of length ℓ' for each ℓ' satisfying $\ell + 1 \le \ell' \le n$ except possibly $\ell' = \ell + 2$.*

(b) *If u and v are distinct vertices in G, then $|N(u) \cap N(v)| \ge \ell - 1$.*

We are now prepared to present the following.

Theorem 3.11 *Let G be a complete multipartite graph of order n that is neither bipartite nor complete. Then G is ℓ-path-pancyclic if and only if $2 \le \ell \le n + 1 - 2\alpha(G)$.*

Proof. If G is ℓ-path-pancyclic, then G is ℓ-path-Hamiltonian and so $\ell \le n + 1 - 2\alpha(G)$ by Observation 3.9. Thus, it remains to show that G must be ℓ-path-pancyclic provided $\ell \le n + 1 - 2\alpha(G)$. Suppose that

$2 \leq \ell \leq n+1-2\alpha(G)$ ($\leq n-3$). Then $\delta(G) = n-\alpha(G) \geq n-(n+1-\ell)/2 = (n+\ell-1)/2$. By Proposition 3.10(a), therefore, it suffices to verify that every ℓ-path lies on an $(\ell+2)$-cycle in G.

Let P be an $x-y$ path of order ℓ. Then by Proposition 3.10(a), there exists an $(\ell+1)$-cycle C that contains P as a subpath, say $C = (P, z, x)$. Let $S = V(G)\backslash V(C)$. If $N(x) \cap N(z) \cap S \neq \emptyset$, say there exists a vertex $v \in S$ that is adjacent to both x and z, then an $(\ell+2)$-cycle (P, z, v, x) results. Hence, we next suppose that $N(x) \cap N(z) \cap S = \emptyset$. By Proposition 3.10(b), therefore, $N(x) \cap N(z) = V(P)\backslash\{x\}$, which implies that $xy \in E(G)$. Now $\deg_{G-S} z = \ell < (n+\ell-1)/2$ and so there exists a vertex $w \in S$ that is adjacent to z. Note then that $wx \notin E(G)$. Since G is a complete multipartite graph and $xy \in E(G)$ while $wx \notin E(G)$, it follows that $wy \in E(G)$, which in turn implies the existence of an $(\ell+2)$-cycle (P, w, z, x). ∎

A graph G of order n is *panconnected* if for every two vertices u and v, there is a $u-v$ path of length ℓ for every integer ℓ with $d(u,v) \leq \ell \leq n-1$. It is obvious that every panconnected graph is Hamiltonian-connected, but not conversely. In the case of complete t-partite graphs where $t \geq 3$, more can be said. In the following theorem, Williamson [18] in 1977 showed that (a)–(c) are equivalent. The fact that the statements (c) and (d) are equivalent is a direct consequence of Theorem 3.11.

Theorem 3.12 *Let G be a complete t-partite graph of order n where $t \geq 3$. The following statements are equivalent:*

(a) *The graph G is panconnected;*

(b) *The graph G is Hamiltonian-connected;*

(c) $\alpha(G) \leq (n-1)/2$.

(d) *The graph G is edge-pancyclic.*

3.3 Powers of Cycles

Some 40–50 years ago, there was a great deal of research activity involving Hamiltonian properties of powers of graphs. For a connected graph G and a positive integer r, the *rth power* G^r of G is the graph whose vertex set equals $V(G)$ and $E(G^r) = \{uv : 1 \leq d_G(u,v) \leq r\}$. The graphs G^2 and G^3 are called the *square* and *cube* of G, respectively.

In 1960, Sekanina [17] proved that the cube of every connected graph is Hamiltonian-connected and, consequently, the cube of a connected graph is Hamiltonian if its order is at least 3. In the 1960s, it was conjectured independently by Nash-Williams [14] and Plummer (see [9, p.139]) that the square of every 2-connected graph is Hamiltonian. Fleischner [12] verified this conjecture in 1974 and, in the same year, Chartrand, Hobbs, Jung, Kapoor and Nash-Williams [6] proved that the square of every 2-connected graph is Hamiltonian-connected by using Fleischner's result. Thus the square of every Hamiltonian graph is Hamiltonian-connected. Williamson [1] in 1975 showed that the cube of a connected graph is panconnected.

For a connected graph G of order $n \geq 4$ and an integer k with $1 \leq k \leq n-3$, the graph G is *k-Hamiltonian* if $G - S$ is Hamiltonian for every set S of k vertices of G and *k-Hamiltonian-connected* if $G - S$ is Hamiltonian-connected for every set S of k vertices of G. If the order of a graph G is at least 4, then Chartrand and Kapoor [7] showed that the cube of G is 1-Hamiltonian. Also, it was verified in [5] that G^r is $(2r-3)$-Hamiltonian-connected if G is a Hamiltonian graph of order n and $2 \leq r \leq (n+1)/2$.

The Hamiltonian cycle extension numbers of the powers of an n-cycle has been determined for each integer $n \geq 3$.

Theorem 3.13 [5] *For positive integers r and $n \geq 3$,*

$$\text{hce}(C_n^r) = \begin{cases} n & \text{if } r = 1 \text{ or } r \geq \lfloor n/2 \rfloor \\ 2r-1 & \text{if } 2 \leq r \leq \lfloor n/2 \rfloor - 1. \end{cases}$$

Next, we investigate those powers of cycles that are ℓ-path-pancyclic. For a vertex v in a graph G, the closed neighborhood of v is denoted by $N_G[v] = N_G(v) \cup \{v\}$. Two vertices u and v in a connected graph G are *twins* if u and v have the same neighbors in $V(G) - \{u, v\}$. If u and v are adjacent, they are referred to as *adjacent twins* (or *true twins*); while if u and v are nonadjacent, they are *nonadjacent twins* (or *false twins*). The following observation is useful to us.

Observation 3.14 *For positive integers r and $n \geq 3$, if C_n^r is not complete, then C_n^r does not contain any true twins.*

Theorem 3.15 *Let k and n be integers with $2 \leq k \leq n-1$. If*

(i) $r \geq \lfloor \frac{n}{2} \rfloor$ *and* $2 \leq k \leq n-1$ *or*

(ii) $2 \leq r \leq \lfloor \frac{n}{2} \rfloor - 1$ *and* $2 \leq k \leq 4r - n + 1$,

then C_n^r is ℓ-path-pancyclic.

Proof. Let $G = C_n^r$. If $r \geq \lfloor \frac{n}{2} \rfloor$, then $G = K_n$ and so G is ℓ-path-pancyclic for $2 \leq \ell \leq n-1$. Thus, we may assume that $2 \leq r \leq \lfloor \frac{n}{2} \rfloor - 1$. Since G is $(2r)$-regular, it follows that $\delta(G) = 2r$. We show that if $2r \geq (n + \ell - 1)/2$ for some integer $\ell \geq 2$ (or $2 \leq \ell \leq 4r - n + 1$), then G is ℓ-path-pancyclic. By Theorem 3.3, if $\delta(G) = 2r \geq (n + \ell)/2$ (or $2 \leq \ell \leq 4r - n$), then G is ℓ-path-pancyclic. Thus, we may assume that

$2r = (n+\ell-1)/2$ and so $\ell = 4r-n+1$ and ℓ and n are of opposite parity. Now by Proposition 3.10(a), it suffices to show that every ℓ-path in G lies on an $(\ell+2)$-cycle. We apply an argument similar to the one used in the proof of Theorem 3.11.

Assume, to the contrary, that there is a path $P = (u_1, u_2, \ldots, u_\ell)$ of order ℓ that does not lie on any cycle of length $\ell+2$ in G. Since $\delta(G) = (n+\ell-1)/2$, it follows by Proposition 3.10(a) that P lies on a cycle of length $\ell+1$ in G. Let $(u_1, u_2, \ldots, u_\ell, w, u_1)$ be such a cycle. If there is $v \in V(G) - (V(P) \cup \{w\})$ such that v is adjacent to both u_1 and w or v is adjacent to both u_ℓ and w, then P lies on the cycle $(u_1, u_2, \ldots, u_\ell, w, v, u_1)$ or $(u_1, u_2, \ldots, u_\ell, v, w, u_1)$ of length $\ell+2$. Thus, we may assume that no vertex in $V(G) - (V(P) \cup \{w\})$ is adjacent to both u_1 and w or to both u_ℓ and w.

Since $\delta(G) = (n+\ell-1)/2$, it follows by Proposition 3.10(b) that $|N(u_1) \cap N(w)| \geq \ell-1$ and $|N(u_\ell) \cap N(w)| \geq \ell-1$. This implies that $N(u_1) \cap N(w) = \{u_2, u_3, \ldots, u_\ell\}$ and $N(u_\ell) \cap N(w) = \{u_1, u_2, \ldots, u_{\ell-1}\}$. Hence $wu_i \in E(G)$ for $1 \leq i \leq \ell$, $u_1u_i \in E(G)$ for $2 \leq i \leq \ell$ and $u_\ell u_i \in E(G)$ for $1 \leq i \leq \ell-1$. In particular, $u_1u_\ell \in E(G)$. Let $X = V(G) - (V(P) \cup \{w\})$. Since $\delta(G) = (n+\ell-1)/2$ and u_1 is adjacent to exactly ℓ vertices in $V(P) \cup \{w\}$, it follows that u_1 is adjacent to at least $(n-\ell-1)/2$ vertices in X. Let $X' \subseteq X$ such that u_1 is adjacent to every vertex in X' and let $X'' = X - X'$. Similarly, w is adjacent to the ℓ vertices of P and so w is adjacent to at least $(n-\ell-1)/2$ vertices in X. Since there is no vertex in X that is adjacent to both u_1 and w or to both u_ℓ and w and $|X| = n-\ell-1$, it follows that (i) u_1 is adjacent to exactly $(n-\ell-1)/2$ vertices in X and so $|X'| = (n-\ell-1)/2 = |X''|$, (ii) w is not adjacent to any vertex in X' and so w is adjacent to every vertex in X'' and (iii) u_ℓ and u_1 have exactly the same neighbors in X, namely $N_X(u_1) = N_X(u_\ell) = X'$. Since u_1 and u_ℓ have the same neighbors in $V(G) - X$, it follows that $N_G[u_1] = N_G[u_\ell] = X' \cup V(P) \cup \{w\}$. Because $u_1u_\ell \in E(G)$, it follows that u_1 and u_ℓ are true twins of G, which is impossible by Observation 3.14. ■

3.4 On Relationships between 2-Path Pancyclic Graphs and Hamiltonian-Connected Graphs

There are many cubic Hamiltonian-connected graphs. For example, the prism (the Cartesian product of a cycle and P_2) of order $n \equiv 2 \pmod 4$ and the Möbius ladder (an even cycle with the two in each pair of antipodal vertices joined) of order $n \equiv 0 \pmod 4$ are graphs that are 3-regular and Hamiltonian-connected. However, there is only one cubic graph that is 2-path-pancyclic.

Proposition 3.16 *A 3-regular graph G is 2-path-pancyclic if and only if $G = K_4$.*

Proof. Let G be a 3-regular 2-path-pancyclic graph. It suffices to show that $G = K_4$. Let $uv \in E(G)$. Then in order for this edge uv to belong to a triangle, $N(u) \cap N(v) \neq \emptyset$.

If $|N(u) \cap N(v)| = 2$, then either $G = K_4$ or uv belongs to no 4-cycle, depending on whether the two vertices in $N(u) \cap N(v)$ are adjacent or not. Thus, suppose next that $N(u) = \{v, u_1, w\}$ and $N(v) = \{u, v_1, w\}$. Since $\deg w = 3$, it follows that w is adjacent to at most one of u_1 and v_1, say $u_1w \notin E(G)$. Then $N(u) \cap N(u_1) = \emptyset$ and so the edge uu_1 belongs to no triangle. Consequently, G is not 2-path-pancyclic if $|N(u) \cap N(v)| = 1$. This completes the proof. ∎

Note that, for each integer $n \geq 5$, the graph C_n^2 is 4-regular and 2-path-pancyclic. Also, the wheels (the join of a cycle and K_1) of order $n \geq 4$ show that there are many graphs that are 2-path-pancyclic with minimum degree 3.

A graph G is called *vertex-traceable* if every vertex is the initial vertex of a Hamiltonian path in G.

Proposition 3.17 *The join of a graph G and K_1 is 2-path-pancyclic if and only if* (i) *G is vertex-traceable and* (ii) *every edge in G belongs to a Hamiltonian path in G.*

Proof. Let $H = G \vee K_1$ be the graph obtained from a graph G of order $n \geq 2$ by adding a new vertex x and joining x to each vertex in G. Observe that the condition (i) is necessary in order for each edge incident with x to lie on a Hamiltonian cycle in H. Similarly, the condition (ii) is necessary in order for each edge in G to lie on a Hamiltonian cycle in H. Therefore, it remains to verify that the conditions (i) and (ii) are sufficient for H to be 2-path-pancyclic. Suppose that G satisfies the conditions (i) and (ii) and let $e \in E(H)$.

Case 1. $e \notin E(G)$. Then $e = vx$ for some $v \in V(G)$. By (i), let $(v_1, v_2, \ldots, v_n)$ be a Hamiltonian path in G with $v = v_1$. Then for $3 \leq \ell \leq n+1$, there is an ℓ-cycle $(x, v_1, v_2, \ldots, v_{\ell-1}, x)$ in H containing the edge e.

Case 2. $e \in E(G)$. Then by (ii), let $(v_1, v_2, \ldots, v_n)$ be a Hamiltonian path which e belongs to, say $e = v_tv_{t+1}$ for some t $(1 \leq t \leq n-1)$. Then for each $\ell \in \{3, 4, \ldots, n+1\}$, the cycle

$$C = \begin{cases} (x, v_{t-\ell+3}, v_{t-\ell+4}, \ldots, v_{t+1}, x) & \text{if } 3 \leq \ell \leq t+2 \\ (x, v_1, v_2, \ldots, v_{\ell-1}, x) & \text{if } t+3 \leq \ell \leq n+1 \end{cases}$$

is an ℓ-cycle in H containing the edge e. ∎

Observe that, the conditions (i) and (ii) on a graph G in Proposition 3.17 are necessary and sufficient for the graph $G \vee K_1$ to be Hamiltonian-connected. Hence, for example, the join of the Petersen graph and an isolated vertex is Hamiltonian-connected as well as 2-path-pancyclic. Also, a Hamiltonian graph satisfies the conditions (i) and (ii), which provides with us the following corollary.

Corollary 3.18 *If G is Hamiltonian, then $G \vee K_1$ is Hamiltonian-connected and 2-path-pancyclic.*

Corollary 3.19 *If G is a graph of order n with $\delta(G) \geq (n+1)/2$ and $\Delta(G) = n-1$, then G is Hamiltonian-connected and 2-path-pancyclic.*

As we have already seen, there are many graphs that are Hamiltonian-connected but not 2-path-pancyclic. The Möbus ladder of order $n \equiv 0 \pmod 4$ and graphs of the form $G \,\square\, H$, where G is a Hamiltonian-connected graph of order at least 3 and H is a nontrivial traceable graph, are examples of such graphs.

Observation 3.20 *Let G be a graph of order n.*

(a) *For $3 \leq n \leq 5$, the graph G is 2-path-pancyclic if and only if G is Hamiltonian-connected.*

(b) *For $n = 6$, the graph G is 2-path-pancyclic if and only if G is Hamiltonian-connected except for the three graphs $G_1 = \overline{C_6}$, $G_2 = \overline{P_6}$, and $G_3 = \overline{2P_3}$. (These graphs are shown in* Figure 6. *Note that each edge in bold belongs to no triangle.)*

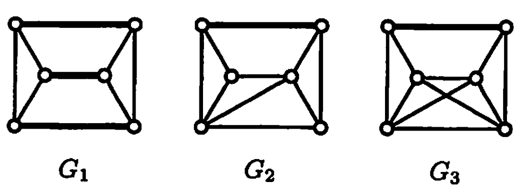

Figure 6: The three Hamiltonian-connected graphs of order 6 that are not 2-path-pancyclic

For integers $n_1, n_2 \geq 3$, let $G = (K_{n_1} + K_{n_2}) \vee 2K_1$. Then G is a graph of order $n_1 + n_2 + 2$ (≥ 8) that is 2-path-pancyclic. However, G is not 3-connected and so cannot be Hamiltonian-connected. Hence, there are graphs that are 2-path-pancyclic but not Hamiltonian-connected as well as there are graphs that are Hamiltonian-connected but not 2-path-pancyclic. Note that this graph G is not 3-path-pancyclic. We conclude this section with the following two questions.

Problem 3.21 *If G is a 2-path-pancyclic graph, then under what conditions is G Hamiltonian-connected?*

Problem 3.22 *If G is a ℓ-path-pancyclic graph for some integer $k \geq 3$, is G also Hamiltonian-connected?*

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