On Color Frames of Stars and Generalized Matching Numbers

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Abstract

A red-blue coloring of a graph G is an edge coloring of G in which every edge of G is colored red or blue. Let F be a connected graph of size 2 or more with a red-blue coloring, at least one edge of each color, where some blue edge of F is designated as the root of F. Such an edge-colored graph F is called a color frame. An F-coloring of a graph G is a red-blue coloring of G in which every blue edge of G is the root edge of a copy of F in G. The F-chromatic index $\chi'_F(G)$ of G is the minimum number of red edges in an F-coloring of G. A minimal F-coloring of G is an F-coloring with the property that if any red edge of G is re-colored blue, then the resulting red-blue coloring of G is not an F-coloring of G. The maximum number of red edges in a minimal F-coloring of G is the upper F-chromatic index $\chi''_F(G)$ of G. For integers k and m with $1 \le k < m$ and $m \ge 3$, let $S_{k,m}$ be the color frame of the star $K_{1,m}$ of size m such that $S_{k,m}$ has exactly k red edges and $m-k$ blue edges.

For a positive integer k, a set X of edges of a graph G is a Δ_k-set if $\Delta(G[X]) = k$, where $G[X]$ is the subgraph of G induced by X. The maximum size of a Δ_k-set in G is referred to as the k-matching number of G and is denoted by $\alpha'_k(G)$. A Δ_k-set X is maximal if $X \cup \{e\}$ is not a Δ_k-set for every $e \in E(G) - X$. The minimum size of a maximal Δ_k-set of G is the lower k-matching number of G and is denoted by $\alpha''_k(G)$. In this paper, we consider $S_{k,m}$-colorings of a graph and study relations between $S_{k,m}$-colorings and Δ_k-sets in graphs. Bounds are established for the $S_{k,m}$-chromatic indexes $\chi'_{S_{k,m}}(G)$ and $\chi''_{S_{k,m}}(G)$ of a graph G in terms of the k-matching numbers $\alpha'_k(G)$ and $\alpha''_k(G)$ of the graph. Other results and questions are also presented.

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1 Introduction

An area of graph theory that has received increased attention during recent decades is that of domination. In 1999 a new way of looking at domination was introduced by Chartrand, Haynes, Henning and Zhang [2] that encompassed several of the best known domination parameters in the literature. This new view of domination was based on a concept introduced by Rashidi [19] in 1994. A graph G whose vertex set $V(G)$ is partitioned is a *stratified graph.* If $V(G)$ is partitioned into k subsets, then G is *k-stratified.* In particular, the vertex set of a 2-stratified graph is partitioned into two subsets. Typically, the vertices of one subset in a 2-stratified graph are considered to be colored red and those in the other subset are colored blue. A *red-blue coloring* of a graph G is an assignment of colors to the vertices of G, where each vertex is colored either red or blue. In a red-blue coloring, all vertices of G may be colored the same. A red-blue coloring in which at least one vertex is colored red and at least one vertex is colored blue thereby produces a 2-stratification of G. Let F be a 2-stratified graph in which some blue vertex ρ is designated as the root of F. The graph F is then said to *be rooted at ρ.* Since F is 2-stratified, F contains at least two vertices, at least one of each color. There may be blue vertices in F in addition to the root. By an *F-coloring* of a graph G, we mean a red-blue coloring of G such that for every blue vertex u of G, there is a copy of F in G with ρ at u. Therefore, every blue vertex u of G belongs to a copy F' of F rooted at u. A red vertex v in G is said to *F-dominate* a vertex u if $u = v$ or there exists a copy F' of F rooted at u and containing the red vertex v. The set S of red vertices in a red-blue coloring of G is an *F-dominating set* of G if every vertex of G is F-dominated by some vertex of S, that is, this red-blue coloring of G is an F-coloring. The minimum number of red vertices in an F-dominating set is called the *F-domination number* $\gamma_F(G)$ of G. An F-dominating set with $\gamma_F(G)$ vertices is a *minimum F-dominating set.* The F-domination number of every graph G is defined since $V(G)$ is an F-dominating set. This concept provides a generalization of domination and has been studied in many articles (see [1, 6, 7] and [8] - [12] for example).

An edge version of this concept was introduced by Chartrand in 2011 [13]. In this context, we refer to a red-blue coloring of a nonempty graph G as an edge coloring of G in which every edge is colored red or blue. Let F be a connected graph of size 2 or more with a red-blue coloring, at least one edge of each color. One of the blue edges of F is designated as the *root edge* of F. The *underlying graph* of F is the graph H obtained by removing the colors assigned to the edges of F. In this case, F is called a *color frame* of H.

For a color frame F, an *F-coloring* of a graph G is a red-blue coloring of G in which every blue edge of G is the root edge of a copy of F in G. If G contains no subgraph isomorphic to F, then the only F-coloring of G is that in which every edge of G is red. The *F-chromatic index* $\chi'_F(G)$ of G is the minimum number of red edges in an F-coloring of G. Since the edge coloring of G that assigns red to every edge is an F-coloring of G, the number $\chi'_F(G)$ exists for every color frame F and every graph G. An F-coloring of G having exactly $\chi'_F(G)$ red edges is called a *minimum F-coloring* of G. For a given color frame F, a *minimal F-coloring* of a graph G is an F-coloring with the property that if any red edge of G is re-colored blue, then the resulting red-blue coloring of G is not an F-coloring of G. Obviously, every minimum F-coloring is minimal but the converse is not true in general (as we will soon see). The maximum number of red edges in a minimal F-coloring of G is the *upper F-chromatic index* $\chi''_F(G)$ of G. Since every minimum F-coloring of G is minimal, $\chi'_F(G) \leq \chi''_F(G)$. These concepts have been studied in [3, 13, 14, 15].

Among the concepts that are fundamental in graph theory is that of matchings. Lovász and Plummer have written a book [18] devoted to the theory of matchings. A set of edges in a graph G is *independent* if no two edges in the set are adjacent in G. The edges in an independent set of edges of G form a *matching* in G. A matching of maximum size in G is a *maximum matching.* The *matching number* $\alpha'(G)$ of G is the number of edges in a maximum matching of G. The number $\alpha'(G)$ is also referred to as the *edge independence number* of G. A matching M in a graph G is a *maximal matching* of G if M is not a proper subset of any other matching in G. While every maximum matching is maximal, a maximal matching need not be a maximum matching. The minimum number of edges in a maximal matching of G is called the *lower matching number* (or *lower edge independence number*) of G and is denoted by $\alpha''(G)$. Necessarily, $\alpha''(G) \leq \alpha'(G)$ for every graph G.

The concepts of matching number and lower matching number have been generalized in [15] as follows. For a positive integer k, a set X of edges of a graph G is a *Δ_k-set* if $\Delta(G[X]) = k$, where $G[X]$ is the subgraph of G induced by X. The maximum size of a Δ_k-set in G is referred to as the *Δ_k-number* or *k-matching number* of G and is denoted by $\alpha'_k(G)$. In particular, $\alpha'_1(G) = \alpha'(G)$ is the matching number of G. A *maximum Δ_k-set* in G is a Δ_k-set of size $\alpha'_k(G)$. Thus the maximum Δ_1-set of G is a maximum matching of G. A Δ_k-set is *maximal* if $\Delta(G[X \cup \{e\}]) > k$ for every edge $e \in E(G) - X$. The minimum size of a maximal Δ_k-set of G is referred to as the *lower Δ_k-number* or *lower k-matching number* of G and is denoted by $\alpha''_k(G)$. In particular, $\alpha''_1(G) = \alpha''(G)$ is the lower matching number of G. Since every maximum Δ_k-set is maximal, $\alpha''_k(G) \leq \alpha'_k(G)$ for every graph G. An edge-induced subgraph H of a graph G is called

Y_1 and Y_2 and shown in Figure 2. The color frame Y_1 of a claw has exactly one red edge while Y_2 has exactly two red edges. In Y_1, there are therefore two blue edges and in Y_2 only one blue edge. By symmetry, we can choose either of the two blue edges in Y_1 as the root edge, while in Y_2, the only blue edge is the root edge of Y_2.

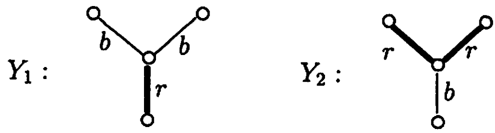

Figure 2: The two color frames of the claw $K_{1,3}$

In general, for integers k and m with $1 \leq k < m$ and $m \geq 3$, let $S_{k,m}$ denote the color frame of the star $K_{1,m}$ of size m having exactly k red edges and $m-k$ blue edges, one of which is the root edge of $S_{k,m}$. In particular, if $m = 3$, then $S_{k,m}$ is one of the two color frames Y_1 and Y_2 of a claw $K_{1,3}$ shown in Figure 2. Hence the star $K_{1,m}$ has $m-1$ color frames for each integer $m \geq 3$. In this paper, we investigate the relationship among the four parameters $\chi'_{S_{k,m}}(G)$, $\chi''_{S_{k,m}}(G)$, $\alpha'_k(G)$ and $\alpha''_k(G)$ of a graph G where $1 \leq k < m$ and $m \geq 3$ and so extend the results established in [3, 15] on $\chi'_F(G)$ and $\chi''_F(G)$ for each color frame $F \in \{Y_1, Y_2\}$.

It will be convenient to introduce some additional definitions and notation. For an F-coloring c of a graph G, let $E_{c,r}$ denote the set of red edges of G and $E_{c,b}$ the set of blue edges of G. (We also use E_r and E_b for $E_{c,r}$ and $E_{c,b}$, respectively, when the coloring c under consideration is clear.) Thus $\{E_{c,r}, E_{c,b}\}$ is a partition of the edge set $E(G)$ of G when $E_{c,b} \neq \emptyset$. Furthermore, let $G_{c,r} = G[E_{c,r}]$ denote the *red subgraph* induced by $E_{c,r}$ and $G_{c,b} = G[E_{c,b}]$ the *blue subgraph* induced by $E_{c,b}$. (Similarly, we also use G_r and G_b for $G_{c,r}$ and $G_{c,b}$, respectively, when the coloring c under consideration is clear.) Thus $\{G_{c,r}, G_{c,b}\}$ is a decomposition of G. If G is a disconnected graph with components $G_1, G_2, \ldots, G_p$ where $p \geq 2$, then $\chi'_F(G) = \chi'_F(G_1) + \chi'_F(G_2) + \cdots + \chi'_F(G_p)$. Thus, it suffices to consider only nontrivial connected graphs. We refer to the books [4, 5] for graph theory notation and terminology not described in this paper.

2 Comparing $\chi'_{S_{k,m}}(G)$ with $\alpha''_k(G)$

For integers k and m with $1 \leq k < m$ and $m \geq 3$, the color frame $S_{k,m}$ of $K_{1,m}$ has exactly k red edges and $m-k$ blue edges, one of which is the root edge of $S_{k,m}$. If G is a connected graph with maximum degree $\Delta(G) < m$, then the only $S_{k,m}$-coloring of G is the one that assigns the color red to every edge of G and so $\chi'_{S_{k,m}}(G)$ is the size of G. On the other hand, if G contains a vertex u with $\deg u \geq m$ and $uv_1, uv_2, \ldots, uv_m \in E(G)$, then the red-blue coloring that assigns the color blue to the edges uv_i for $1 \leq i \leq m-k$ and the color red to all other edges is an $S_{k,m}$-coloring of G. This leads to the following observation.

Observation 2.1 *Let k and m be integers with $1 \leq k < m$ and $m \geq 3$. Then $\chi'_{S_{k,m}}(G) = |E(G)|$ if and only if $\Delta(G) < m$ for every connected graph G.*

First, we show that $\chi'_{S_{k,m}}(G)$ is bounded above by $\alpha''_k(G)$ for every connected graph G having minimum degree $\delta(G) \geq m$.

Theorem 2.2 *Let k and m be integers with $1 \leq k < m$ and $m \geq 3$. If G is a connected graph with $\delta(G) \geq m$, then*

$$\chi'_{S_{k,m}}(G) \leq \alpha''_k(G). \tag{1}$$

Proof. Let $F = S_{k,m}$ and X a maximal Δ_k-set of G with $|X| = \alpha''_k(G)$. Define the red-blue coloring c of G that assigns red to each edge in X and blue to the remaining edges of G. Let $e = uv$ be a blue edge of G. Since X is maximal, $\Delta(G[X \cup \{e\}]) > k$ and so either u or v is incident with exactly k red edges, say the former. Since $\deg_G u \geq m$, it follows that u is incident with at least $m-k$ blue edges. Hence e belongs to a copy of F and so c is an F-coloring. Therefore, $\chi'_F(G) \leq |X| = \alpha''_k(G)$.

The condition "$\delta(G) \geq m$" in Theorem 2.2 is necessary. For example, let G be an $(m-1)$-regular bipartite graph of order $n \geq 4$ where $m \geq 3$. By Observation 2.1 $\chi'_{S_{k,m}}(G) = |E(G)| = (m-1)n/2$. By Kónig's theorem [17], the graph G is 1-factorable. Hence, for each integer k with $1 \leq k \leq m-2$, it follows that $\alpha'_k(G) = kn/2$ and so $\alpha''_k(G) \leq kn/2$. Therefore, $\chi'_{S_{k,m}}(G) > \alpha''_k(G)$.

The following is a consequence of the proof of Theorem 2.2.

Corollary 2.3 *Let k and m be integers with $1 \leq k < m$ and $m \geq 3$. If X is a maximal Δ_k-set of a connected graph G with $\delta(G) \geq m$, then the red-blue coloring of G that assigns red to each edge in X and blue to the remaining edges of G is an $S_{k,m}$-coloring of G.*

The converse of Corollary 2.3 is not true in general. For example, Figure 3 shows a (minimal) Y_1-coloring c of a graph, where Y_1 is the color frame of a claw with exactly one red edge, but $E_{c,r}$ is not even a Δ_1-subgraph (a matching) of the graph.

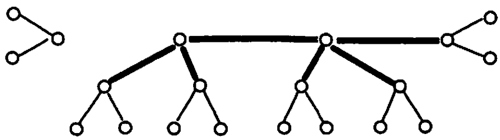

Figure 3: A Y_1-coloring of a graph

On the other hand, it is possible that $\chi'_{S_{k,m}}(G) = \alpha''_k(G)$ even when $\delta(G) < m$. In order to show this, we now describe a class of connected graphs. For a graph H and a positive integer p, the *p-corona* $cor_p(H)$ of H is that graph obtained from H by adding p pendant edges at each vertex of H. In particular, the graph $cor_1(H) = cor(H)$ is the *corona* of H. If H has order n and size m, then the order of $cor_p(H)$ is $(p+1)n$ and the size of $cor_p(H)$ is $pn + m$.

Theorem 2.4 *Let k and m be integers with $1 \le k < m$ and $m \ge 3$. If G is the p-corona of an n-cycle where $p \ge m-2$ and $n \ge 3$, then*

$$\chi'_{S_{k,m}}(G) = \alpha''_k(G) = \begin{cases} \lceil n/2 \rceil & \text{if } k = 1 \\ (k-1)n & \text{if } k \ge 2. \end{cases}$$

Proof. Let $F_k = S_{k,m}$ where $1 \le k < m$ and $m \ge 3$ and let $G = cor_p(C_n)$ where $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ for some integers $p \ge m-2$ and $n \ge 3$. Then $1 \le k \le p+1$. For each i with $1 \le i \le n$, let $X_i = \{u_{i,1}v_i, u_{i,2}v_i, \ldots, u_{i,p}v_i\}$ be the set of the p pendant edges of G at v_i. There are two cases, according to whether $k = 1$ or $k \ge 2$.

Case 1. $k = 1$. For each even integer $n \ge 4$, define a red-blue coloring c of G with

$$E_{c,r} = \{v_i v_{i+1} : i \text{ is odd}, 1 \le i \le n-1\}; \qquad (2)$$

while for each odd integer $n \ge 3$, define a red-blue coloring c of G with

$$E_{c,r} = \{v_i v_{i+1} : i \text{ is odd}, 1 \le i \le n-2\} \cup \{u_{n,1}v_n\}. \qquad (3)$$

Since c is an F_1-coloring of G in each case, $\chi'_{F_1}(G) \le |E_{c,r}| = \lceil n/2 \rceil$.

Next, we show that $\chi'_{F_1}(G) \ge \lceil n/2 \rceil$. Assume, to the contrary, that G has an F_1-coloring c^* using at most $\lceil n/2 \rceil - 1$ red edges. Thus the size of the red subgraph G^*_r induced by c^* is at most $\lceil n/2 \rceil - 1$. Since G^*_r contains no isolated vertices and $n/2 > \lceil n/2 \rceil - 1$, the order of G^*_r is at most $n-1$ and so there is at least one vertex v belonging to C_n such that $v \notin V(G^*_r)$, say $v = v_1$. However then, the blue edge $u_{1,1}v_1$ does not belong to any copy of F_1, which is impossible. Thus $\chi'_{F_1}(G) \ge \lceil n/2 \rceil$ and so $\chi'_{F_1}(G) = \lceil n/2 \rceil$.

Case 2. $k \geq 2$. Thus $2 \leq k \leq p+1$. For each integer $n \geq 3$, define a red-blue coloring c^* of G with

$$E_{c^*,r} = \begin{cases} E(C_n) & \text{if } k = 2 \\ E(C_n) \cup \{u_{i,j}v_i : 1 \leq i \leq n, 1 \leq j \leq k-2\} & \text{if } k \geq 3 \end{cases} \quad (4)$$

Since c^* is an F_k-coloring of G, it follows that $\chi'_{F_k}(G) \leq |E_{c^*,r}| = (k-1)n$. Next, we show that $\chi'_{F_k}(G) \geq (k-1)n$. Let c be a minimum F_k-coloring of G. First, suppose that there is some j with $1 \leq j \leq n$ such that $X_j \cup \{v_jv_{j+1}\}$ contains at most $k-2$ red edges. If X_j contains a blue edge f, then f does not belong to any copy of F_k and so each edge in X_j must be red. Since $|X_j| = p \geq k-1$, this is impossible. Thus, each set $X_i \cup \{v_iv_{i+1}\}$ contains at least $k-1$ red edges for $1 \leq i \leq n$ and so $\chi'_{F_k}(G) = |E_{c,r}| \geq (k-1)n$. Therefore, $\chi'_{F_k}(G) = (k-1)n$.

It remains to determine $\alpha''_k(G)$. Since $\alpha''_1(G) = \alpha''(G) = \lceil n/2 \rceil$, we may assume that $k \geq 2$. Since the set $E_{c^*,r}$ described in (4) is a maximal Δ_k-set of size $(k-1)n$, it follows that $\alpha''_k(G) \leq \chi'_{F_k}(G) = (k-1)n$. Next, we show that $\alpha''_k(G) \geq (k-1)n$. Assume, to the contrary, that $\alpha''_k(G) \leq (k-1)n-1$ and let X be a maximal Δ_k-set of G with $|X| = \alpha''_k(G)$. Let $H = G[X]$ and so $\Delta(H) = k$. Suppose first that there is a vertex $v \in V(C_n)$ such that $v \notin V(H)$. Let e be a pendant edge of G that is incident with v. Then $\Delta(G[X \cup \{e\}]) = \Delta(H) = k$, which contradicts the fact that X is a maximal Δ_k-set of G. Hence $V(C_n) \subseteq V(H)$.

First, suppose that $\deg_H v = k$ for each $v \in V(C_n)$. Because each vertex of C_n is incident in H with at most two edges of C_n, it follows that $|X| \geq n + (k-2)n = (k-1)n$, which is a contradiction. Hence there is $v \in V(C_n)$ such that $\deg_H v \leq k-1$. We consider two cases.

Case 1. $2 \leq k \leq p$. Let $v \in V(C_n)$ such that $\deg_H v \leq k-1$. Then $\deg_H v \leq p-1$ and so there is a pendant edge $f \notin X$ that is incident with v. However then, $\Delta(G[X \cup \{f\}]) = \Delta(H) = k$, which is a contradiction.

Case 2. $k = p+1$. If there is $v \in V(C_n)$ such that $\deg_H v \leq k-1$ and there is a pendant edge $f \notin X$ that is incident with v, then, as in Case 1, $\Delta(G[X \cup \{f\}]) = \Delta(H) = k$, which is a contradiction. This implies for each $v \in V(C_n)$ that $\deg_H v \geq k-1$ and all pendant edges incident with v belong to X. However then, $|X| \geq pn = (k-1)n$, which is impossible.

Therefore, $\alpha''_k(G) \geq (k-1)n$ and so $\alpha''_k(G) = (k-1)n$ for $k \geq 2$. ■

Next, we show that the equality in (1) holds for all connected graphs when $k = m-1$.

Theorem 2.5 *For an integer $m \geq 3$, if G is a connected graph with $\delta(G) \geq m$, then $\chi'_{S_{m-1,m}}(G) = \alpha''_{m-1}(G)$.*

Proof. Let $F = S_{m-1,m}$. By Theorem 2.2, it remains to show that $\chi'_F(G) \geq \alpha''_{m-1}(G)$. Among all minimum F-colorings of G, let c be one such that the sum of the degrees of the vertices of degree m or more in the resulting red subgraph is minimum. Let $G_{c,r} = G[E_{c,r}]$ be the red subgraph of G induced by $E_{c,r}$. First, we show that $E_{c,r}$ is a Δ_{m-1}-set. Since c is an F-coloring of G, it follows that $\Delta(G_{c,r}) \geq m-1$. We now show that $\Delta(G_{c,r}) = m-1$.

Assume, to the contrary, that the graph $G_{c,r}$ contains a vertex v for which $\deg_{G_{c,r}} v = \ell \geq m$, where $vv_1, vv_2, \ldots, vv_\ell$ are the edges in $G_{c,r}$ incident with v. Since c is a minimum F-coloring, the red-blue coloring c^* of G obtained from c by changing the color of vv_ℓ to blue is not an F-coloring of G. First, we claim that v_ℓ is incident with at least one blue edge; for otherwise, assume that v_ℓ is incident only with red edges. Since $\delta(G) \geq m$, the blue edge vv_ℓ in the red-blue coloring c^* belongs to a copy of F, which implies that c^* is an F-coloring of G, a contradiction. Thus v_ℓ is incident with at least one blue edge, as claimed. Also, since c^* is not an F-coloring, there is a blue edge e incident with v_ℓ such that e belongs to a copy F that contains the red edge vv_ℓ but e does not belong to any other copy of F. Hence there are edges uv_ℓ and $v_\ell w_i$ $(1 \leq i \leq m-2)$, where $v \notin \{u, w_1, w_2, \ldots, w_{m-2}\}$, such that (1) uv_ℓ is blue and u is incident with at most $m-2$ red edges in $G_{c,r}$ and (2) $v_\ell w_i$ is red for $1 \leq i \leq m-2$ and $\deg_{G_{c,r}} v_\ell = m-1$. This is illustrated in Figure 4 (where $uu_1, uu_2, \ldots, uu_p$ are red edges in $G_{c,r}$, $p \leq m-2$ and it is possible that $p = 0$). The red-blue coloring c' obtained from c by interchanging the colors of vv_ℓ and $v_\ell u$ is also a minimum F-coloring of G. In the red subgraph $G_{c',r}$ of G induced by $E_{c',r}$, it follows that $\deg_{G_{c',r}} v_\ell = m-1$, $\deg_{G_{c',r}} u \leq m-1$ and $\deg_{G_{c',r}} v = \ell - 1 \geq m-1$. Thus the number of vertices of degree m or more in $G_{c',r}$ is at most the number of such vertices in $G_{c,r}$. Since the sum of degrees of the vertices of degree m or more in $G_{c',r}$ is at least one less than that of $G_{c,r}$, this contradicts the defining property of c. Thus, $\Delta(G_{c,r}) = m-1$ and so $E_{c,r}$ is a Δ_{m-1}-set.

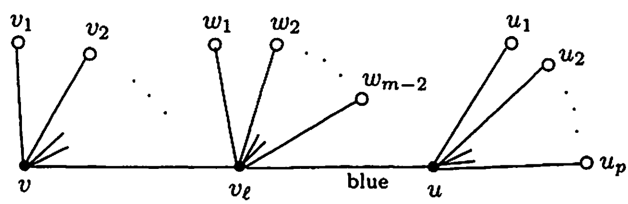

Figure 4: A step in the proof of Theorem 2.5

Next, we show that $E_{c,r}$ is a maximal Δ_{m-1}-set. Let $e \in E(G) - E_{c,r}$. Since c is an F-coloring, the blue edge e belongs to copy of F in G and so e

is adjacent with $m-1$ red edges in $E_{c,r}$. This implies that $\Delta(G[X \cup \{e\}]) > m-1$ and so $E_{c,r}$ is maximal. Hence $\chi'_F(G) = |E_{c,r}| \geq \alpha''_{m-1}(G)$. Therefore, $\chi'_F(G) = \alpha''_{m-1}(G)$. ■

As a consequence of Theorem 2.5, if G is a connected graph with $\delta(G) \geq 3$ and Y_2 is the color frame of the claw $K_{1,3}$ having exactly two red edges, then $\chi'_{Y_2}(G) = \alpha''_2(G)$, which was verified in [3]. There are reasons to make the following conjecture.

Conjecture 2.6 *Let k and m be integers with $1 \leq k < m$ and $m \geq 3$. If G is a connected graph with $\delta(G) \geq m$, then $\chi'_{S_{k,m}}(G) = \alpha''_k(G)$.*

3 Comparing $\chi''_{S_{k,m}}(G)$ with $\alpha'_k(G)$

First, we show that $\chi''_{S_{k,m}}(G)$ is bounded below by $\alpha'_k(G)$ where $1 \leq k < m$ and $m \geq 3$.

Theorem 3.1 *Let k and m be integers with $1 \leq k < m$ and $m \geq 3$. If G is a connected graph with $\delta(G) \geq m$, then*

$$\chi''_{S_{k,m}}(G) \geq \alpha'_k(G). \quad (5)$$

Proof. Let $F = S_{k,m}$ and X a maximum Δ_k-set of G. Then $|X| = \alpha'_k(G)$. Since X is maximum, X is maximal. By Corollary 2.3, the red-blue coloring c of G that assigns red to each edge in X and blue to the remaining edges of G is an F-coloring of G. It remains to show that c is minimal. Assume, to the contrary, that c is not minimal. Then there is a proper subset X' of X such that the red-blue coloring c' with $E_{c',r} = X'$ is an F-coloring of G. Let $f = xy \in X - X'$ be a blue edge in c'. Since f belongs to a copy of F, it follows that either x or y is incident with at least k red edges in X'. However then, $\Delta(G[X']) \geq k$ and so $\Delta(G[X]) > k$, which is impossible. Therefore, c is a minimal F-coloring and so $\chi''_{S_{k,m}}(G) \geq |X| = \alpha'_k(G)$.

The discussion appeared after Theorem 2.2 also shows that the condition "$\delta(G) \geq m$" in Theorem 3.1 is necessary. The following is a consequence of the proof of Theorem 3.1.

Corollary 3.2 *Let k and m be integers with $1 \leq k < m$ and $m \geq 3$. If X is a maximal Δ_k-set of a connected graph G with $\delta(G) \geq m$, then the red-blue coloring of G that assigns red to each edge in X and blue to the remaining edges of G is a minimal $S_{k,m}$-coloring of G.*

The converse of Corollary 3.2 is not true in general as the graph Figure 3 shows, where the Y_1-coloring is minimal but $E_{c,r}$ is not even a Δ_1-subgraph (or a matching) of the graph. Next, we show that it is possible for $\chi''_{S_{k,m}}(G) = \alpha'_k(G)$ even when $\delta(G) < m$.

Theorem 3.3 *Let k and m be integers with $1 \le k < m$ and $m \ge 3$. If G is the p-corona of an n-cycle where $p \ge m-2$ and $n \ge 3$, then*

$$\chi''_{S_{k,m}}(G) = \alpha'_k(G) = \begin{cases} kn & \text{if } 1 \le k \le p \\ kn - \lceil n/2 \rceil & \text{if } k = p+1. \end{cases}$$

Proof. Let $F_k = S_{k,m}$ where $1 \le k < m$ and $m \ge 3$ and let $G = cor_p(C_n)$ where $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ for some integers $p \ge m-2$ and $n \ge 3$. For each i with $1 \le i \le n$, let $X_i = \{u_{i,1}v_i, u_{i,2}v_i, \ldots, u_{i,p}v_i\}$ be the set of the p pendant edges of G at v_i. Since $k \le m-1$ and $p \ge m-2$, it follows that $1 \le k \le p+1$. We consider two cases, according to whether $1 \le k \le p$ or $k = p+1$

Case 1. $1 \le k \le p$. Since the red-blue coloring c^* of G with

$$E_{c^*,r} = \{u_{i,j}v_i : \ 1 \le i \le n, 1 \le j \le k\}$$

is a minimal F_k-coloring of G, it follows that $\chi''_{F_k}(G) \ge |E_{c^*,r}| = kn$. To show that $\chi''_{F_1}(G) \le kn$, we consider two subcases, according to whether $k = 1$ or $k \ge 2$.

Subcase 1.1. $k = 1$. Among all minimal F_1-colorings of G having exactly $\chi''_{F_1}(G)$ red edges, let c be one such that the red subgraph G_r induced by c has the largest edge independence number. We claim that $E_{c,r}$ is an independent set of edges in G. It suffices to show that each vertex of C_n is incident with exactly one red edge in G_r. First, suppose that there is $v_i \in V(C_n)$ where $1 \le i \le n$ such that v_i is incident with no red edge of G_r. Then any blue edge in X_i does not belong to any copy of F_1, which is impossible. Next, suppose that there is $v_j \in V(C_n)$ where $1 \le j \le n$ such that v_j is incident with at least two red edges of G_r, say $j = 1$. If there is an edge $e \in X_1$ such that e is red, then the red-blue coloring obtained from c by changing the color of e to blue is a F_1-coloring of G with fewer red edges, which is a contradiction. Thus v_1 is incident with exactly two red edges, namely, v_nv_1 and v_1v_2. If either v_n or v_2 is incident with two or more red edges, say v_2 is incident with two or more red edges, then the red-blue coloring obtained from c by changing the color of v_1v_2 to blue is a F_1-coloring of G with fewer red edges, which is again a contradiction. Thus v_1v_2 is the only red edge incident with v_2. Then the coloring c' of G obtained from c by exchanging the colors of v_1v_2 and $u_{2,1}v_2$ is a minimal F_1-colorings of G having exactly $\chi''_{F_1}(G)$ red edges. However then, the red subgraph induced by c' has a larger edge independence number than that of G_r, which is impossible. Therefore, every vertex of C_n is incident with exactly one red edge in G_r. This implies that $E_{c,r}$ is an independent set of edges in G and so $\chi''_{F_1}(G) = |E_{c,r}| \le \alpha'(G) = n$. Therefore, $\chi''_{F_1}(G) = n$.

Subcase 1.2. $k \geq 2$. Let c be a minimal F_k-colorings of G having exactly $\chi''_{F_k}(G)$ red edges and let G_r be the red subgraph induced by c. We claim that $\deg_{G_r} v = k$ for every vertex $v \in V(C_n)$; for otherwise, we may assume, without loss of generality, that $\deg_{G_r} v_1 \neq k$. First, suppose that $\deg_{G_r} v_1 \leq k-1$. Since $k \leq p$, there is a blue edge in X_1 and so this blue edge does not belong to any copy of F_k in G, a contradiction. Next, suppose that $\deg_{G_r} v_1 \geq k+1$. Since $k+1 \geq 3$, it follows that X_1 contains at least one red edge, say $u_{1,1}v_1$ is red. However then, the red-blue coloring obtained from c by changing the color of $u_{1,1}v_1$ to blue is an F_k-coloring of G with fewer red edges, a contradiction. Hence, as claimed, $\deg_{G_r} v = k$ for every vertex $v \in V(C_n)$. By the structure of G, the largest possible size of G_r is kn and so $\chi''_{F_k}(G) \leq kn$. Therefore, $\chi''_{F_k}(G) = kn$.

Case 2. $k = p+1$. Thus $k = m-1$, $p = m-2$ and the order of G is kn. For an even integer $n \geq 4$, define a red-blue coloring c of G with

$$E_{c,r} = \{v_iv_{i+1} : i \text{ is odd}, 1 \leq i \leq n-1\} \cup \{u_{i,j}v_i : 1 \leq i \leq n, 1 \leq j \leq k-1\}$$

and so $|E_{c,r}| = (k-1)n + n/2 = kn - n/2$. For an odd integer $n \geq 3$, define a red-blue coloring c of G with

$$\begin{aligned} E_{c,r} &= \{v_iv_{i+1} : i \text{ is odd}, 1 \leq i \leq n\} \cup \{u_{1,j}v_1 : 1 \leq j \leq k-2\} \\ &\quad \cup \{u_{i,j}v_i : 2 \leq i \leq n, 1 \leq j \leq k-1\}. \end{aligned}$$

Then $|E_{c,r}| = (k-1)(n-1)+(k-2)+(n+1)/2 = (k-1)n+(n-1)/2$. Since c is a minimal F_k-coloring, $\chi''_{F_k}(G) \geq |E_{c,r}| = (k-1)n + \lfloor n/2 \rfloor = kn - \lceil n/2 \rceil$.

Next, we show that $\chi''_{F_k}(G) \leq kn - \lceil n/2 \rceil$. Assume, to the contrary, that

$$\chi''_{F_k}(G) = t \geq kn - \lceil n/2 \rceil + 1. \tag{6}$$

Let c^* be a minimal F_k-coloring of G having exactly t red edges and let G_r be the red subgraph induced by c^*. Thus the size of G_r is t. First, suppose that G_r contains a vertex v such that $\deg_{G_r} v = \deg_G v = k+1$, say $v = v_1$ and v_1w_i is red for $1 \leq i \leq k+1$. Since $k = p+1 \geq 2$, it follows that $k+1 \geq 3$. Thus some edge v_1w_i $(1 \leq i \leq k+1)$ is a pendant edge of G, say $v_1w_1 = v_1u_{1,1} \in X_1$. Then the red-blue coloring obtained from c^* by changing the color of $v_1u_{1,1}$ to blue is an F_k-coloring, which is impossible. Thus $\deg_{G_r} v \leq k$ for every vertex v of G_r. Since (i) the order n_r of G_r is at most the order of G, namely $n_r \leq (p+1)n = kn$ and (ii) at most n vertices in G_r have degree k and the remaining vertices of G_r are end-vertices, it follows that the size t of G_r is at most $1/2[kn + (k-1)n] = kn - n/2$. By (6), $kn - \lceil n/2 \rceil + 1 \leq t \leq kn - n/2$ or $(n+2)/2 \leq \lceil n/2 \rceil$, which is impossible. Therefore, $\chi''_{F_k}(G) = kn - \lceil n/2 \rceil$.

It remains to determine $\alpha'_k(G)$. For $1 \leq k \leq p$, the set $E_{c^*,r}$ defined in Case 1 is a Δ_k-set of size kn, it follows that $\alpha'_k(G) \geq |E_{c^*,r}| = kn$.

For $k = p+1$, the set $E_{c,r}$ defined in Case 2 is a Δ_k-set of size $kn - \lceil n/2 \rceil$. It follows that $\alpha'_k(G) \geq |E_{c,r}| = kn - \lceil n/2 \rceil$. Next, we show that $\alpha'_k(G) \leq \chi''_{F_k}(G)$. Let X be a Δ_k-set with $|X| = \alpha'_k(G)$ and let $H = G[X]$. Since X is a maximum Δ_k-set, it follows that H contains every vertex of C_n and $\Delta(H) = \deg_H v$ for some $v \in V(C_n)$. Furthermore, every edge in H is incident with some vertex v of C_n. If $1 \leq k \leq p$, then $|X| \leq \sum_{v \in V(C_n)} \deg_H v \leq kn = \chi''_{F_k}(G)$. Thus, we may assume that $k = p+1$. In this case, we show that $|X| \leq kn - \lceil n/2 \rceil$. (Here, we apply an argument similar to the one used in Case 2 to show that $\chi''_{F_k}(G) \leq kn - \lceil n/2 \rceil$.) Assume, to the contrary, that

$$|X| = t \geq kn - \lceil n/2 \rceil + 1. \tag{7}$$

Since (i) $\deg_H v \leq k$ for every vertex v of H, (ii) the order n_H of H is at most the order of G, namely $n_H \leq (p+1)n = kn$ and (iii) at most n vertices in H have degree k and the remaining vertices of H are end-vertices, it follows that the size t of H is at most $1/2[kn + (k-1)n] = kn - n/2$. By (7), $kn - \lceil n/2 \rceil + 1 \leq t \leq kn - n/2$ or $(n+2)/2 \leq \lceil n/2 \rceil$, which is impossible. Hence $\alpha'_k(G) = |X| \leq kn - \lceil n/2 \rceil$ when $k = p+1$. Therefore, $\alpha'_k(G) \leq \chi''_{F_k}(G)$ and so $\alpha'_k(G) = \chi''_{F_k}(G)$.

Although it was conjectured (in Conjecture 2.6) that if G is a connected graph with $\delta(G) \geq m$, then $\chi'_{S_{k,m}}(G) = \alpha''_k(G)$, this is not the case for $\chi''_{S_{k,m}}(G)$ and $\alpha'_k(G)$. In fact, the value of $\chi''_{S_{k,m}}(G) - \alpha'_k(G)$ can be arbitrarily large. For $k = 1, 2$, it was shown in [15] that $\chi''_F(G) - \alpha'_k(G)$ can be arbitrarily large if F is a color frame of the claw $K_{1,3}$. This is also possible for $k \geq 3$. For example, the value of $\chi''_{S_{m-1,m}}(G) - \alpha'_{m-1}(G)$ can be arbitrarily large for a connected graph G, as we show next. For integers m and t where $3 \leq m < t$, let H and H' be two copy of the complete bipartite graph $K_{m-1,t}$, where the partite sets of H are $U = \{u_1, u_2, \ldots, u_{m-1}\}$ and $V = \{v_1, v_2, \ldots, v_t\}$ and the partite sets of H' are $U' = \{u'_1, u'_2, \ldots, u'_{m-1}\}$ and $V' = \{v'_1, v'_2, \ldots, v'_t\}$. The graph $H_{m-1,t}$ is obtained from H and H' by adding the t new vertices $w_1, w_2, \ldots, w_t$ and joining each w_i to $v_i \in V$ and to $v'_i \in V'$ for $i = 1, 2, \ldots, t$. The order of $H_{m-1,t}$ is $3t + 2(m-1)$.

Define a red-blue coloring c of $H_{m-1,t}$ with $E_{c,r} = E(H) \cup E(H')$. Since c is a minimal $S_{m-1,m}$-coloring of $H_{m-1,t}$, it follows that $\chi''_{S_{m-1,m}}(H_{m-1,t}) \geq |E_{c,r}| = 2(m-1)t$. Next, we show that $\alpha'_{m-1}(H_{m-1,t}) \leq 2t + 2(m-1)^2$. Let X be a maximum Δ_{m-1}-set of $H_{m-1,t}$ and let $F = H_{m-1,t}[X]$ denote the subgraph of $H_{m-1,t}$ induced by X. Since (i) the size of the subgraph $F[X - (E(H) \cup E(H'))]$ of F induced by $X - (E(H) \cup E(H'))$ is at most $2t$ and (ii) the set X is a Δ_{m-1}-set, it follows that the size of F is at most $2t + 2(m-1)^2$. This implies that $\alpha'_{m-1}(H_{m-1,t}) = |X| \leq 2t + 2(m-1)^2$. Thus $\chi''_{S_{m-1,m}}(H_{m-1,t}) - \alpha'_{m-1}(H_{m-1,t}) \geq 2(m-1)t - 2t - 2(m-1)^2$. Now let $t = m + a$ where a is an arbitrarily large positive integer. Then

$\chi''_{S_{m-1,m}}(H_{m-1,t}) - \alpha'_{m-1}(H_{m-1,t}) \geq 2a(m-2) - 2$ and so the value of $\chi''_{S_{m-1,m}}(H_{m-1,t}) - \alpha'_{m-1}(H_{m-1,t})$ can be arbitrarily large.

4 Intermediate Value Problems

For integers k and m with $1 \leq k < m$ and $m \geq 3$, we saw that $\chi'_{S_{k,m}}(G) \leq \chi''_{S_{k,m}}(G)$ and $\alpha''_k(G) \leq \alpha'_k(G)$ for every graph G. It was shown in [16] that if G is a graph and α is an integer with $\alpha''_1(G) = \alpha''(G) \leq \alpha \leq \alpha'(G) = \alpha'_1(G)$, then G contains a maximal matching of size α. This brings up the following question.

Problem 4.1 *For integers k and m with $1 \leq k < m$ and $m \geq 3$, let $S_{k,m}$ be the color frame of the star $K_{1,m}$.*

(a) If G is a connected graph of order at least 4 and χ is an integer with $\chi'_{S_{k,m}}(G) \leq \chi \leq \chi''_{S_{k,m}}(G)$, is there a minimal $S_{k,m}$-coloring of G using exactly χ red edges?

(b) If G is a connected graph of order at least 4 and α is an integer with $\alpha''_k(G) \leq \alpha \leq \alpha'_k(G)$, is there a maximal Δ_k-set of G having exactly α edges?

We show that if G is the p-corona of an n-cycle where $p \geq m-2$ and $n \geq 3$, then Problem 4.1 has an affirmative answer.

Theorem 4.2 *Let k and m be integers with $1 \leq k < m$ and $m \geq 3$ and let G be the p-corona of an n-cycle where $p \geq m-2$ and $n \geq 3$.*

(a) For each integer χ with $\chi'_{S_{k,m}}(G) \leq \chi \leq \chi''_{S_{k,m}}(G)$, there is a minimal $S_{k,m}$-coloring of G using exactly χ red edges.

(b) For each integer α with $\alpha''_k(G) \leq \alpha \leq \alpha'_k(G)$, there is a maximal Δ_k-set of G having exactly α edges.

Proof. We first verify (a). Let χ be an integer with $\chi'_{S_{k,m}}(G) \leq \chi \leq \chi''_{S_{k,m}}(G)$. We show that there is a minimal F_k-coloring of G using exactly χ red edges. By Theorems 2.4 and 3.3. we may assume that $\chi \neq \chi'_{S_{k,m}}(G)$ and $\chi \neq \chi''_{S_{k,m}}(G)$. We consider three cases, according to whether $k = 1$, $2 \leq k \leq p$ or $k = p + 1$.

Case 1. $k = 1$. Then $\chi'_{S_{k,m}}(G) = \lceil n/2 \rceil$ and $\chi''_{S_{k,m}}(G) = n$. Let $\chi = \lceil n/2 \rceil + i$ with $1 \leq i \leq \lceil n/2 \rceil - 1$. First, suppose that $n \geq 4$ is even and so $\lceil n/2 \rceil = n/2$. Let c_0 be the red-blue coloring of G such that $E_{c_0,r}$ is the set in (2). Let c_1 be the red-blue coloring of G obtained from c_0

by changing the color of v_1v_2 to blue and changing the colors of $u_{1,p}v_1$ and $u_{2,p}v_2$ to red, that is, $E_{c_1,r} = (E_{c_0,r} - \{v_1v_2\}) \cup \{u_{1,p}v_1, u_{2,p}v_2\}$ and so $|E_{c_1,r}| = |E_{c_0,r}| + 1$. In general, for each i with $2 \le i \le n/2 - 1$, the red-blue coloring c_i is obtained from c_{i-1} by changing the color of $v_{2i-1}v_{2i}$ to blue and changing the colors of $u_{2i-1,p}v_{2i-1}$ and $u_{2i,p}v_{2i}$ to red; that is, $E_{c_i,r} = (E_{c_{i-1},r} - \{v_{2i-1}v_{2i}\}) \cup \{u_{2i-1,p}v_{2i-1}, u_{2i,p}v_{2i}\}$ and so $|E_{c_i,r}| = |E_{c_{i-1},r}| + 1$. It can be verified that each coloring c_i is a minimal F_1-coloring with exactly $n/2 + i$ red edges for $1 \le i \le n/2 - 1$.

Next, suppose that $n \ge 3$ is odd and so $\lceil n/2 \rceil = (n+1)/2$. Let c_0 be the red-blue coloring of G such that $E_{c_0,r}$ is the set in (3). For each i with $1 \le i \le (n-1)/2$, the red-blue coloring c_i is obtained from c_{i-1} by changing the color of $v_{2i-1}v_{2i}$ to blue and changing the colors of $u_{2i-1,p}v_{2i-1}$ and $u_{2i,p}v_{2i}$ to red and so $|E_{c_i,r}| = |E_{c_{i-1},r}| + 1$. It can be verified that each coloring c_i is a minimal F_1-coloring with exactly $(n+1)/2 + i$ red edges for $1 \le i \le (n-1)/2$.

Case 2. $2 \le k \le p$. Then $\chi'_{S_{k,m}}(G) = (k-1)n$ and $\chi''_{S_{k,m}}(G) = kn$. Let c_0 be the red-blue coloring of G such that $E_{c_0,r}$ is the set in (4). For each i with $1 \le i \le n$, the red-blue coloring c_i is obtained from c_{i-1} by changing the color of v_iv_{i+1} to blue and changing the colors of a blue edge in X_i and a blue edge of X_{i+1} to red. Thus $|E_{c_i,r}| = |E_{c_{i-1},r}| + 1$ for $1 \le i \le n$. It can be verified that each coloring c_i is a minimal F_k-coloring with exactly $(k-1)n + i$ red edges for $1 \le i \le n$.

Case 3. $k = p + 1$. Then $\chi'_{S_{k,m}}(G) = (k-1)n$ and $\chi''_{S_{k,m}}(G) = kn - \lceil n/2 \rceil = (k-1)n + \lfloor n/2 \rfloor$. Let c_0 be the red-blue coloring of G such that $E_{c_0,r}$ is the set in (4). For each i with $1 \le i \le \lfloor n/2 \rfloor$, the red-blue coloring c_i is obtained from c_{i-1} by changing the color of $v_{2i-1}v_{2i}$ to blue and changing the colors of $u_{2i-1,p}v_{2i-1}$ and $u_{2i,p}v_{2i}$ to red; that is, $E_{c_i,r} = (E_{c_{i-1},r} - \{v_{2i-1}v_{2i}\}) \cup \{u_{2i-1,p}v_{2i-1}, u_{2i,p}v_{2i}\}$ and so $|E_{c_i,r}| = |E_{c_{i-1},r}| + 1$. It can be verified that each coloring c_i is a minimal F_1-coloring with exactly $(k-1)n + i$ red edges for $1 \le i \le \lfloor n/2 \rfloor$.

Next, we verify (b). Let α be an integer with $\alpha''_k(G) \le \alpha \le \alpha'_k(G)$ where $1 \le k < m$. We show that there is a maximal Δ_k-set of G having exactly α edges. By Theorems 2.4 and 3.3. we may assume that $\alpha \ne \alpha''_k(G)$ and $\alpha \ne \alpha'_k(G)$. Since the result is true if $k = 1$, we may assume $k \ge 2$. We consider two cases, according to whether $2 \le k \le p$ or $k = p + 1$.

Case 1. $2 \le k \le p$. Then $\alpha''_k(G) = (k-1)n$ and $\alpha'_k(G) = kn$. Let X_0 be the set $E_{c^*,r}$ in (4). Then X_0 is a maximal Δ_k-set of G having $(k-1)n$ edges. For each i with $1 \le i \le n$, the set X_i is obtained from X_{i-1} by removing the edge v_iv_{i+1} and adding the edges $u_{i,k}v_i$ and $u_{i+1,k-1}v_{i+1}$. Thus $|X_i| = |X_{i-1}| + 1$ for $1 \le i \le n$. It can be verified that each set X_i is a maximal Δ_k-set of G having exactly $(k-1)n + i$ edges for $1 \le i \le n$.

Case 2. $k = p + 1$. Then $\alpha''_k(G) = (k-1)n$ and $\alpha'_k(G) = kn - \lceil n/2 \rceil = (k-1)n + \lfloor n/2 \rfloor$. Let X_0 be the set $E_{c^*,r}$ in (4). Then X_0 is a maximal Δ_k-set of G having $(k-1)n$ edges. For each i with $1 \le i \le \lfloor n/2 \rfloor$, the set X_i is obtained from X_{i-1} by removing the edge $v_{2i}v_{2i+1}$ and adding two edges $u_{2i,p}v_{2i}$ and $u_{2i+1,p}v_{2i+1}$; that is, $X_i = (X_{i-1} - \{v_{2i}v_{2i+1}\}) \cup \{u_{2i,p}v_{2i}, u_{2i+1,p}v_{2i+1}\}$ and so $|X_i| = |X_{i-1}| + 1$. It can be verified that each X_i is a maximal Δ_k-set of G with exactly $(k-1)n + i$ edges for $1 \le i \le \lfloor n/2 \rfloor$. ■

For a connected graph G and a color frame F, if $\chi'_F(G) = a$ and $\chi''_F(G) = b$, then $a \le b$ by the definitions of the F-chromatic index and upper F-chromatic index of G. It can be shown that there are infinitely many pairs a, b of positive integers with $a \le b$ for which there exists a connected graph G such that $\chi'_F(G) = a$ and $\chi''_F(G) = b$ where F is a color frame of a star. Thus we conclude this paper with another question.

Problem 4.3 *For integers k and m with $1 \le k < m$ and $m \ge 3$, let $S_{k,m}$ be the color frame of the star $K_{1,m}$. Determine all pairs a, b of positive integers with $a \le b$ for which there exists a connected graph G such that $\chi'_{S_{k,m}}(G) = a$ and $\chi''_{S_{k,m}}(G) = b$.*

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