Proper-Path Colorings in Graph Operations

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ABSTRACT

Let G be an edge-colored connected graph. A path P is a proper path in G if no two adjacent edges of P are colored the same. An edge coloring is a proper-path coloring of G if every pair u, v of distinct vertices of G are connected by a proper $u - v$ path in G. The minimum number of colors required for a proper-path coloring of G is the proper connection number $\text{pc}(G)$ of G. We study proper-path colorings in those graphs obtained by some well-known graph operations, namely line graphs, powers of graphs, coronas of graphs and vertex or edge deletions. Proper connection numbers are determined for all iterated line graphs and powers of a given connected graph. For a connected graph G, sharp lower and upper bounds are established for the proper connection number of (i) the k-iterated corona of G in terms of $\text{pc}(G)$ and k and (ii) the vertex or edge deletion graphs $G - v$ and $G - e$ where v is a non-cut-vertex of G and e is a non-bridge of G in terms of $\text{pc}(G)$ and the degree of v. Other results and open questions are also presented.

Key Words: edge coloring, proper-path coloring, graph operations.
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1 Introduction

A *rainbow coloring* of a connected graph G is an edge coloring c of G with the property that for every two vertices u and v of G, there exists a $u-v$ *rainbow path* (no two edges of the path are colored the same). In this case, G is *rainbow-connected* (*with respect to* c). The minimum number of colors needed for a rainbow coloring of G is referred to as the *rainbow connection number* of G and denoted by $\text{rc}(G)$. There is a related concept concerning rainbow colorings. Let c be a rainbow coloring of a connected graph G. For two vertices u and v of G, a *rainbow* $u-v$ *geodesic* in G is a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between u and v (the length of a shortest $u-v$ path in G). The graph G is *strongly rainbow-connected* if G contains a rainbow $u-v$ geodesic for every two vertices u and v of G. In this case, the coloring c is called a *strong rainbow coloring* of G. The minimum number of colors needed for a strong rainbow coloring of G is referred to as the *strong rainbow connection number* $\text{src}(G)$ of G. Thus $\text{rc}(G) \leq \text{src}(G)$ for every connected graph G. These concepts were introduced and studied by Chartrand, Johns, McKeon and Zhang in [3, 4]. In recent years, this topic has been studied by many and there is now a book [8] on this subject.

The most-studied edge colorings of a graph G are proper edge colorings in which every two adjacent edges of G are assigned distinct colors. The minimum number of colors needed in a proper coloring of G is referred to as the *chromatic index* of G, denoted by $\chi'(G)$. One property that a properly edge-colored graph G has is that for every two vertices u and v, each $u-v$ path of G is properly colored. However, if we are primarily concerned with a graph G containing a properly colored $u-v$ path for every two vertices u and v of G, then it is possible that this can be accomplished using fewer than $\chi'(G)$ colors.

Inspired by rainbow colorings and proper colorings in graphs, the concepts of proper-path colorings and strong proper-path colorings were introduced and studied [1]. Let G be an edge-colored connected graph, where adjacent edges may be colored the same. A path P in G is *properly colored* or, more simply, P is a *proper path* in G if no two adjacent edges of P are colored the same. An edge coloring c is a *proper-path coloring* of a connected graph G if every pair u,v of distinct vertices of G are connected by a proper $u-v$ path in G. If k colors are used, then c is referred to as a *proper-path k-coloring*. The minimum k for which G has a proper-path k-coloring is called the *proper connection number* $\text{pc}(G)$ of G. A proper-path coloring using $\text{pc}(G)$ colors is referred to as a *minimum proper-path coloring*. Since every rainbow coloring and every proper coloring is a proper-path coloring, it follows that $\text{pc}(G)$ exists. If G is a nontrivial connected graph of order

n and size m, then

$$1 \leq \mathrm{pc}(G) \leq \min\{\chi'(G), \mathrm{rc}(G)\} \leq m. \tag{1}$$

Furthermore, $\mathrm{pc}(G) = 1$ if and only if $G = K_n$ and $\mathrm{pc}(G) = m$ if and only if $G = K_{1,m}$ is a star of size m.

As with rainbow colorings and strong rainbow colorings, there is an analogous concept of proper-path colorings, which was introduced in [1]. Let c be a proper-path coloring of a nontrivial connected graph G. For two vertices u and v of G, a *proper* $u - v$ *geodesic* in G is a proper $u - v$ path of length $d(u, v)$. If there is a proper $u - v$ geodesic for every two vertices u and v of G, then c is called a *strong proper-path coloring* of G or a *strong proper-path k-coloring* if k colors are used. The minimum number of colors needed to produce a strong proper-path coloring of G is called the *strong proper connection number* $\mathrm{spc}(G)$ of G. A strong proper-path coloring using $\mathrm{spc}(G)$ colors is referred to as a *minimum strong proper-path coloring* . In general, if G is a nontrivial connected graph, then $1 \leq \mathrm{pc}(G) \leq \mathrm{spc}(G) \leq \chi'(G)$. Since every strong rainbow coloring of G is a strong proper-path coloring of G, it follows that $\mathrm{spc}(G) \leq \mathrm{src}(G)$. Therefore, if G is a nontrivial connected graph of order n and size m, then

$$1 \leq \mathrm{spc}(G) \leq \min\{\chi'(G), \mathrm{src}(G)\} \leq m. \tag{2}$$

Similarly, $\mathrm{spc}(G) = 1$ if and only if $G = K_n$ and $\mathrm{spc}(G) = m$ if and only if $G = K_{1,m}$ is the star of size m. In [1] the numbers $\mathrm{pc}(G)$ and $\mathrm{spc}(G)$ were determined for several well-known classes of graphs G and relationships among these five edge colorings (namely, proper-path colorings, strong proper-path colorings, rainbow colorings, strong rainbow colorings and proper edge colorings) were investigated. Furthermore, several realization theorems were established for the five edge coloring parameters (namely $\mathrm{pc}(G)$, $\mathrm{spc}(G)$, $\mathrm{rc}(G)$, $\mathrm{src}(G)$ and $\chi'(G)$) of a connected graph G. By (1) and (2), if G is a nontrivial connected graph of size m, then $\mathrm{pc}(G) \leq m$ and $\mathrm{spc}(G) \leq m$. As we mentioned above, the star $K_{1,m}$ of size m is the only nontrivial connected graph of size m having proper connection number and strong proper connection number m. This is also the case for the rainbow connection numbers of graphs. In [9], all graphs of size m having rainbow connection numbers $m-2$ and $m-3$ have been characterized by Li, Sun and Zhao; while in [7], all connected graphs of size m having proper connection number or strong proper connection number $m - 1$, $m - 2$ or $m - 3$ have been characterized. Furthermore, the proper-path colorings in the joins and Cartesian products of two graphs as well as the permutation graph of a graph have been studied in [1]. In this paper, we determine the proper connection numbers of iterated line graphs, powers of graphs and iterated corona graphs. Moreover, we study how the proper connection number of a graph can be affected by deleting a vertex or an edge from the graph and establish sharp lower and upper bounds $\text{pc}(G - v)$ and $\text{pc}(G - e)$ in terms of $\text{pc}(G)$ where v is a non-cut-vertex of G and e is a non-bridge of G. We refer to the books [5, 6] for graph theory notation and terminology not described in this paper.

2 Iterated Line Graphs

The most familiar graph operation of a graph is that of the line graph. The *line graph* $L(G)$ of a graph G is that graph whose vertices can be put in one-to-one correspondence with the edges of G in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. We determine the proper connection number of $L(G)$ of every connected graph G of order at least 3. In order to do this, we first present an additional definition. For a connected graph G and two sets X and Y of vertices of G, the *distance* $d(X, Y)$ *between* X *and* Y is defined as

$$d(X, Y) = \min\{d(x, y) : \ x \in X \text{ and } y \in Y\}.$$

Thus $d(X, Y) = 0$ if and only if $X \cap Y \neq \emptyset$.

Theorem 2.1 *For each connected graph* G *of order at least* 3,

$$\text{pc}(L(G)) \leq 2.$$

Proof. Let G be a connected graph of order $n \geq 3$ with $V(G) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$ and T a rooted spanning tree of G where the root of T is v_0. For each d where $0 \leq d \leq e_T(v_0)$, let $L_d = \{v \in V(G) : d_T(v_0, v) = d\}$. Thus, $L_0 = \{v_0\}$. First, we define a vertex coloring $c : V(G) \to \{1, 2\}$ of G as follows. For $0 \leq k \leq n - 1$ and $0 \leq d \leq e_T(v_0)$, let

$$c(v_k) = \begin{cases} 1 & \text{if } v_k \in L_d \text{ and } d \text{ is odd} \\ 2 & \text{if } v_k \in L_d \text{ and } d \text{ is even.} \end{cases}$$

Observe that each vertex of $L(G)$ can be represented as $v_i v_j$ for some i and j where $0 \leq i \neq j \leq n - 1$ since it corresponds to an edge $v_i v_j$ in G and two vertices $v_i v_j$ and $v_k v_\ell$ of $L(G)$ are adjacent if and only if (exactly) one of these four conditions occurs: $i = k$, $i = \ell$, $j = k$ or $j = \ell$.

Next, we define an edge coloring $c_L : E(L(G)) \to \{1, 2\}$ of $L(G)$ by $c_L(e) = c(v_j)$ where, without loss of generality, $e = xy$ in which $x = v_i v_j$ and $y = v_j v_k$ are edges of G. It remains to show that c_L is a proper-path 2-coloring of $L(G)$. For $x, y \in V(L(G))$, we show that there is a properly colored $x - y$ path in $L(G)$. We may assume x and y are nonadjacent. Let $x = v_i v_j$ and $y = v_k v_\ell$, where then v_i, v_j, v_k and v_ℓ are distinct vertices of G. Without loss of generality, for $X = \{v_i, v_j\}$ and $Y = \{v_k, v_\ell\}$ we let $d_T(X,Y) = d_T(v_j, v_k) \geq 1$. Then there is a unique $v_j - v_k$ path P' in T that does not contain v_i and v_ℓ.

Let P be a $v_i - v_\ell$ path of G obtained from the path P' in T by joining v_i to v_j and joining v_ℓ to v_k. Thus $P = (v_i = w_1, v_j = w_2, w_3, \ldots, w_{q-1} = v_k, w_q = v_\ell)$ for some integer $q \geq 4$. Moreover, we may assume for $2 \leq t \leq q-1$ that

$$c(w_t) = \begin{cases} 1 & \text{if } t \text{ is odd} \\ 2 & \text{if } t \text{ is even.} \end{cases} \tag{3}$$

Then $L(P) = (e_1, e_2, \ldots, e_{q-1})$ is a path of order $q-1$ in $L(G)$, where $e_i = w_i w_{i+1}$ $(1 \leq i \leq q-1)$. In particular, $e_1 = x$ and $e_{q-1} = y$. Observe that e_1 and e_2 are both incident with w_2 and so the edge e_1e_2 in $L(G)$ is colored 2 by c_L; that is $c_L(e_1e_2) = c(w_2) = 2$ by (3). Next, e_2 and e_3 are both incident with w_3 and so $c_L(e_2e_3) = c(w_3) = 1$ again by (3). This is illustrated in Figure 1 for $q = 8$, where the edges in $L(G)$ are indicated by dash lines. In general,

$$c_L(e_ie_{i+1}) = \begin{cases} 2 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even.} \end{cases}$$

Since the edges of $L(P)$ are colored alternatively by 1 and 2, it follows that $L(P)$ is a properly colored $x - y$ path in $L(G)$. Therefore, $\text{pc}(L(G)) \leq 2$.

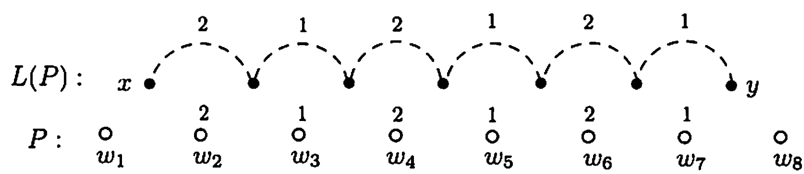

Figure 1: Illustrating the proper-path coloring c_L in the proof of Theorem 2.1

For a connected graph G, the line graph $L(G)$ of G is complete if and only if G is a star or G is a triangle. Since the complete graph of order $n \geq 2$ is the only connected graph of order n with proper connection number 1, the following is a consequence of Theorem 2.1.

Corollary 2.2 *If G is a connected graph of order $n \geq 3$, then*

$$\text{pc}(L(G)) = \begin{cases} 1 & \textit{if } G \in \{K_3, K_{1,n-1}\} \\ 2 & \textit{otherwise.} \end{cases}$$

There is a more general concept in line graphs. For a nonempty graph G, we write $L^0(G)$ to denote G and $L^1(G)$ to denote $L(G)$. For an integer $k \geq 2$, the *k-iterated line graph* $L^k(G)$ is defined as $L(L^{k-1}(G))$, where $L^{k-1}(G)$ is assumed to be nonempty. In order to determine the proper connection number of each iterated line graph of a graph, we first present some useful information in line graphs. A graph H is called a *line graph* if there exists a graph G such that $H = L(G)$. A natural question to ask is whether a given graph H is a line graph. Several characterizations of line graphs have been obtained, perhaps the best known of which is a 1970 forbidden subgraph characterization due to Beineke.

Theorem 2.3 [2] *A graph H is a line graph if and only if none of the graphs of Figure 2 is isomorphic to an induced subgraph of H.*

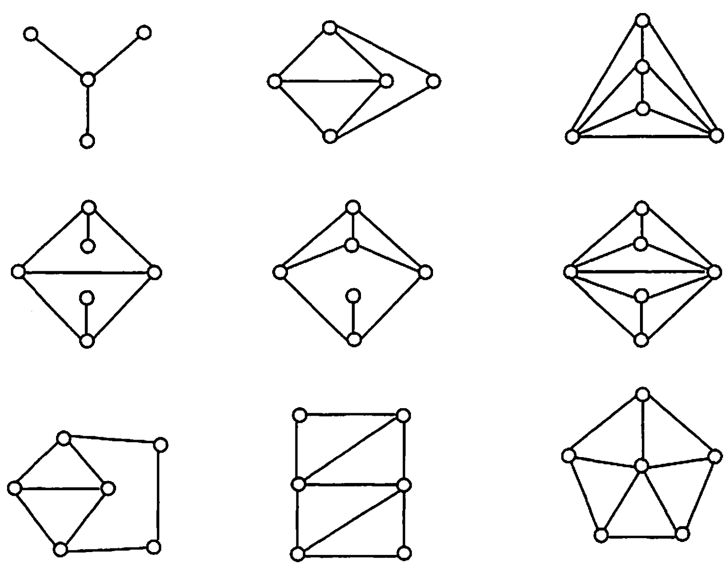

Figure 2: The induced subgraphs not contained in any line graph

By Theorem 2.3, the star $K_{1,n-1}$ of order $n \geq 4$ is not a line graph. Furthermore, $L(K_{1,n-1}) = K_{n-1}$ and $L^k(K_3) = K_3$ for each positive integer k. It then follows by Corollary 2.2 that if $G \neq K_3$, then $L^k(G) = 2$ for each integer $k \geq 2$. Therefore, the following corollary is a consequence of Theorem 2.3 and Corollary 2.2, which provides the exact value of $\mathrm{pc}(L^k(G))$ for every connected graph G and each $k \geq 1$.

Corollary 2.4 *If G is a connected graph of order $n \geq 3$ and k is a positive integer, then*

$$\mathrm{pc}(L^k(G)) = \begin{cases} 1 & \text{if either (i) } G \in \{K_3, K_{1,3}\} \text{ and } k \geq 1 \\ & \text{or (ii) } G = K_{1,n-1} \text{ where } n \neq 4 \text{ and } k = 1 \\ 2 & \text{otherwise.} \end{cases}$$

3 Powers of Graphs

For a connected graph G and a positive integer k, the *kth power* G^k of G is that graph whose vertex set is $V(G)$ such that uv is an edge of G^k if $1 \leq d_G(u, v) \leq k$. The graph G^2 is called the *square* of G and G^3 is the *cube* of G. In order to determine the proper connection numbers of powers of graphs, we first present the following useful information. A fundamental property of the chromatic number or chromatic index is that if H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$ and $\chi'(H) \leq \chi'(G)$. For the proper connection number, the situation is different.

Proposition 3.1 [1] *If G is a nontrivial connected graph and H is a connected spanning subgraph of G, then* $\mathrm{pc}(G) \leq \mathrm{pc}(H)$. *Furthermore,*

$$\mathrm{pc}(G) \leq \min\{\Delta(T) : T \text{ is a spanning tree of } G\}.$$

A *Hamiltonian path* in a graph G is a path containing every vertex of G and a graph having a Hamiltonian path is often called a *traceable graph.* The following is an immediate consequence of Proposition 3.1.

Corollary 3.2 [1] *If G is a traceable graph that is not complete, then* $\mathrm{pc}(G) = 2$.

Theorem 3.3 *For each connected graph G of order at least 3,*

$$\mathrm{pc}(G^2) \leq 2.$$

Proof. Let G be a connected graph of order $n \geq 3$ and T a spanning tree of G. Thus T is a spanning tree of G^2 as well. Moreover, T^2 is a spanning subgraph of G^2. Define an edge coloring $c : E(G^2) \to \{1, 2\}$ of G^2 by

$$c(e) = \begin{cases} 1 & \text{if } e \in E(T) \\ 2 & \text{if } e \in E(G^2) - E(T). \end{cases}$$

If we can show, for any two vertices x and y of T^2, that there is a properly colored $x - y$ path in T^2, then the result follows since T^2 is a spanning subgraph of G^2.

Let x and y be two vertices of T^2 and $P = (x = v_0, v_1, v_2, \ldots, y = v_d)$ a unique $x - y$ path in T (and so in T^2) of length d. We may assume that x and y are nonadjacent in T^2 and so $d \geq 3$. We claim that there is an $x - y$ path P' in T^2 obtained from P that is properly colored.

- If $d \equiv 0 \pmod 3$, say $d = 3t$ for some positive integer t, then $P' = (x = v_0, v_1, v_3, v_4, v_6, \ldots, v_{3(t-1)}, v_{3t-2}, v_{3t} = y)$ is an $x - y$ path in T^2 whose edges are colored alternatively by 1 and 2.
- If $d \equiv 1 \pmod 3$, say $d = 3t + 1$ for some positive integer t, then $P' = (x = v_0, v_1, v_3, v_4, v_6, v_7, \ldots, v_{3(t-1)}, v_{3t-2}, v_{3t}, v_{3t+1} = y)$ is an $x - y$ path in T^2 whose edges are colored alternatively by 1 and 2.
- If $d \equiv 2 \pmod 3$, say $d = 3t + 2$ for some positive integer t, then $P' = (x = v_0, v_2, v_3, v_5, v_6, v_8, \ldots, v_{3(t-1)}, v_{3t-1}, v_{3t}, v_{3t+2} = y)$ is an $x - y$ path in T^2 whose edges are colored alternatively by 1 and 2.

In any case, P' is a properly colored $x - y$ path in T^2 and so in G^2. Hence $\mathrm{pc}(G^2) \leq 2$. ■

In 1960 Sekanina [10] proved that the cube of every connected graph G is Hamiltonian-connected and, consequently, G^3 is Hamiltonian if its order is at least 3. Furthermore, the k-power G^k of a connected graph G is complete if and only if $\mathrm{diam}(G) \leq k$. Thus, the following is a consequence of Corollary 3.2 and Theorem 3.3.

Corollary 3.4 *Let $k \geq 2$ be an integer. If G is a connected graph of order at least 3, then $\mathrm{pc}(G^k) \leq 2$. Furthermore, $\mathrm{pc}(G^k) = 1$ if and only if $\mathrm{diam}(G) \leq k$.*

4 Iterated Corona Graphs

For a given graph G, the *corona* $\mathrm{cor}(G)$ of G is obtained from G by adding a pendant edge to each vertex of G. Thus, if the order of G is n, then the order of $\mathrm{cor}(G)$ is $2n$. As with iterated line graphs, there is a more general concept in coronas of graphs. For a nonempty graph G, we write $\mathrm{cor}^0(G)$ to denote G and $\mathrm{cor}^1(G)$ to denote $\mathrm{cor}(G)$. For an integer $k \geq 2$, the *k-iterated corona graph* $\mathrm{cor}^k(G)$ is defined as $\mathrm{cor}(\mathrm{cor}^{k-1}(G))$. The following two results will be useful.

Proposition 4.1 [1] *Let G be a nontrivial connected graph containing bridges. If b is the maximum number of bridges incident with a single vertex in G, then $\mathrm{pc}(G) \geq b$.*

Proposition 4.2 [1] *If T is a nontrivial tree, then $\mathrm{pc}(T) = \chi'(T) = \Delta(T)$, where $\Delta(T)$ is the maximum degree of T.*

With the aid of Proposition 4.1 and the property of trees in Proposition 4.2, we will see that iterated corona graphs have relatively large proper connection numbers. First, we introduce an additional definition. For a proper-path coloring c of a graph G, the *restriction* c_H of c to a subgraph H of G is the coloring defined by $c_H(e) = c(e)$ for every edge e of H.

Theorem 4.3 *If G is a nontrivial connected graph and k is a positive integer, then*

$$\max\{\text{pc}(G), k\} \leq \text{pc}(\text{cor}^k(G)) \leq \text{pc}(G) + k. \tag{4}$$

Proof. Let G be a connected graph of order $n \geq 2$. In the graph $\text{cor}^k(G)$, each vertex is incident with at least k bridges. It then follows by Proposition 4.1 that $\text{pc}(\text{cor}^k(G)) \geq k$. Furthermore, for every two vertices of u and v in $\text{cor}^k(G)$ such that $u, v \in V(G)$, each $u - v$ path lies completely in G. This implies that the restriction of a proper-path coloring of $\text{cor}^k(G)$ to G must be a proper-path coloring of G. Hence $\text{pc}(\text{cor}^k(G)) \geq \text{pc}(G)$. Therefore, the lower bound in (4) holds.

To show that $\text{pc}(\text{cor}^k(G)) \leq \text{pc}(G) + k$, let c_G be a minimum proper-path coloring of G using the colors $1, 2, \ldots, \text{pc}(G)$ and $F = \text{cor}^k(G) - E(G)$. Then F is a forest consisting of n components $T_1, T_2, \ldots, T_n$. The maximum degree of each component T_i is k. By Proposition 4.2, $\text{pc}(T_i) = k$. Let c_{T_i} be a minimum proper-path coloring of T_i using the colors $\text{pc}(G)+1, \text{pc}(G)+2, \ldots, \text{pc}(G) + k$. Now, define the coloring c of $\text{cor}^k(G)$ by

$$c(e) = \begin{cases} c_G(e) & \text{if } e \in E(G) \\ c_{T_i}(e) & \text{if } e \in E(T_i) \text{ for } 1 \leq i \leq n. \end{cases}$$

Since c is a proper-path coloring of $\text{cor}^k(G)$ using exactly $\text{pc}(G) + k$ colors, it follows that $\text{pc}(\text{cor}^k(G)) \leq \text{pc}(G) + k$.

Both upper and lower bounds in Theorem 4.3 are sharp. For example, if G is a tree, then $\text{cor}^k(G)$ is a tree with $\Delta(\text{cor}^k(G)) = \Delta(G) + k$. Hence $\text{pc}(\text{cor}^k(G)) = \text{pc}(G) + k$ by Proposition 4.2 and so the upper bound in Theorem 4.3 is sharp. Furthermore, there are connected graphs G that is not a tree for which $\text{pc}(\text{cor}^k(G)) = \text{pc}(G) + k$ for each integer $k \geq 1$. For example, the 3-regular graph G of Figure 3 has $\text{pc}(G) = 3$ (a proper-path 3-coloring is shown in the figure) and $\text{pc}(\text{cor}^k(G)) = 3 + k$ for each positive integer k.

To show that the lower bound in Theorem 4.3 is sharp, we determine $\text{pc}(\text{cor}^k(K_n))$ for each complete graph K_n of order $n \geq 3$ and each positive integer k.

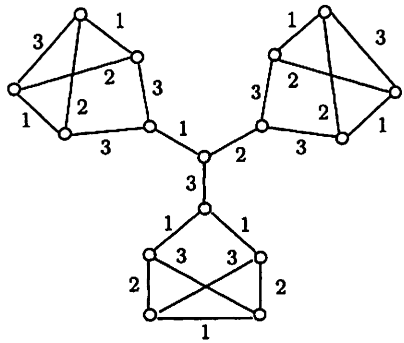

Figure 3: A 3-regular graph G with $\mathrm{pc}(G) = 3$

Theorem 4.4 *For integers $k \geq 1$ and $n \geq 3$,*

$$\mathrm{pc}(\mathrm{cor}^k(K_n)) = \begin{cases} k+1 & \text{if either } k = 1 \text{ or } k = 2 \text{ and } n = 3 \\ k & \text{if either } k = 2 \text{ and } n \geq 4 \text{ or } k \geq 3. \end{cases} \tag{5}$$

Proof. In the graph $\mathrm{cor}^k(K_n)$, let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$ where $k \geq 1$ and $n \geq 3$. Let G be the k-corona of K_n; that is, G is obtained from K_n by adding exactly k pendant edges at each vertex of of K_n. Since $\mathrm{pc}(T) = \Delta(T)$ for each tree T, every proper-path coloring of G can be extended to a proper-path coloring of $\mathrm{cor}^k(K_n)$ with the same set of colors and so $\mathrm{pc}(G) \geq \mathrm{pc}(\mathrm{cor}^k(K_n))$. On the other hand, if x and y are vertices of $\mathrm{cor}^k(K_n)$ such that $x, y \in V(G)$, then every $x - y$ path lies completely in G. Hence the restriction of a proper-path coloring of $\mathrm{cor}^k(K_n)$ to G is a proper-path coloring of G and so $\mathrm{pc}(G) \leq \mathrm{pc}(\mathrm{cor}^k(K_n))$ and so $\mathrm{pc}(G) = \mathrm{pc}(\mathrm{cor}^k(K_n))$. Thus, it suffices to show that $\mathrm{pc}(G)$ satisfies the formula given in (5). For $1 \leq i \leq n$, let $v_{i,1}, v_{i,2}, \ldots, v_{i,k}$ be the k end-vertices adjacent to u_i in G. Thus, every proper-path coloring of G must assign distinct k colors to these k pendant edges at each vertex of K_n. We consider three cases, according to whether $k = 1$, $k = 2$ or $k \geq 3$.

Case 1. $k = 1$. Since $\mathrm{pc}(G) \geq 2$, it suffices to find a proper-path 2-coloring of G. The coloring that assigns the color 1 to each pendant edge of G and the color 2 to the remaining edges of G is a proper-path 2-coloring of G and so $\mathrm{pc}(G) = 2$.

Case 2. $k = 2$. First, suppose that $n = 3$. The coloring that assigns the colors 1 and 2 to the two pendant edges at each vertex of K_3 and the color 3 to each edge of K_3 is a proper-path 3-coloring of G and so $\mathrm{pc}(G) \leq 3$. If there were a proper-path 2-coloring c of G, then c must assign two edges of K_3 the same color, say $c(u_1u_2) = c(u_2u_3) = 1$. Furthermore, we may assume that $c(u_2v_{2,1}) = 1$. However then, there is neither a proper $v_{2,1} - u_1$ path nor a proper $v_{2,1} - u_3$ path in G. Thus $\text{pc}(G) = 3$.

Next, suppose that $n \geq 4$. Since $\text{pc}(G) \geq 2$, it suffices to find a proper-path 2-coloring of G. Let $C = (u_1, u_2, \cdots, u_n, u_1)$ be a Hamiltonian cycle of K_n. Define a coloring c by assigning (1) the colors 1 and 2 to the two pendant edges at each vertex of K_n, (2) the color 1 to each edge of C and (3) the color 2 to the remaining edges of K_n. Since c is a proper-path 2-coloring of G, it follows that $\text{pc}(G) = 2$.

Case 3. $k \geq 3$. Since $\text{pc}(G) \geq k$ by Proposition 4.1, it suffices to find a proper-path k-coloring of G. The coloring defined on Case 2 can be extended to a proper-path k-coloring of G; that is, we assign (1) the colors $1, 2 \ldots, k$ to the k pendant edges at each vertex of K_n, (2) the color 1 to each edge of a Hamiltonian cycle of K_n and (3) the color 2 to the remaining edges of K_n. Therefore, $\text{pc}(G) = k$.

Every connected graph G that we considered so far has the property that either $\text{pc}(\text{cor}^k(G)) = \max\{\text{pc}(G), k\}$ or $\text{pc}(\text{cor}^k(G)) = \text{pc}(G) + k$. Furthermore, we know of no connected graphs G for which $\max\{\text{pc}(G), k\} < \text{pc}(\text{cor}^k(G)) < \text{pc}(G) + k$. Thus, we are left with the following question.

Problem 4.5 *Are there connected graphs G and integers $k \geq 3$ for which*

$$\max\{\text{pc}(G), k\} < \text{pc}(\text{cor}^k(G)) < \text{pc}(G) + k?$$

5 Vertex or Edge Deletions

Let G be a connected graph of order at least 3. For each vertex v of G and each edge e of G, it is known that

$$\chi(G) - 1 \leq \chi(G - v) \leq \chi(G) \text{ and } \chi(G) - 1 \leq \chi(G - e) \leq \chi(G).$$

However, this is not the case for the proper connection number of a graph in general. In order to show this, we first present a useful observation.

Observation 5.1 *If T is a nontrivial tree with maximum degree Δ and having n_1 end-vertices, then $\Delta \leq n_1$.*

Theorem 5.2 *Let G be a connected graph of order at least 3. If v is a non-cut-vertex of G, then*

$$\text{pc}(G) - 1 \leq \text{pc}(G - v) \leq \text{pc}(G) + \deg v. \tag{6}$$

Proof. Suppose that $\text{pc}(G - v) = a$ and $\deg v = d$. First, observe that if c is a proper-path coloring of $G - v$ using the colors $1, 2, \ldots, a$, then c can be extended to a proper-path coloring of G by assigning the color $a + 1$ to each edge incident with v in G. Thus, $\text{pc}(G) \leq \text{pc}(G - v) + 1$, establishing the lower bound.

To verify the upper bound, let $c_G : E(G) \to \{1, 2, \ldots, k\}$ be a minimum proper-path coloring of G and let $N(v)$ be the neighborhood of v, where then $|N(v)| = d$. Since $G - v$ is connected, there is a tree T of minimum order in $G - v$ such that $N(v) \subseteq V(T)$. Necessarily, each end-vertex of T belongs to $N(v)$. Thus, if the number of end-vertices of T is n_1, then $n_1 \leq d$. Now, let $\Delta = \Delta(T)$ be the maximum degree of T. By Proposition 4.2 and Observation 5.1, it follows that $\chi'(T) = \text{pc}(T) = \Delta \leq n_1 \leq d$. Let $c_T : E(T) \to \{k+1, k+2, \ldots, k+\Delta\}$ be a proper edge coloring of T. Define an edge coloring $c : E(G - v) \to \{1, 2, \ldots, k + \Delta\}$ of $G - v$ by

$$c(e) = \begin{cases} c_G(e) & \text{if } e \in E(G - v) - E(T) \\ c_T(e) & \text{if } e \in E(T). \end{cases} \tag{7}$$

It remains to show that c is a proper-path coloring of $G - v$. Let x and y be two nonadjacent vertices of $G - v$. We show that there is a properly colored $x - y$ path in $G - v$. Since c_G is a proper-path coloring of G, there is an $x - y$ path in G that is properly colored by the edge coloring c_G of G. We consider two cases.

Case 1. *There is an $x - y$ path P in G that does not contain v and is properly colored by c_G.* If $E(P) \cap E(T) = \emptyset$, then P is an $x - y$ path in $G - v$ that is properly colored by c. Thus, we may assume that $E(P) \cap E(T) \neq \emptyset$. We now divide the path P into a finite number of blocks $A_1, B_1, A_2, B_2, \ldots$ for which

$$P = (A_1, B_1, A_2, B_2, \ldots)$$

where each block is a subpath of P such that $E(A_i) \subseteq E(G - v) - E(T)$ for each $i \geq 1$ and $E(B_j) \subseteq E(T)$ for each $j \geq 1$ (or $E(A_i) \subseteq E(T)$ for each $i \geq 1$ and $E(B_j) \subseteq E(G - v) - E(T)$ for each $j \geq 1$). Since P is properly colored by c_G and T is properly colored by c_T, it follows by the definition of c in (7) that each of the blocks A_i and B_j is properly colored by c in $G - v$. Furthermore, the colors of edges in A_i belong to $\{1, 2, \ldots, k\}$ and the colors of edges in B_j belong to $\{k + 1, k + 2, \ldots, k + \Delta\}$. Therefore, P is properly colored by c and so P is a proper $x - y$ path in $G - v$.

Case 2. *Every $x - y$ path in G that is properly colored by c_G contains the vertex v.* Let Q be an $x - y$ path in G that is properly colored by c_G. Thus Q contains a subpath (u, v, w) where $u, w \in N(v)$. Let W be the $x - y$ walk in $G - v$ obtained from Q by replacing the subpath (u, v, w) by the $u - w$ path R in T. Let $Q_{x,u}$ be the $x - u$ subpath of Q and $Q_{w,y}$ be the $w - y$ subpath of Q. Furthermore, let u' be the first vertex (from x to u) that belongs to $V(Q_{x,u}) \cap V(R)$ and let w' be the last vertex (from w to

y) that belong to $V(Q_{w,y}) \cap V(R)$. Then $u' \neq w'$ where it is possible that $u = u'$ or $w = w'$. Now let $Q_{x,u'}$ be the $x - u'$ subpath of Q, let $Q_{w',y}$ be the $w' - y$ subpath of Q and let $R_{u',w'}$ be the $u' - w'$ subpath of R. Now the path $P = (Q_{x,u'}, R_{u',w'}, Q_{w',y})$ is an $x - y$ path in $G - v$. Since the colors of edges in $Q_{x,u'}$ and $Q_{w',y}$ belong to $\{1, 2, \ldots, k\}$ and the colors of edges in $R_{u',w'}$ belong to $\{k+1, k+2, \ldots, k+\Delta\}$, it follows that P is properly colored by c and so P is a proper $x - y$ path in $G - v$.

Therefore, the edge coloring $c : E(G - v) \to \{1, 2, \ldots, k + \Delta\}$ defined in (7) is a proper-path coloring of $G - v$ and so $\text{pc}(G - v) \leq k + \Delta \leq \text{pc}(G) + \deg v$. ■

Both lower and upper bounds in (6) are sharp. For example, let $G = K_{1,t}$ be the star of order $t + 1 \geq 3$ and let v be an end-vertex of G. Since $\text{pc}(G) = t$ and $\text{pc}(G - v) = t - 1 = \text{pc}(G) - 1$, it follows that the lower bound is sharp. For the upper bound in (6), we start with the complete bipartite graph $K_{2,t}$ of order $2 + t \geq 4$ where u and v are the vertices of degree t in $K_{2,t}$. It was shown in [1] that if G is a complete multipartite graph that is neither a complete graph nor a tree, then $\text{pc}(G) = 2$. Thus $\text{pc}(K_{2,t}) = 2$. The graph H is then obtained from $K_{2,t}$ by adding two pendant edges at the vertex u of degree t in $K_{2,t}$. It can be shown that $\text{pc}(H) = 2$. In fact, a proper-path 2-coloring of H can be obtained from a proper-path 2-coloring of $K_{2,t}$ (using the colors 1 and 2) by assigning the colors 1 and 2 to the two pendant edges incident with the vertex u in H. Then $H - v = K_{1,t+2}$. Since $\deg_H v = t$ and $\text{pc}(H - v) = t + 2$, it follows that $\text{pc}(H - v) = \text{pc}(H) + \deg_H v$. Therefore, the upper bound in (6) is sharp. Furthermore, strict equalities are also possible in (6). For example, let $F = K_{2,t}$ where $t \geq 3$ and so $\text{pc}(F) = 2$. Now let v be a vertex of degree t in F. Then $F - v = K_{1,t}$ and so $\text{pc}(F - v) = t$. Therefore, $\text{pc}(F) < \text{pc}(F - v) = t = \text{pc}(F) + \deg v - 2 < \text{pc}(F) + \deg v$.

Theorem 5.3 *Let G be a connected graph of order at least 3. If e is an edge of G that is not a bridge, then*

$$\text{pc}(G) \leq \text{pc}(G - e) \leq \text{pc}(G) + 2. \tag{8}$$

Proof. Since $G - e$ is a spanning subgraph of G for each nonbridge e of G, it follows by Proposition 3.1 that $\text{pc}(G) \leq \text{pc}(G - e)$, establishing the lower bound. It remains to verify the upper bound. Suppose that $\text{pc}(G) = k$ and $e = uv$ where $u, v \in V(G)$. Let $c_G : E(G) \to \{1, 2, \ldots, k\}$ be a minimum proper-path coloring of G. Since e is not a bridge, there is a $u - v$ path P in $G - e$. Let $c_P : E(P) \to \{k+1, k+2\}$ be a proper edge coloring of P. Define an edge coloring $c : E(G - e) \to \{1, 2, \ldots, k+2\}$ of $G - e$ by

$$c(e) = \begin{cases} c_G(e) & \text{if } e \in E(G - e) - E(P) \\ c_P(e) & \text{if } e \in E(P). \end{cases}$$

Applying an argument similar to one used in the proof of Theorem 5.2 where the tree T is replaced by the $u - v$ path P, it can be shown that c is a proper-path coloring of $G - e$ and so $\mathrm{pc}(G - e) \leq k + 2 = \mathrm{pc}(G) + 2$. ■

Both lower and upper bounds in (8) are sharp. For example, $\mathrm{pc}(C_n) = \mathrm{pc}(C_n - e) = 2$ for $n \geq 4$ and any edge e in C_n. Furthermore, if $G = K_{1,t} + e$ where $t \geq 5$, then $\mathrm{pc}(G) = t - 2$ and so $\mathrm{pc}(G - e) = t = \mathrm{pc}(G) + 2$.

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