Characterizations of Highly Path-Hamiltonian Graphs

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Abstract

A Hamiltonian graph G is said to be ℓ-path-Hamiltonian, where ℓ is a positive integer less than or equal to the order of G, if every path of order ℓ in G is a subpath of some Hamiltonian cycle in G. The Hamiltonian cycle extension number of G is the maximum positive integer ℓ for which every path of order ℓ or less is a subpath of some Hamiltonian cycle in G. If the order of G equals n, then it is known that $\text{hce}(G) = n$ if and only if G is a cycle or a regular complete bipartite graph (when n is even) or a complete graph. We present a complete characterization of Hamiltonian graphs of order n that are ℓ-path-Hamiltonian for each $\ell \in \{n-3, n-2, n-1, n\}$.

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1 Introduction

We refer to the book [4] for graph theory notation and terminology not described in this paper. A Hamiltonian graph G of order $n \geq 3$ is said to be *ℓ-path-Hamiltonian* for some integer ℓ, where $1 \leq \ell \leq n$, if for every path P of order ℓ, there exists a Hamiltonian cycle C in G such that P is a path on C. Certainly, a graph is Hamiltonian if and only if it is 1-path-Hamiltonian. The largest positive integer ℓ for which a graph G is i-path-Hamiltonian for $1 \le i \le \ell$ is the *Hamiltonian cycle extension number* $\text{hce}(G)$ of G. Hence, $1 \le \text{hce}(G) \le n$ for every Hamiltonian graph of order n. Furthermore, $\text{hce}(G) = 1$ if and only if G contains an edges that lies on no Hamiltonian cycle of G.

For each integer $n \ge 3$, it is not difficult to see that if G equals the n-cycle C_n or the complete graph K_n or, when $n \ge 4$ is even, the regular complete bipartite graph $K_{n/2,n/2}$, then a path of any possible length in G can be extended to a Hamiltonian cycle in G. That is, $\text{hce}(G) = n$ for each of these graphs G. In fact, Chartrand and Kronk [3] showed that these are the only Hamiltonian graphs possessing this property. Let

$$\mathcal{A}_n = \begin{cases} \{C_n, K_n\} & \text{if } n \text{ is odd} \\ \{C_n, K_n, K_{n/2,n/2}\} & \text{if } n \text{ is even.} \end{cases} \tag{1}$$

Theorem 1.1 [3] *Let G be a graph of order $n \ge 3$. Then* $\text{hce}(G) = n$ *if and only if $G \in \mathcal{A}_n$.*

By definition, if G is a graph of order $n \ge 3$ with $\text{hce}(G) = n$, then G is ℓ-path-Hamiltonian for $1 \le \ell \le n$. In particular, G is n-path-Hamiltonian. It then follows by Theorem 1.1 that if $G \in \mathcal{A}_n$, then G is n-path-Hamiltonian.

In this work, we characterize (i) all graphs of order $n \ge 3$ that are ℓ-path-Hamiltonian for each $\ell \in \{n-2, n-1, n\}$ and (ii) all graphs of order $n \ge 4$ that are $(n-3)$-path-Hamiltonian.

2 The First Characterization

In this section, we determine all graphs of order $n \ge 3$ that are ℓ-path-Hamiltonian for each $\ell \in \{n-2, n-1, n\}$. Recall that, if $G \in \mathcal{A}_n$, where $\mathcal{A}_n$ is the set defined in (1), then G is n-path-Hamiltonian. It turns out that the converse of this statement also holds, that is, the graphs in $\mathcal{A}_n$ are the only graphs of order n that are n-path-Hamiltonian. In fact, a Hamiltonian graph G of order n belongs to $\mathcal{A}_n$ if and only if G is ℓ-path-Hamiltonian for $n-2 \le \ell \le n$, as we will show.

We first show that if G is an n-path-Hamiltonian graph of order n, then $G \in \mathcal{A}_n$. Although this fact can be derived from the proof of Theorem 1.1 provided in [3], we present an independent proof here. The following lemma will be useful.

Lemma 2.1 *Let G be an n-path-Hamiltonian graph of order n with a Hamiltonian cycle $C = (v_1, v_2, \ldots, v_n, v_1)$.*

(a) *If $v_1 v_{a+1} \in E(G)$ for some integer a $(2 \leq a \leq n-2)$, then $v_i v_{a+i} \in E(G)$ for $1 \leq i \leq n$, where the subscripts are expressed modulo n. (In other words, G is a circulant.)*

(b) *If $G \neq C$, then $v_1 v_4 \in E(G)$.*

(c) *If $v_1 v_{a+1} \in E(G)$ for some integer a $(2 \leq a \leq n-4)$, then $v_1 v_{a+3} \in E(G)$.*

Proof. For (a), suppose that $v_1 v_{a+1} \in E(G)$. Then there is an n-path $(v_2, v_3, \ldots, v_{a+1}, v_1, v_n, v_{n-1}, \ldots, v_{a+2})$ in G connecting v_2 and v_{a+2}. Since G is n-path-Hamiltonian, $v_2 v_{a+2} \in E(G)$. By the same argument, one can show that $v_{i+1} v_{a+i+1} \in E(G)$ whenever $v_i v_{a+i} \in E(G)$ for $1 \leq i \leq n-1$.

Next we verify (b). If $G \neq C$, then $v_1 v_{a+1} \in E(G)$ for some a, where $2 \leq a \leq n-2$. By (a), we may assume that $v_2 v_{a+2}, v_3 v_{a+3} \in E(G)$. Then G contains an n-path $(v_1, v_n, v_{n-1}, \ldots, v_{a+3}, v_3, v_2, v_{a+2}, v_{a+1}, \ldots, v_4)$, which implies that $v_1 v_4 \in E(G)$.

For (c), suppose that $v_1 v_{a+1} \in E(G)$ for some integer a $(2 \leq a \leq n-4)$. By (a), it follows that $v_2 v_{a+2}, v_a v_n \in E(G)$. One can then construct an n-path

$$\begin{array}{ll} (v_1, v_3, v_4, v_2, v_n, v_{n-1}, \ldots, v_5) & \text{if } a = 2 \\ (v_1, v_{a+1}, v_{a+2}, v_2, v_3, \ldots, v_a, v_n, v_{n-1}, \ldots, v_{a+3}) & \text{otherwise} \end{array}$$

connecting v_1 and v_{a+3}. Consequently, $v_1 v_{a+3} \in E(G)$.

By Lemma 2.1, if G is an n-path-Hamiltonian graph of order n containing a Hamiltonian cycle $C = (v_1, v_2, \ldots, v_n, v_1)$ as a proper subgraph, then $v_i v_j \in E(G)$ for each pair i, j of integers where $1 \leq i, j \leq n$ and $i \not\equiv j \pmod 2$. In particular, $v_1 v_i \in E(G)$ for each even integer i $(2 \leq i \leq n)$. Thus, if n is even, then G contains $K_{n/2,n/2}$ as a spanning subgraph. Furthermore, G is complete if and only if $v_1 v_3 \in E(G)$ or $v_1 v_{n-1} \in E(G)$. We are now prepared to establish the following result.

Theorem 2.2 *Let G be a graph of order $n \geq 3$. If G is n-path-Hamiltonian, then $G \in \mathcal{A}_n$.*

Proof. Let G be an n-path-Hamiltonian graph of order n. We may also assume that G is neither C_n nor $K_{n/2,n/2}$. It remains to show that $G = K_n$. Let $C = (v_1, v_2, \ldots, v_n, v_1)$ be a Hamiltonian cycle in G. If n is odd, then

$v_1v_{n-1} \in E(G)$ and so G is complete. Thus, assume next that n is even and G contains $K_{n/2,n/2}$ as a proper spanning subgraph. Then v_1v_{a+1} for some even integer a with $2 \le a \le n-2$, which in turn implies that $v_1v_i \in E(G)$ for $a \le i \le n$. In particular, $v_1v_{n-1} \in E(G)$ and so G is complete.

It was shown in [2] that there is no Hamiltonian graph of order n whose Hamiltonian cycle extension number equals either $n-2$ or $n-1$. In fact, if G is $(n-2)$-path-Hamiltonian graph of order $n \ge 3$, then consider an $(n-1)$-path $P = (v_1, v_2, \ldots, v_{n-1})$ in G and the vertex $v_n \in V(G)\backslash V(P)$. The $(n-2)$-path $P - v_1$ must lie on a Hamiltonian cycle in G and so $v_1v_n \in E(G)$. One can similarly show that $v_{n-1}v_n \in E(G)$ by considering the path $P - v_{n-1}$. Hence, P can be extended to a Hamiltonian cycle in G and so G is $(n-1)$-path-Hamiltonian. It is also straightforward to show that G is n-path-Hamiltonian if G is $(n-1)$-path-Hamiltonian. Consequently, we have the following.

Observation 2.3 [2] *If G is a graph of order $n \ge 3$ that is either $(n-2)$-path-Hamiltonian or $(n-1)$-path-Hamiltonian, then G is n-path-Hamiltonian.*

As a consequence of Theorems 1.1 and 2.2 with Observation 2.3, we obtain the following characterization of all Hamiltonian graphs of order n that are ℓ-path-Hamiltonian for $n-2 \le \ell \le n$.

Theorem 2.4 *A graph G of order $n \ge 3$ is ℓ-path-Hamiltonian for $n-2 \le \ell \le n$ if and only if $G \in \mathcal{A}_n$.*

Corollary 2.5 *For a graph G of order $n \ge 3$, the following are equivalent.*

(a) *The graph G equals C_n or K_n or $K_{n/2,n/2}$ (when n is even).*

(b) *The graph G is n-path-Hamiltonian.*

(c) *The graph G is $(n-1)$-path-Hamiltonian.*

(d) *The graph G is $(n-2)$-path-Hamiltonian.*

(e) *The graph G is i-path-Hamiltonian for $1 \le i \le n$, that is,* $\text{hce}(G) = n$.

3 The Second Characterization

In this section, we characterize all graphs of order $n \ge 4$ that are $(n-3)$-path-Hamiltonian. For each integer $n \ge 4$, let $\mathcal{B}_n$ be the set of graphs G of order n that are $(n-3)$-path-Hamiltonian. It then follows by Corollary 2.5 that $\mathcal{A}_n \subseteq \mathcal{B}_n$. Therefore, we basically need to determine the set $\mathcal{B}_n\backslash\mathcal{A}_n$.

Clearly, $\mathcal{B}_4 \backslash \mathcal{A}_4 = \{K_{2,1,1}\}$. Now for $n \geq 5$, what can we say? First, we already have the following.

Proposition 3.1 [2] *If G is a graph of order $n \geq 4$ and $\delta(G) = n - 2$, then* $\text{hce}(G) = n - 3$.

For this reason, we should investigate graphs G of order n for which $2 \leq \delta(G) \leq n - 3$. Suppose that $G \in \mathcal{B}_5 \backslash \mathcal{A}_5$. If $v \in V(G)$ and $\deg v = 2$, say $N(v) = \{x, y\}$, then $xy \notin E(G)$ since otherwise (x, y) is a 2-path that lies on no Hamiltonian cycle in G. Consequently, $\mathcal{B}_5 \backslash \mathcal{A}_5 = \{P_3 + P_2, K_{2,2,1}, K_{2,1,1,1}\}$. Here, the graph $P_3 + P_2$ is the *union*, not the join, of P_3 and P_2.

To continue our search for $(n - 3)$-path-Hamiltonian graphs of order n in general, let us first state an observation, which is elementary but useful.

Observation 3.2 [2] *If G is a Hamiltonian graph with $\delta(G) = 2 < \Delta(G)$, then* $\text{hce}(G) \leq 2$.

We next state a few of the best-known sufficient conditions for a graph to be traceable or Hamiltonian or Hamitonian-connected in terms of its order. In the following theorem, (b) and (c) are both due to Ore [5, 6] while (a) is an immediate consequence of (b).

Theorem 3.3 *For a graph G of order $n \geq 3$, let $\sigma = \min\{\deg u + \deg v : uv \notin E(G)\}$.* (a) *If $\sigma \geq n - 1$, then G is traceable.* (b) *If $\sigma \geq n$, then G is Hamiltonian.* (c) *If $\sigma \geq n + 1$, then G is Hamiltonian-connected.*

With the aid of Theorem 3.3, we obtain another result concerning graphs in $\mathcal{B}_n \backslash \mathcal{A}_n$ and their minimum degree.

Observation 3.4 *Let $n \geq 6$. If $G \in \mathcal{B}_n \backslash \mathcal{A}_n$, then either $\delta(G) = n - 2$ or $3 \leq \delta(G) \leq (n + 1)/2$.*

Proof. By Observation 3.2, let us assume that $n \geq 8$ and $n/2 + 1 \leq \delta(G) \leq n - 3$. We show that G is not $(n - 3)$-path-Hamiltonian. Since $\delta(G) \leq n - 3$, there is a 3-set $S \subseteq V(G)$ such that $G[S]$ is disconnected. Now let $G' = G - S$. Then $\delta(G') \geq \delta(G) - 3 \geq n/2 + 1 - 3 = (n' - 1)/2$, where $n' = n - 3$ is the order of G'. This implies that G' contains an $(n - 3)$-path which cannot be extended to a Hamiltonian cycle in G. The result now follows.

If G is a graph of order $n \geq 5$ containing an $(n - 3)$-path P, then the subgraph induced by the three vertices not belonging to P contains P_3 or hce(G) $< n-3$. In general, if G is not $(n-3)$-path-Hamiltonian, then G contains an $(n-3)$-path that cannot be a subpath of a Hamiltonian cycle in G. The following observation categorizes such paths into five types.

Observation 3.5 *A Hamiltonian graph G of order $n \geq 5$ is not $(n-3)$-path Hamilotnian if and only if there is an $(n-3)$-path P in G that is of one of the five types described in* Figure 1, *where P is shown in bold and edges not belonging to G are shown as dashed line segments.*

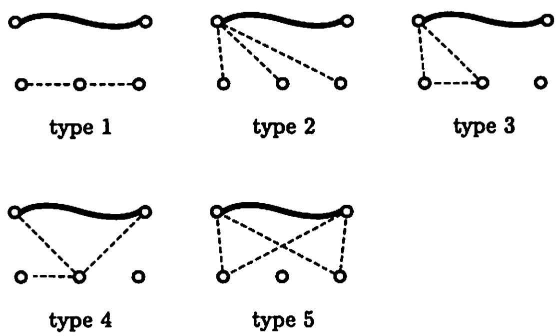

Figure 1: The five types of $(n-3)$-paths in Observation 3.5

For example, if G is a Hamiltonian graph of order $n \geq 5$ containing a vertex v such that $\deg v \leq n-3$ and $G-v$ contains a $u-w$ Hamiltonian path (u, P, w), where neither u nor w is adjacent to v, then P is an $(n-3)$-path of type 1, implying that G is not $(n-3)$-path-Hamiltonian. In particular, if $G-v$ is Hamiltonian-connected, then G is not $(n-3)$-path-Hamiltonian.

As another example, let G be a Hamiltonian graph of order $n \geq 7$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$, where $\deg v_1 \leq \deg v_2 \leq \cdots \leq \deg v_n$. Let us define $\delta_i(G) = \deg v_i$ for $1 \leq i \leq n$. Thus, $\delta_1(G) = \delta(G)$ and $\delta_n(G) = \Delta(G)$. If $\delta_1(G) \leq n-4$ and $\delta_2(G) \geq n/2+2$, then G is not $(n-3)$-path-Hamiltonian. To see this, let S be a 3-set such that $S \cap N[v_1] = \emptyset$. If H is the graph of order $n-4$ (≥ 3) obtained from G by deleting the four vertices in $S \cup \{v_1\}$, then $\delta(H) \geq \delta_2(G) - 4 \geq (n-4)/2$ and so H is Hamiltonian. This in turn implies that G contains an $(n-3)$-path whose vertex set equals $V(G) \backslash S$ with v_1 as one of the two end-vertices. Note that this path is of type 2. The following is a generalization of this fact.

Observation 3.6 *Let n and ℓ be integers satisfying $n \geq 7$ and $3 \leq \ell \leq n-3$. If G is a Hamiltonian graph of order n with $\delta(G) \leq \ell - 1$ and*

$\delta_2(G) \geq n - (\ell - 1)/2$, *then G is not ℓ-path-Hamiltonian.*

For $n = 6, 7$, the set $\mathcal{B}_n \backslash \mathcal{A}_n$ can be determined fairly easily. Let us first consider the set $\mathcal{B}_6$. By Proposition 3.1 and Observation 3.2, we may assume that $G \in \mathcal{B}_6$ and $\delta(G) = 3$. Then $G \neq K_{3,3}$. One can then verify that if G is either $\overline{C_6}$ ($= C_3 \square P_2$, the Cartesian product of C_3 and P_2) or $\overline{2P_3}$, then G is 3-path-Hamiltonian. If $G = \overline{C_3 + P_3}$, then G contains a 3-path of type 3. Otherwise, G contains a 3-path of type 1. Thus, $\mathcal{B}_6 \backslash \mathcal{A}_6 = \{\overline{C_6}, \overline{2P_3}, K_{2,2,2}, K_{2,2,1,1}, K_{2,1,1,1,1}\}$.

Similarly, to determine the set $\mathcal{B}_7$, we may consider those graphs G with $3 \leq \delta(G) \leq 4$. We will show that if $G \in \mathcal{B}_7$, then $\delta(G) \neq 3$. We first state another lemma.

Lemma 3.7 *Let G be an $(n-3)$-path-Hamiltonian graph of order $n \geq 5$ with a Hamiltonian cycle $(v_1, v_2, \ldots, v_n, v_1)$.*

(a) *If neither v_1v_a nor v_1v_{a+1} is an edge in G for some $a \in \{3, 4, \ldots, n-2\}$, then none of the edges v_2v_{a+2}, v_2v_n, $v_{a-1}v_{a+2}$, $v_{a-1}v_n$ is contained in G. (Thus, if $v_1v_a \in E(G)$ for some $a \in \{3, 4, \ldots, n-2\}$, then at least one of $v_{a-1}v_{n-1}$ and $v_{a-1}v_n$ is an edge in G.)*

(b) *For $n \geq 7$, if v_1v_3 is an edge in G, then at least one of v_1v_{n-3} and v_1v_{n-2} is an edge in G. (Thus,* $\deg v_1 \geq 4$.*)*

Proof. For (a), the statement must hold in order to avoid type-1 paths in G. For (b), assume, to the contrary, that v_1v_3 is an edge in G while neither v_{n-3} nor v_{n-2} is. Then by (a), it follows that $\{v_2v_{n-1}, v_2v_n\} \cap E(G) = \emptyset$. However then, $(v_1, v_3, v_4, v_5, \ldots, v_{n-2})$ is an $(n-3)$-path of type 1 in G. This contradicts the fact that G is $(n-3)$-path-Hamiltonian. ∎

We are prepared to show that a graph G of order 7 cannot be 4-path-Hamiltonian if G contains a vertex whose degree is less than 4.

Lemma 3.8 *Let G be a graph of order 7. If G is 4-path-Hamiltonian, then $\delta(G) \neq 3$.*

Proof. Let G be a 4-path-Hamiltonian graph with a Hamiltonian cycle $(v_1, v_2, \ldots, v_7, v_1)$ and assume, to the contrary, that $\deg v_1 = 3$. We show that G must contain a path of type 1 or type 3. By Lemma 3.7(b), we may assume that $N(v_1) = \{v_2, v_4, v_7\}$. Then $N(v_7) \cap \{v_2, v_4\} = \emptyset$ to avoid 4-paths of type 1. In addition, at most one of v_2v_4 and v_3v_7 belongs to $E(G)$. Since $v_4v_7 \notin E(G)$ while $\deg v_7 \geq 3$, it follows by Lemma 3.7(b) again that

$v_3v_7 \in E(G)$. Consequently, $v_2v_4 \notin E(G)$. Then $v_2v_6 \in E(G)$ since the path (v_5, v_4, v_3, v_7) cannot be of type 1. However then, either (v_2, v_6, v_5, v_3) or (v_1, v_2, v_6, v_7) is a 4-path of type 3, depending on whether v_3 and v_5 are adjacent or not. Thus, if $\delta(G) = 3$, then G cannot be 4-path-Hamiltonian.

As a consequence, if G is a 4-path-Hamiltonian graph of order 7, then $\delta(G) \in \{4, 5\}$. In fact, there is only one 4-path-Hamiltonian graph of order 7 and minimum degree 4, as we verify next.

Proposition 3.9 *Let G be a graph of order 7. Then $G \in \mathcal{B}_7$ if and only if either $\delta(G) = 5$ or $G = \overline{C_4 + C_3}$.*

Proof. It is straightforward to verify that $\overline{C_4 + C_3}$ is 4-path-Hamiltonian. By Lemma 3.8, it suffices to prove that if $\delta(G) = 4$ and $\overline{G} \neq C_4 + C_3$, then G is not 4-path-Hamiltonian. Let G be a graph of order 7 and $\delta(G) = 4$. If $\overline{G}$ is neither $C_4 + C_3$ nor $2C_3 + P_1$, then G contains a 4-path of type 1. We also see that if $\overline{G} = 2C_3 + P_1$, then G contains a 4-path of type 3. The result now follows. ∎

Hence,

$$\begin{aligned}
\mathcal{B}_4 \backslash \mathcal{A}_4 &= \{K_{2,1,1}\} \\
\mathcal{B}_5 \backslash \mathcal{A}_5 &= \{\overline{P_3 + P_2}, K_{2,2,1}, K_{2,1,1,1}\} \\
\mathcal{B}_6 \backslash \mathcal{A}_6 &= \{\overline{C_6}, \overline{2P_3}, K_{2,2,2}, K_{2,2,1,1}, K_{2,1,1,1,1}\} \\
\mathcal{B}_7 \backslash \mathcal{A}_7 &= \{\overline{C_4 + C_3}, K_{2,2,2,1}, K_{2,2,1,1,1}, K_{2,1,1,1,1,1}\}.
\end{aligned}$$

These graphs are not only $(n-3)$-path-Hamiltonian but i-path-Hamiltonian for each positive integer i less than or equal to $n - 3$. Hence, $\text{hce}(G) = n - 3$ for each of the graph G listed above. Note also that for $n = 6, 7$, if $G \in \mathcal{B}_n \backslash \mathcal{A}_n$, then $\delta(G)$ equals either $\lfloor (n+1)/2 \rfloor$ or $n - 2$. This does not hold for $n \geq 8$, which we discuss in the following subsections.

3.1 Graphs of Order n and Minimum Degree $\lfloor (n+1)/2 \rfloor$

In this subsection, we show that if G is an $(n-3)$-path-Hamiltonian graph of order $n \geq 8$, then $\delta(G) \neq \lfloor (n+1)/2 \rfloor$.

The *closure* G^* of a graph G of order n is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n (in the resulting graph at each stage) until no such pair remains. The following is a consequence of a well-known theorem in 1976 by Bondy and Chvátal [1].

Theorem 3.10 *A graph is Hamiltonian if and only if its closure is Hamiltonian.*

For two disjoint subsets V and V' of $V(G)$, let $E[V, V']$ denote the set of edges joining a vertex in V and a vertex in V'. Also, for convenience, let K_0 be the "null graph" (the "graph" of order 0 and size 0).

Theorem 3.11 *Let G be a graph of order $n \geq 8$ with $\delta(G) = \lfloor (n+1)/2 \rfloor$. Then G is $(n-3)$-path-Hamiltonian if and only if n is even and $G = K_{n/2,n/2}$.*

Proof. Let G be a graph of order $n \geq 8$ with $\delta(G) = \delta = \lfloor (n+1)/2 \rfloor$ and assume that either $G \neq K_{\delta,\delta}$ or n is odd. Our goal is to show that G is not $(n-3)$-path-Hamiltonian by finding an $(n-3)$-path that is of one of the five types in Observation 3.5.

Select a vertex x_1 with $\deg x_1 = \delta$ and, among the vertices adjacent to x_1, let x_2 be one whose degree is the minimum. If $\deg x_2 \geq \delta + 1$, then $\delta(G - x_1) \geq \delta \geq n/2$ and so $G - x_1$ is Hamiltonian-connected. Consequently, the graph G contains an $(n-3)$-path of type 1. Thus, for the rest of the proof, we assume that $\deg x_2 = \delta$.

Let H be the graph of order $n-2$ obtained from G by deleting x_1 and x_2. For each integer $i = \{0, 1, 2\}$, let $V_i = \{v \in V(H) : |N_G(v) \cap \{x_1, x_2\}| = i\}$. Thus, $\{V_0, V_1, V_2\}$ is a partition of $V(H)$ and $\deg_H v = \deg_G v - i \geq \delta - i$ if $v \in V_i$. If n is even, then $0 \leq |V_0| = |V_2| \leq \delta - 1$. If n is odd, then $V_2 \neq \emptyset$ and $0 \leq |V_0| = |V_2| - 1 \leq \delta - 2$. In either case, if V_0 is nonempty and H is Hamiltonian, say $v \in V_0$ and C is a Hamiltonian cycle in H, then $P = C - v$ is a path of type 1 in G. Thus, let us suppose that H is not Hamiltonian or $V_0 = \emptyset$.

Case 1. H is not Hamiltonian. Then the closure H^* of H is not Hamiltonian, neither. Suppose that u and w are not adjacent in H^*. Then $\deg_H u + \deg_H w \leq n - 3 = 2\delta - 3$. Recall that $\deg_H v \geq \delta - i$ for each $v \in V_i$. Thus, if n is even, then we may assume that either (i) $u \in V_1$ and $w \in V_2$ or (ii) $u, w \in V_2$. Similarly, if n is odd, then $u, w \in V_2$.

Subcase 1.1. n is even. Since H^* is not Hamiltonian it must be that, in H, (i) none of V_0, V_1, and V_2 is nonemepty, (ii) V_2 is independent, and (iii) $E[V_1, V_2]$ is empty. Thus, the degree of each vertex belonging to $V_1 \cup V_2$ is exactly δ in G. Since $\delta = \deg_G v \leq |V_0| + 2$ for each $v \in V_2$, it follows that $|V_0| = |V_2| = \delta - 2$ and $|V_1| = 2$. Furthermore, every vertex in V_0 is adjacent to every vertex in $V_1 \cup V_2$. Thus, if we write $V_0 = \{u_1, u_2, \ldots, u_{\delta-2}\}$, $V_1 = \{v_1, v_2\}$, and $V_2 = \{w_1, w_2, \ldots, w_{\delta-2}\}$, where

$x_iv_i \in E(G)$ for $i = 1, 2$, then $(v_1, u_1, w_1, u_2, w_2, \ldots, u_{\delta-3}, w_{\delta-3}, x_2, w_{\delta-2})$ is a type-1 path in G since x_1 is adjacent to neither v_2 nor $u_{\delta-2}$.

Subcase 1.2. n is odd. Then in H, it must be that V_1 is empty and V_2 is independent. Thus, $|V_2| = |V_0| + 1 = \delta - 1 < \deg_G v$ for every vertex v in V_2 and so V_0 is not independent. Write $V_0 = \{u_1, u_2, \ldots, u_{\delta-2}\}$ and $V_2 = \{w_1, w_2, \ldots, w_{\delta-1}\}$. Also, without loss of generality, suppose that $u_1u_2 \in E(G)$. Then $(w_1, x_1, x_2, w_2, u_1, u_2, Q)$, where

$$Q = \begin{cases} K_0 & \text{if } n = 9 \\ (w_3, u_3, w_4, u_4, \ldots, w_{\delta-3}, u_{\delta-3}) & \text{if } n \geq 11, \end{cases}$$

is a type-3 path.

Case 2. H is Hamiltonian and $V_0 = \emptyset$.

Subcase 2.1. n is even. Then $|V_1| = n - 2$ and $V_2 = \emptyset$. Among the δ vertices adjacent to x_1, some have degree δ in G. Let N' be the set of such vertices. Obviously, $x_2 \in N'$. Let $N'' = N_G(x_1)\backslash N'$ and $U = V(G)\backslash N_G[x_1]$. If there exists a vertex $x_3 \in N'$ such that x_1 and x_3 belong to a common triangle, then let H' be the graph obtained from G by deleting x_1 and x_3. Then, by considering the closure of H', one can verify, as done earlier, that G contains an $(n-3)$-path of type 1 regardless of whether or not H' is Hamiltonian. Thus, suppose that N' is independent and either N'' is empty or $E[N', N'']$ is empty.

If $N'' = \emptyset$, then U is not independent since $G \neq K_{\delta,\delta}$. Let $U = \{u_1, u_2, \ldots, u_{\delta-1}\}$ and, without loss of generality, suppose that $u_1u_2 \in E(G)$. Also, let $N' = \{w_1, w_2, \ldots, w_\delta\}$. Then $(w_1, x_1, w_2, u_1, u_2, Q)$, where

$$Q = \begin{cases} K_0 & \text{if } n = 8 \\ (w_3, u_3, w_4, u_4, \ldots, w_{\delta-2}, u_{\delta-2}) & \text{if } n \geq 10, \end{cases}$$

is a type-3 path.

If N'' is nonempty, then let $C = (v_1, v_2, \ldots, v_{n-1}, v_1)$ be a Hamiltonian cycle in H. If two vertices in U are adjacent in C, say $v_{n-2}, v_{n-1} \in U$, then the path $(v_1, v_2, \ldots, v_{n-3})$ is of type 1. Otherwise, we may assume that $U = \{v_1, v_3, \ldots, v_{n-3}\}$ and $v_{n-2}, v_{n-1} \in N''$. If $v_1v_3 \notin E(G)$, then $(x_1, v_4, v_5, \ldots, v_{n-1})$ is an $(n-3)$-path of type 3. Similarly, G contains an $(n-3)$-path of type 3 if $v_3v_5 \notin E(G)$. Also, if $v_2v_4 \in E(G)$, then $(v_2, v_4, v_5, \ldots, v_{n-1})$ is of type 1. Hence, suppose that v_3 is adjacent to both v_1 and v_5 while v_2 is not adjacent to v_4. Then

$$P = \begin{cases} (v_4, v_5, \ldots, v_{n-1}, v_2) & \text{if } v_2v_{n-1} \in E(G) \\ (v_1, v_3, v_5, v_6, \ldots, v_{n-2}, x_1) & \text{otherwise} \end{cases}$$

is of type 1.

Subcase 2.2. n is odd. Then $|V_1| = n-3$ and $|V_2| = 1$. Let $V_1' = V_1 \cap N(x_1)$ and $V_1'' = V_1 \cap N(x_2)$. Also, let $C = (v_1, v_2, \ldots, v_{n-2}, v_1)$ be a Hamiltonian cycle in H. Without loss of generality, assume that $v_{n-2} \in V_2$. First, suppose that $v_1 \in V_1'$.

If there exists an integer a $(1 \leq a \leq n-4)$ such that both v_a and v_{a+1} belong to V_1', then $(v_{a+2}, v_{a+3}, \ldots, v_{n-2}, x_1, Q)$, where

$$Q = \begin{cases} K_0 & \text{if } a = 1 \\ (v_1, v_2, \ldots, v_{a-1}) & \text{if } 2 \leq a \leq n-4, \end{cases}$$

is a type-1 path.

Thus, let us next suppose that V_1' is independent with respect to C. If V_1'' is also independent with respect to C, then we may assume that $V_1' = \{v_1, v_3, \ldots, v_{n-4}\}$ and $V_1'' = \{v_2, v_4, \ldots, v_{n-3}\}$. Then consider the $(n-3)$-path P given by

$$P = \begin{cases} (x_2, v_{n-2}, v_1, v_3, v_4, \ldots, v_{n-4}) & \text{if } v_1v_3 \in E(G) \\ (v_3, v_4, \ldots, v_{n-2}, x_1) & \text{otherwise} \end{cases}$$

and note that P is of either type 1 or type 3.

Finally, suppose that, with respect to C, the set V_1' is independent but V_1'' is not. Let a be the smallest positive integer such that $\{v_a, v_{a+1}\} \subseteq V_1''$. Due to the symmetry of the graph, we may assume that $1 \leq a \leq (n-3)/2$. If $a = 1$, then the $(n-3)$-path $(v_3, v_4, \ldots, v_{n-2}, x_2)$ is of type 1. If $2 \leq a \leq (n-3)/2$, then $v_{a-1} \in V_1'$. If $v_{a-1}v_{a+2} \in E(G)$ or $v_{a+2} \in V_1''$, then there exists an $(n-3)$-path P whose vertex set equals $V(G) \backslash \{x_1, v_a, v_{a+1}\}$. Otherwise, $v_{a-1}v_{a+2} \notin E(G)$ and $v_{a+2} \in V_1'$. Then $v_{a+3} \in V_1''$. In this case, the path P' given by

$$P' = \begin{cases} (v_{a-1}, v_a, v_{a+1}, v_{a+3}, \ldots, v_{n-2}, v_1, v_2, \ldots, v_{a-2}) & \text{if } v_{a+1}v_{a+3} \in E(G) \\ (v_{a+4}, v_{a+5}, \ldots, v_{n-2}, v_1, v_2, \ldots, v_{a-1}, v_a, x_2, v_{a+1}) & \text{otherwise} \end{cases}$$

is of type 3.

As desired, G is not $(n-3)$-path-Hamiltonian.

By Corollary 2.5 and Theorem 3.11 with Proposition 3.1, therefore, if G is an $(n-3)$-path-Hamiltonian graph of order $n \geq 8$, then (i) $G \in \mathcal{A}_n$ or (ii) $\delta(G) = n-2$ or (iii) $3 \leq \delta(G) \leq (n-1)/2$. We next show that (iii) never occurs.

3.2 Graphs of Order n and Minimum Degree Less Than $\lfloor (n+1)/2 \rfloor$

Recall that, if G is a graph of order $n \geq 8$ belonging to $\mathcal{B}_n \backslash \mathcal{A}_n$, then $\delta(G) = n-2$ or $3 \leq \delta(G) \leq (n-1)/2$. In this subsection, we pay attention to graphs G of order $n \geq 8$ with $3 \leq \delta(G) \leq (n-1)/2$. More precisely, we will show that such G cannot be $(n-3)$-path-Hamiltonian. Let us begin with graphs with minimum degree 3.

Proposition 3.12 *If G is a graph of order $n \geq 8$ and $\delta(G) = 3$, then G is not $(n-3)$-path-Hamiltonian.*

Proof. We prove that G is not $(n-3)$-path-Hamiltonian by showing that G must contain a path of type 1 or type 3. Let $C = (v_1, v_2, \ldots, v_n, v_1)$ be a Hamiltonian cycle in G and, without loss of generality, suppose that $\deg v_n = \delta(G) = 3$. Let $N = N(v_n) = \{v_1, v_a, v_{n-1}\}$ for some $a \in \{2, 3, \ldots, n-2\}$. By the symmetry of the graph, we may assume that $a \leq n/2$. Thus, $2 \leq a \leq n-4$ since $n \geq 8$ (and $a = n-4$ only if $n = 8$). We consider the following two cases.

Case 1. $C[N] = P_1 + P_2$. Then $a = 2$. We show that G contains a type-1 path. Assume, to the contrary, that this is not the case. Let $\{x, y\} = \{v_{n-2}, v_{n-1}\}$. Since $(x, y, v_1, v_2, \ldots, v_{n-5})$ cannot be a type-1 path, v_1 is adjacent to neither v_{n-2} nor v_{n-1}. However then, another type-1 path $(v_n, v_2, v_3, \ldots, v_{n-3})$ results. Thus, this case is impossible. (Note that this is also immediate by Lemma 3.7(b).)

Case 2. $C[N] = 3P_1$. Then $3 \leq a \leq n-4$. We show that G contains a path of type 1 or type 3. Again, assume that this is not the case. Then we first claim that N is an independent set in G. By the fact that neither $(v_{n-1}, v_1, v_2, \ldots, v_{n-4})$ nor $(v_1, v_2, \ldots, v_a, v_{n-1}, v_{n-2}, \ldots, v_{a+3})$ is a type-1 path, it follows that $v_1v_{n-1}, v_av_{n-1} \notin E(G)$. Similarly, if $a \geq 4$, then $v_1v_a \notin E(G)$ as $(v_{n-1}, v_{n-2}, \ldots, v_a, v_1, v_2, \ldots, v_{a-3})$ is not a type-1 path. If $a = 3$, then by the fact that $v_1v_{n-1} \notin E(G)$ and the existence of the path $(v_n, v_3, v_4, \ldots, v_{n-2})$, which is not of type 1, we have $v_2v_{n-1} \in E(G)$. Thus, $v_1v_a = v_1v_3 \notin E(G)$ in order for avoiding the path $(v_{n-1}, v_2, v_1, v_3, v_4, \ldots, v_{n-4})$ in G, which is of type 1. As claimed, therefore, no two vertices in $N = \{v_1, v_a, v_{n-1}\}$ are adjacent in G.

Since neither $(v_{n-2}, v_{n-3}, \ldots, v_{a+1}, v_2, v_3, \ldots, v_a)$ nor $(v_1, v_2, \ldots, v_{a-1}, v_{n-2}, v_{n-3}, \ldots, v_{a+1})$ is of type 3, it follows that $v_2v_{a+1}, v_{a-1}v_{n-2} \notin E(G)$. This in turn implies that $v_1v_{a+1}, v_{a-1}v_{n-1} \in E(G)$ since neither of the paths $(v_3, v_4, \ldots, v_a, v_n, v_{n-1}, v_{n-2}, \ldots, v_{a+2})$ nor $(v_{n-3}, v_{n-4}, \ldots, v_a, v_n, v_1,$

$v_2, \ldots, v_{a-2})$ is of type 1. However then, a type-1 path $(v_{a+4}, v_{a+5}, \ldots, v_{n-1}, v_{a-1}, v_a, v_{a+1}, v_1, v_2, \ldots, v_{a-2})$ is produced. This contradicts the initial assumption.

Thus, a graph G of order $n \geq 8$ and minimum degree 3 must contain a path that is of type 1 or type 3. Hence, G cannot be $(n-3)$-path-Hamiltonian. ∎

By Propositions 3.1 and 3.12 with Observation 3.4, there are exactly four graphs of order 8 that are 5-path-Hamiltonian; namely

$$\begin{aligned}\mathcal{B}_8 \backslash \mathcal{A}_8 &= \{G : \delta(G) = |V(G)| - 2 = 6\} \\ &= \{K_{2,2,2,2}, K_{2,2,2,1,1}, K_{2,2,1,1,1,1}, K_{2,1,1,1,1,1,1}\}.\end{aligned}$$

Finally, let us consider graphs G of order $n \geq 9$ with $4 \leq \delta(G) \leq (n-1)/2$. For a path P, let $\overleftarrow{P}$ denote the reverse of P.

Theorem 3.13 *If G is a Hamiltonian graph of order $n \geq 9$ and $4 \leq \delta(G) \leq (n-1)/2$, then G is not $(n-3)$-path-Hamiltonian.*

Proof. We prove that G is not $(n-3)$-path-Hamiltonian by showing that G must contain a path of type 1–5 described in Observation 3.5. Assume, to the contrary, this is not the case. Let $C = (v_1, v_2, \ldots, v_n, v_1)$ be a Hamiltonian cycle in G and, without loss of generality, suppose that $\deg v_n = \delta(G) = \delta$. Let $A = N(v_n) = \{v_{a_1}, v_{a_2}, \ldots, v_{a_\delta}\}$ and $B = V(G) \backslash N[v_n] = \{v_{b_1}, v_{b_2}, \ldots, v_{b_{n-\delta-1}}\}$, where $1 = a_1 < a_2 < \cdots < a_\delta = n-1$ and $2 \leq b_1 < b_2 < \cdots < b_{n-\delta-1} \leq n-2$. Let $I_A = \{a_1, a_2, \ldots, a_\delta\}$ and $I_B = \{b_1, b_2, \ldots, b_{n-\delta-1}\}$. Also, let β be the largest integer such that β and $\beta - 1$ both belong to I_B. Since $\delta \leq (n-1)/2$, such β always exists and $3 \leq \beta \leq n-2$. Considering the paths

$$\begin{aligned}&(v_{n-1}, v_{n-2}, \ldots, v_{\beta+1}, v_1, v_2, \ldots, v_{\beta-1}), \\ &(v_{\beta+1}, v_{\beta+2}, \ldots, v_{n-1}, v_1, v_2, \ldots, v_{\beta-2}), \\ &(v_1, v_2, \ldots, v_{\beta-2}, v_{\beta+1}, v_{\beta+2}, \ldots, v_{n-1}), \\ &(v_1, v_2, \ldots, v_{\beta-2}, v_{n-1}, v_{n-2}, \ldots, v_{\beta+1}),\end{aligned}$$

none of which is of type 1 in G, we have

- $\{v_1v_{\beta+1}, v_1v_{n-1}, v_{\beta-2}v_{\beta+1}, v_{\beta-2}v_{n-1}, v_{\beta-1}v_n, v_\beta v_n\} \cap E(G) = \emptyset$, (2)
- $v_{\beta+1}v_n \in E(G)$.

We now consider two cases according to whether or not A contains two vertices that are adjacent in C. Define Q_1 and Q_2 by

$$Q_1 = \begin{cases} K_0 & \text{if } \beta = 3 \\ (v_2, v_3, \ldots, v_{\beta-2}) & \text{if } \beta \geq 4 \end{cases}$$
$$Q_2 = \begin{cases} K_0 & \text{if } \beta \geq n-4 \\ (v_{\beta+1}, v_{\beta+2}, \ldots, v_{n-4}) & \text{if } \beta \leq n-5. \end{cases}$$

Thus, Q_1 and Q_2 are subpaths of C when $\beta \geq 4$ and $\beta \leq n-5$, respectively.

Case 1. *$C[A]$ is not empty.* Then let α be the smallest integer such that both α and $\alpha+1$ belong to I_A. Thus, $\alpha \neq 2$. Also, by the symmetry of the graph, we may assume that $\alpha \leq (n-1)/2$. Since $n \geq 9$, it follows that $\alpha \leq n-5$ (and $\alpha = n-5$ if and only if $n = 9$). Define Q_3 and Q_4 by

$$Q_3 = \begin{cases} K_0 & \text{if } \alpha = 1 \\ (v_2, v_3, \ldots, v_\alpha) & \text{if } \alpha \geq 3 \end{cases}$$
$$Q_4 = \begin{cases} K_0 & \text{if } \alpha = n-5\ (=4) \\ (v_{\alpha+1}, v_{\alpha+2}, \ldots, v_{n-5}) & \text{if } \alpha \leq n-6. \end{cases}$$

Thus, Q_3 and Q_4 are subpaths of C when $\alpha \geq 3$ and $\alpha \leq n-6$, respectively. Then $(Q_3, v_n, Q_4, v_{n-4}, v_{n-3})$ cannot be a type-1 path and so $v_1v_{n-2} \in E(G)$. By (2), therefore, $\beta \neq n-3$. If there exists an integer b such that $\{b-2, b-1, b\} \subseteq I_B$, then $(v_b, v_{b+1}, \ldots, v_{n-2}, v_1, v_2, \ldots, v_{b-2})$ is a type-4 path, which cannot occur. Thus, there is no such integer b. In other words, each component in $C[B]$ is either P_1 or P_2. Summarizing,

- $3 \leq \beta \leq n-4$ or $\beta = n-2$,
- $1 \leq a_{i+1} - a_i \leq 3$ for $1 \leq i \leq \delta - 1$ (3)
- $\{v_1v_{n-2}, v_\alpha v_n, v_{\alpha+1}v_n, v_{\beta-2}v_n\} \subseteq E(G)$.

In particular, $n-4 \leq a_{\delta-1} \leq n-2$.

Subcase 1.1. $\alpha = 1$. Then $a_2 = 2$ and $\beta \geq 4$. Also, $Q_4 = (v_2, v_3, \ldots, v_{n-5})$. First, if $a_{\delta-1} = n-2$, then $4 \leq \beta \leq n-4$. Then the path $(v_{\beta+2}, v_{\beta+3}, \ldots, v_{n-2}, v_n, Q_1, v_{\beta-1}, v_\beta)$ is of type 1 since $v_1v_{\beta+1}, v_1v_{n-1} \notin E(G)$ by (2). We therefore assume that $a_{\delta-1} \in \{n-4, n-3\}$. Let $S = \{v_1, v_{a_{\delta-1}}, v_{n-1}\}$.

Subcase 1.1.1. $a_{\delta-1} = n-4$. Then $\beta = n-2$. We first show that S is independent. By (2), it suffices to verify that $v_1v_{n-4} \notin E(G)$. Let $\{x, y\} = \{v_{n-3}, v_{n-2}\}$. If $v_1v_{n-4} \in E(G)$, then G contains a path P given by

$$P = \begin{cases} (v_n, Q_4, x, y) & \text{if } x \in N(v_{n-5}) \\ (v_{n-1}, v_n, v_{n-4}, v_1, v_2, \ldots, v_{n-6}) & \text{if } x, y \notin N(v_{n-5}), \end{cases}$$

which is of type 1. Therefore, as claimed, no two verities in $S = \{v_1, v_{n-4}, v_{n-1}\}$ are adjacent in G. Now

$$P' = \begin{cases} (v_1, Q_4, x, y) & \text{if } x \in N(v_{n-5}) \\ (v_1, v_n, v_2, v_3, \ldots, v_{n-6}, x, y) & \text{if } x \in N(v_{n-6}) \end{cases}$$

is a type-3 path. Since such P' does not exist, no edge joins a vertex in $\{x, y\}$ and a vertex in $\{v_{n-6}, v_{n-5}\}$. However then, a type-5 path $(v_{n-5}, v_{n-4}, v_n, v_1, v_2, \ldots, v_{n-6})$ results, which is a contradiction anyway. We conclude that $a_{\delta-1} \neq n-4$.

Subcase 1.1.2. $a_{\delta-1} = n-3$. Then $4 \leq \beta \leq n-4$. As in Subcase 1.1.1, we first show that S is independent. First, $v_{n-3}v_{n-1} \notin E(G)$ since $(Q_2, v_{n-3}, v_{n-1}, v_{n-2}, v_1, Q_1)$ cannot be a type-1 path. Thus, when $\beta = n-4$, the set $S = \{v_1, v_{n-3}, v_{n-1}\}$ is indeed independent by (2). For $4 \leq \beta \leq n-5$, note first that $v_{n-4}v_{n-1} \in E(G)$ so that (v_{n-2}, v_1, v_n, Q_4) is not a type-1 path. This in turn implies that $v_1v_{n-3} \notin E(G)$ as $(Q_2, v_{n-1}, v_{n-2}, v_{n-3}, v_1, Q_1)$ cannot be a type-1 path. Thus, no two vertices in S are adjacent for $4 \leq \beta \leq n-5$ as well. However, this produces a type-3 path (v_1, v_n, Q_4, v_{n-4}), which is a contradiction.

Thus, Subcase 1.1 never occurs and so $\alpha \neq 1$. Consequently, $v_2v_n \notin E(G)$. By the symmetry of the graph, we may also assume that $v_{n-2}v_n \notin E(G)$.

Subcase 1.2. $\alpha \neq 1$. Then $3 \leq \alpha \leq n-6$ or $\alpha = n-5 = 4$. By (3) and what is verified in Subcase 1.1, we may assume that $a_2 \in \{3, 4\}$ and $a_{\delta-1} \in \{n-4, n-3\}$. If $n = 9$, then $\delta = 4$ and so $\alpha = a_2 = a_3 - 1$, that is, $a_{\delta-1} = n-4$ $(= 5)$. As before, let $S = \{v_1, v_{a_{\delta-1}}, v_{n-1}\}$.

Subcase 1.2.1. $a_{\delta-1} = n-4$. Then $\beta = n-2$ and so $Q_1 = (v_2, v_3, \ldots, v_{n-4})$. Also, $v_{\beta-2}v_{n-1} = v_{n-4}v_{n-1}$ is not an edge in G by (2). Also, $v_1v_{n-4} \notin E(G)$ since $(v_3, v_4, \ldots, v_{n-4}, v_1, v_{n-2}, v_{n-1})$ cannot be a type-1 path. Thus, $S = \{v_1, v_{n-4}, v_{n-1}\}$ is independent. This in turn implies that $v_2v_{n-2} \notin E(G)$, that is, $\{v_2, v_{n-2}, v_n\}$ is independent, in order to avoid a type-3 path $(v_{n-4}, v_{n-3}, v_{n-2}, v_2, v_3, \ldots, v_{n-5})$. Observe then that we obtain a path

$$P = \begin{cases} (Q_1, v_{n-3}, v_{n-1}) & \text{if } v_{n-3}v_{n-1} \in E(G) \\ (v_{n-2}, v_1, Q_1) & \text{otherwise,} \end{cases}$$

which is of either type 1 or type 3.

Subcase 1.2.2. $a_{\delta-1} = n-3$. Then $5 \leq \beta \leq n-4$ and $n \geq 10$. Hence, $3 \leq \alpha \leq n-6$. Then $v_{n-3}v_{n-1} \notin E(G)$ since $(Q_2, v_{n-3}, v_{n-1}, v_{n-2}, v_1, Q_1)$

cannot be a type-1 path. Hence, $v_2v_{n-2} \notin E(G)$, that is, $\{v_2, v_{n-2}, v_n\}$ is independent, as the path $(v_{n-2}, Q_3, v_n, Q_4, v_{n-4})$ is not of type 1. Then $v_1v_{n-3} \notin E(G)$ or a type-3 path $(Q_3, Q_4, v_{n-4}, v_{n-3}, v_1)$ results. Consequently, $S = \{v_1, v_{n-3}, v_{n-1}\}$ must be independent. However then, we obtain a type-3 path $(v_1, Q_3, v_n, Q_4, v_{n-4})$. This is again impossible.

Thus, Subcase 1.2 never occurs. Consequently, Case 1 never occurs and so $C[A]$ must be empty.

Case 2. *$C[A]$ is empty.* Then $a_{i+1} - a_i \geq 2$ for $1 \leq i \leq \delta - 1$. Hence, $a_2 \geq 3$ while $a_{\delta-1} \leq n-3$. For the rest of the proof, let us write $\alpha^* = a_{\delta-1}$.

Define the subpaths Q_5 and Q_6 of C by

$$Q_5 = (v_3, v_4, \ldots, v_{\alpha^*-2}) \qquad \text{if } \alpha^* \geq 5$$

$$Q_6 = \begin{cases} K_0 & \text{if } \alpha^* = n-3 \\ (v_{\alpha^*+1}, v_{\alpha^*+2}, \ldots, v_{n-3}) & \text{if } \alpha^* \leq n-4. \end{cases}$$

Note that Q_5 is always a subpath of C since $\alpha^* = a_{\delta-1} \geq a_2 + 2 \geq 5$ while Q_6 is a subpath of C when $\alpha^* \leq n-4$ In summary,

$$\begin{aligned} &\bullet\ \{v_2v_n, v_{a_2+1}v_n, v_{\alpha^*-1}v_n\} \cap E(G) = \emptyset, \\ &\bullet\ \{v_iv_n : \alpha^* + 1 \leq i \leq n-2\} \cap E(G) = \emptyset, \\ &\bullet\ v_{\beta+2}v_n \notin E(G). \end{aligned} \tag{4}$$

Subcase 2.1. $\alpha^* \leq n-4$. Then $\beta = n-2$. We first show that (i) $S = \{v_1, v_{\alpha^*}, v_{n-1}\}$ is independent and (ii) $v_{\alpha^*-1}v_{n-1} \in E(G)$. By (4), observe that $v_{\alpha^*}v_{n-1} \notin E(G)$ as the path $(v_1, v_2, Q_5, v_{\alpha^*-1}, v_{\alpha^*}, v_{n-1}, v_{n-2}, \ldots, v_{\alpha^*+3})$ is not of type 1. For (i), therefore, it suffices to verify that $v_1v_{\alpha^*} \notin E(G)$.

Subcase 2.1.1. $a_{\delta-2} \leq \alpha^* - 3$. Then $v_{\alpha^*-2}v_n \notin E(G)$ and so clearly $v_1v_{\alpha^*} \notin E(G)$, that is, (i) holds, as $(v_{n-1}, v_{n-2}, \overleftarrow{Q_6}, v_{\alpha^*}, v_1, v_2, \ldots, v_{\alpha^*-3})$ is not a type-1 path. Then, since $(v_1, v_2, Q_5, v_{\alpha^*-1}, v_{n-2}, \overleftarrow{Q_6})$ is not a type-3 path, $v_{\alpha^*-1}v_{n-2} \notin E(G)$. It then follows that (ii) holds so that the path $(\overleftarrow{Q_6}, v_{\alpha^*}, v_n, v_1, v_2, Q_5)$ is not of type 1.

Subcase 2.1.2. $a_{\delta-2} = \alpha^* - 2$. In this case, that (ii) must hold as $(v_2, Q_5, v_n, v_{\alpha^*}, Q_6, v_{n-2})$ is not a type-1 path and $v_1v_{n-1} \notin E(G)$ by (2). Therefore, (i) must hold so that the path $(v_{n-4}, v_{n-5}, \ldots, v_{\alpha^*}, v_1, v_2, Q_5, v_{\alpha^*-1}, v_{n-1})$, which is of type 1, does not exist.

Hence, as claimed, S is independent and $v_{\alpha^*-1}v_{n-1} \in E(G)$. By the fact that S is independent, $v_2v_{n-2} \notin E(G)$ as $(v_{\alpha^*}, Q_6, v_{n-2}, v_2, Q_5, v_{\alpha^*-1})$ cannot be a type-3 path. Thus, $\{v_2, v_{n-2}, v_n\}$ is independent by (4), which in turn implies that $v_{\alpha^*-2}v_{n-3} \notin E(G)$ since $(v_2, Q_5, \overleftarrow{Q_6}, v_{\alpha^*}, v_{\alpha^*-1}, v_{n-1})$ is not a type-3 path, either. However then, the path

$$P = \begin{cases} (v_{n-4}, v_{n-5}, \ldots, v_{\alpha^*-1}, v_{n-1}, v_{n-2}, v_1, v_2, \ldots, v_{\alpha^*-3}) & \text{if } v_1v_{n-2} \in E(G) \\ (Q_5, v_{\alpha^*-1}, v_{n-1}, v_n, v_{\alpha^*}, Q_6) & \text{otherwise} \end{cases}$$

is a type-1 path. As a result, Subcase 2.1 is impossible.

<u>*Subcase* 2.2. $\alpha^* = n-3$.</u> Then $v_{n-4}v_n \notin E(G)$ by (4). In this case, $3 \le \beta \le n-6$ or $\beta = n-4$. Since $(v_2, v_3, \ldots, v_{n-3}, v_n)$ is not a type-1 path, it follows that $v_1v_{n-2} \in E(G)$. Also, $v_{\beta+1}v_{n-1} \notin E(G)$ while $v_{n-4}v_{n-1} \in E(G)$ as neither of the paths $(\overleftarrow{Q_1}, v_1, v_{n-2}, v_{n-1}, Q_2, v_{n-3})$ and $(v_{n-3}, v_{n-2}, v_1, v_2, \ldots, v_{n-5})$ is of type 1. Note therefore that $\{v_1, v_{\beta+1}, v_{n-1}\}$ is independent by (2). Then the path $(v_1, Q_1, v_{\beta-1}, v_\beta, v_{\beta+2}, v_{\beta+3}, \ldots, v_{n-2})$ cannot be of type 3 and so $v_\beta v_{\beta+2} \notin E(G)$, that is, $\{v_\beta, v_{\beta+2}, v_n\}$ is also independent by (4). Then, we obtain a path

$$P = \begin{cases} (v_1, Q_1, v_{\beta-1}, v_{n-3}, v_{n-2}, Q_7) & \text{if } v_{\beta-1}v_{n-3} \in E(G) \\ (v_\beta, Q_2, v_{n-1}, v_{n-2}, v_1, Q_1) & \text{otherwise,} \end{cases}$$

where

$$Q_7 = \begin{cases} K_0 & \text{if } \beta = n-4 \\ (v_{n-1}, v_{n-4}, v_{n-5}, \ldots, v_{\beta+2}) & \text{if } \beta \le n-6, \end{cases}$$

is either a type-1 or type-3 path. We conclude that Subcase 2.2 is impossible, that is, Case 2 never occurs.

We have considered all possible cases. Therefore, if G is a graph of order $n \ge 9$ with $4 \le \delta(G) \le (n-1)/2$, then G must contain a path of type 1–5 and so G cannot be $(n-3)$-path-Hamiltonian.

Let us conclude this section by stating the main result and a corollary.

Theorem 3.14 *A graph G of order $n \ge 4$ is $(n-3)$-path-Hamiltonian if and only if* (i) *G is n-path-Hamiltonian, that is, G equals C_n or K_n or $K_{n/2,n/2}$ (when n is even) or* (ii) $\overline{G} \in \{P_3 + P_2, C_6, 2P_3, C_4 + C_3\}$ *or* (iii) $\delta(G) = n-2$.

For a graph G, every path of order at most $\delta(G)$ can be further extended to a path of order $\delta(G)+1$. Thus, if G is an ℓ-path-Hamiltonian graph, where $1 \le \ell \le \delta(G)+1$, then $\mathrm{hce}(G) \ge \ell$. Since $\delta(G) \ge n-3$ for the graphs G in Theorem 3.14 (ii)(iii), the following also holds.

Corollary 3.15 *A graph G of order $n \ge 4$ is $(n-3)$-path-Hamiltonian if and only if* $\mathrm{hce}(G) \in \{n-3, n\}$.

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