A Note on the 2-Ramsey Numbers of 4-Cycles

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Abstract

A balanced complete bipartite graph is a complete bipartite graph the degrees of whose vertices differ by at most 1. In a red-blue-green coloring of the edges of a graph G, every edge of G is colored red, blue or green. For three graphs F_1, F_2 and F_3, the 2-Ramsey number $R_2(F_1, F_2, F_3)$ of F_1, F_2 and F_3, if it exists, is the smallest order of a balanced complete bipartite graph G for which every red-blue-green coloring of the edges of G results in a red F_1, a blue F_2 or a green F_3. In this note, we show that $20 \leq R_2(C_4, C_4, C_4) \leq 21$.

Key Words: red-blue-green coloring, balanced complete bipartite graph, 2-Ramsey number.

AMS Subject Classification: 05C35, 05C55.

1 Introduction

The *Ramsey number* $R(F, H)$ of two graphs F and H is the smallest positive integer n for which every red-blue coloring (in which every edge is colored red or blue) of the complete graph K_n of order n results in a red F (a subgraph of K_n isomorphic to F each edge of which is colored red) or a blue H. It is well known that $R(F, H)$ exists for every two graphs F and H although $R(F, H)$ has been determined for relatively few pairs F, H of graphs.

For bipartite graphs F and H, the *bipartite Ramsey number* $BR(F, H)$ is defined in [1] as the smallest positive integer r for which every red-blue coloring of the r-regular complete bipartite graph $K_{r,r}$ results in a red F or a blue H. It is known that the bipartite Ramsey number exists for every two bipartite graphs (see [1]). If $BR(F, H) = r$, then there exists a red-blue coloring of $K_{r-1,r-1}$ for which there is neither a red F nor a blue H. Whether there is a red-blue coloring of $K_{r-1,r}$ for which there is neither a red F nor a blue H depends on the graphs F and H. This observation led to the introduction of the k-Ramsey number of two graphs in [2]. A balanced complete k-partite graph, $k \geq 2$, is a complete k-partite graph the degrees of whose vertices differ by at most 1. For two bipartite graphs F and H and an integer k with $2 \leq k \leq R(F,H)$, the *k-Ramsey number* $R_k(F,H)$ of F and H is the smallest order of a balanced complete k-partite graph G for which every red-blue coloring of G results in a red F or a blue H. If $R(F,H) = n$, then $R_n(F,H) = R(F,H)$. For the 4-cycle C_4 of order 4, the Ramsey number is $R(C_4,C_4) = 6$ and the bipartite Ramsey number is $BR(C_4,C_4) = 5$. Furthermore, for $2 \leq k \leq 6 = R(C_4,C_4)$, it was shown in [2] that $R_k(C_4,C_4) = 12 - k$. It is not surprising that $R_k(F,H)$ has been determined for very few pairs F,H of graphs for integers $k \geq 2$ (see [4]).

Ramsey numbers have also been defined for three or more graphs. In particular, for three graphs F_1, F_2 and F_3, the *Ramsey number* $R(F_1,F_2,F_3)$ of F_1, F_2 and F_3 is the smallest positive integer n for which every red-blue-green coloring (in which every edge is colored red, blue or green) of the complete graph K_n of order n results in a red F_1, a blue F_2 or a green F_3. This gives rise to the concept of k-Ramsey number of three (or more) graphs. For three graphs F_1, F_2 and F_3 and an integer k with $2 \leq k \leq R(F_1,F_2,F_3)$, the k-Ramsey number $R_k(F_1,F_2,F_3)$ of F_1, F_2 and F_3, if it exists, is the smallest order of a balanced complete k-partite graph G for which every red-blue-green coloring of the edges of G results in a red F_1, a blue F_2 or a green F_3. In particular, if $k = 2$ and $F_i \cong F$ for some graph F where $i = 1,2,3$, then the 2-Ramsey number $R_2(F,F,F)$ is the smallest order of a balanced complete bipartite graph G for which every red-blue-green coloring of the edges of G results in a monochromatic F (all of whose edges are colored the same). In general, it is very challenging to determine the value of $R_2(F,F,F)$ even when F is a graph of smallest size. In this note, we show that $20 \leq R_2(C_4,C_4,C_4) \leq 21$. We refer to the book [3] for graph theory notation and terminology not described in this paper.

2 The values of $R_2(C_4,C_4,C_4)$

First, we show that $R_2(C_4,C_4,C_4)$ is at least 20.

Theorem 2.1 $R_2(C_4,C_4,C_4) \geq 20$.

Proof. We describe a red-blue-green coloring c of $K_{9,10}$ that avoids a monochromatic C_4, which implies that $R_2(C_4,C_4,C_4) \geq 20$. Let $U = \{u_1,u_2,\ldots,u_9\}$ and $V = \{v_1,v_2,\ldots,v_{10}\}$ be the partite sets of $K_{9,10}$. The edge coloring c of $K_{9,10}$ is defined by the following table, where the $u_i - v_j$ entry in row u_i and column v_j indicates the color of the edge u_iv_j for $1 \leq i \leq 9$ and $1 \leq j \leq 10$.

$K_{9,10}$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
u_1	r	r	r	r	g	g	b	b	g	b
u_2	b	b	g	r	r	r	r	g	g	b
u_3	r	g	g	b	b	g	r	r	r	b
u_4	b	r	b	b	g	r	g	b	r	g
u_5	g	g	b	r	b	b	g	r	g	r
u_6	g	b	r	b	r	b	b	g	r	g
u_7	g	r	g	g	g	b	r	b	b	r
u_8	b	g	r	g	b	r	b	r	b	g
u_9	r	b	b	g	r	g	g	g	b	r

A red-blue-green coloring of $K_{9,10}$

Next, we describe the structures of the red, blue and green subgraphs G_r, G_b and G_g of $K_{9,10}$ produced by this edge coloring c. Figure 1 shows a spanning subgraph G of size 30 in $K_{9,10}$, where each solid vertex is a vertex in U and each empty vertex is a vertex in V. In fact, each of the resulting red, blue and green subgraphs G_r, G_b and G_g is isomorphic to the graph G of Figure 1. To illustrate this fact, we label a vertex $u \in U$ by a triple (u_p, u_q, u_s), $1 \le p, q, s \le 9$, and a vertex $v \in V$ by a triple (v_p, v_q, v_s), $1 \le p, q, s \le 10$, such that (1) the label (u_p, u_q, u_s) of $u \in U$ indicates that $u = u_p$ in G_r, $u = u_q$ in G_b and $u = u_s$ in G_g and (2) the label (v_p, v_q, v_s) of $v \in V$ indicates that $v = v_p$ in G_r, $v = v_q$ in G_b and $v = v_s$ in G_g. Furthermore, if a vertex $u \in U$ is labeled by u_p, then $u = u_p$ in each of G_r, G_b, G_g. Similarly, a vertex $v \in V$ labeled v_p indicates that $v = v_p$ in each of G_r, G_b, G_g.

The table below lists the red-neighborhood, blue-neighborhood and green-neighborhood $N_R(u)$ of each vertex $u \in U$. Observe that $N_R(u) \cup N_B(u) \cup N_G(u) = V$ for each $u \in U$.

$N_R(u_1) = \{v_1, v_2, v_3, v_4\}$	$N_B(u_1) = \{v_7, v_8, v_{10}\}$	$N_G(u_1) = \{v_5, v_6, v_9\}$
$N_R(u_2) = \{v_4, v_5, v_6, v_7\}$	$N_B(u_2) = \{v_1, v_2, v_{10}\}$	$N_G(u_2) = \{v_3, v_8, v_9\}$
$N_R(u_3) = \{v_7, v_8, v_9, v_1\}$	$N_B(u_3) = \{v_4, v_5, v_{10}\}$	$N_G(u_3) = \{v_2, v_3, v_6\}$
$N_R(u_4) = \{v_2, v_6, v_9\}$	$N_B(u_4) = \{v_1, v_3, v_4, v_8\}$	$N_G(u_4) = \{v_5, v_7, v_{10}\}$
$N_R(u_5) = \{v_4, v_8, v_{10}\}$	$N_B(u_5) = \{v_3, v_5, v_6\}$	$N_G(u_5) = \{v_1, v_2, v_7, v_9\}$
$N_R(u_6) = \{v_3, v_5, v_9\}$	$N_B(u_6) = \{v_2, v_4, v_6, v_7\}$	$N_G(u_6) = \{v_1, v_8, v_{10}\}$
$N_R(u_7) = \{v_2, v_7, v_{10}\}$	$N_B(u_7) = \{v_6, v_8, v_9\}$	$N_G(u_7) = \{v_1, v_3, v_4, v_5\}$
$N_R(u_8) = \{v_3, v_6, v_8\}$	$N_B(u_8) = \{v_1, v_5, v_7, v_9\}$	$N_G(u_8) = \{v_2, v_4, v_{10}\}$
$N_R(u_9) = \{v_1, v_5, v_{10}\}$	$N_B(u_9) = \{v_2, v_3, v_9\}$	$N_G(u_9) = \{v_4, v_6, v_7, v_8\}$

Since no two vertices in U have two common neighbors in G, it follows that G is C_4-free and so G_r, G_b and G_g are C_4-free. Therefore, there is no monochromatic C_4 in this edge-colored $K_{9,10}$ and so $R_2(C_4, C_4, C_4) \ge 20$. ■

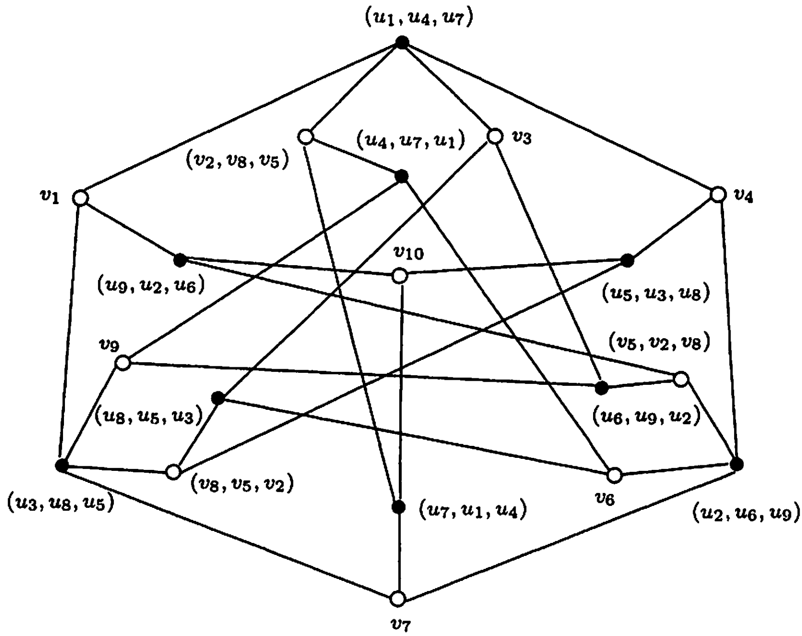

Figure 1: A spanning subgraph G of size 30 in $K_{9,10}$

Next, we show that $R_2(C_4, C_4, C_4)$ is at most 21.

Theorem 2.2 $R_2(C_4, C_4, C_4) \leq 21$.

Proof. We show that every red-blue-green coloring of $K_{10,11}$ results in a monochromatic C_4, which implies that $R_2(C_4, C_4, C_4) \leq 21$. Let

$$U = \{u_1, u_2, \ldots, u_{11}\} \text{ and } V = \{v_1, v_2, \ldots, v_{10}\}$$

be partite sets of $K_{11,10}$. Assume, to the contrary, that there is a red-blue-green coloring c of $K_{10,11}$ that avoids a monochromatic C_4. Let m_r, m_b and m_g be the sizes of the resulting red, blue and green subgraphs G_r, G_b and G_g, respectively, where say $m_r \geq m_b \geq m_g$. Thus

$$m_r = \sum_{i=1}^{11} \deg_{G_r} u_i \geq \left\lceil \frac{110}{3} \right\rceil = 37.$$

In what follows, we show that G_r contains C_4 as a subgraph, producing a contradiction.

Suppose, without loss of generality, that $\deg_{G_r} u_1 \geq \deg_{G_r} u_2 \geq \cdots \geq \deg_{G_r} u_{11}$. Thus, $\deg_{G_r} u_1 \geq \lceil \frac{37}{11} \rceil = 4$. Since there is no red C_4 in G_r, it follows that

$$|N_R(u_i) \cap N_R(u_j)| \leq 1 \text{ for } 1 \leq i < j \leq 11. \tag{1}$$

This implies that

$$\textstyle\sum_{i=1}^{3} \deg_{G_r} u_i \leq |V| + 3 = 13 \tag{2}$$

and

$$\textstyle\sum_{i=1}^{4} \deg_{G_r} u_i \leq |V| + \binom{4}{2} = 16. \tag{3}$$

Since $\lceil \frac{37-16}{7} \rceil = 3$, it follows that $\deg_{G_r} u_5 \geq 3$ and so

$$\deg_{G_r} u_i \geq 3 \text{ for } 2 \leq i \leq 5. \tag{4}$$

If $\deg_{G_r} u_1 \geq 8$, then $\sum_{i=1}^{3} \deg_{G_r} u_i \geq 8 + 3 + 3 = 14$ by (4), which contradicts (2). Thus $\deg_{G_r} u_1 = 4, 5, 6, 7$ and so there are four cases to consider. First, we make an observation. If $\sum_{i=1}^{3} \deg_{G_r} u_i = 13$, then $N_R(u_1) \cup N_R(u_2) \cup N_R(u_3) = V$ and each of the following conditions (i), (ii) and (iii) hold in G_r:

(i) Since $|V| = 10$, it follows by (1) that $|N(u_i) \cap N(u_j)| = 1$ for $1 \leq i < j \leq 3$.

(ii) If $\deg_{G_r} u_3 = 3$, then $\deg_{G_r} u_i = 3$ for $4 \leq i \leq 11$, as $m_r \geq 37$.

(iii) No vertex of degree 3 or more in G_r is adjacent to the vertex in $N(u_i) \cap N(u_j)$ for $1 \leq i < j \leq 3$. To see this, let $u \in U$ such that $N_R(u)$ contains the vertex $v \in N(u_1) \cap N(u_2)$ say. Thus, $(N_R(u) - \{v\}) \cap N_R(u_i) = \emptyset$ for $i = 1, 2$ by (1). Since $N_R(u)$ contains at most one vertex in $N_R(u_3) = V - [N_R(u_1) \cup N_R(u_2)]$, it follows that $\deg_{G_r} u \leq 2$.

We are now prepared to consider these four cases.

Case 1. $\deg_{G_r} u_1 = 7$. Then $\deg_{G_r} u_2 = \deg_{G_r} u_3 = 3$ by (2) and (4) and so $\sum_{i=1}^{3} \deg_{G_r} u_i = 13$. Hence $\deg_{G_r} u_i = 3$ for $2 \leq i \leq 11$ by (ii). We may assume, without loss generality, that $N_R(u_1) = \{v_1, v_2, \ldots, v_7\}$, $N_R(u_2) = \{v_7, v_8, v_9\}$ and $N_R(u_3) = \{v_9, v_{10}, v_1\}$. Since

$$\deg_{G_r} u_4 = \deg_{G_r} u_5 = 3,$$

it follows by (1) that each of u_4 and u_5 is adjacent in G_r to exactly one vertex in $N_R(u_i)$ for each $i = 1, 2, 3$ but not adjacent to any of v_1, v_7, v_9 in G_r. This implies that each of u_4 and u_5 is adjacent in G_r to $v_8 \in N_R(u_2)$

and is adjacent to $v_{10} \in N_R(u_3)$. However then, $\{v_8, v_{10}\} \subseteq N_R(u_4) \cap N_R(u_5)$ and results in a red C_4, which is a contradiction.

Case 2. $\deg_{G_r} u_1 = 6$. Since $\deg_{G_r} u_2 \geq \lceil \frac{37-6}{10} \rceil = 4$, it follows that $\deg_{G_r} u_i = 3$ for $i = 3, 4, 5$ by (2) and (4) and $\deg_{G_r} u_2 = 4$. We may assume, without loss generality, that $N_R(u_1) = \{v_1, v_2, \ldots, v_6\}$, $N_R(u_2) = \{v_6, v_7, v_8, v_9\}$ and $N_R(u_3) = \{v_9, v_{10}, v_1\}$. Since $\deg_{G_r} u_i = 3$ for $i = 4, 5, 6$, it follows by (1) that each of u_4, u_5, u_6 is adjacent in G_r to exactly one vertex in $N_R(u_i)$ in G_r for $i = 1, 2, 3$ but not adjacent in G_r to any of v_1, v_6, v_9 in G_r. This implies that at least two of u_4, u_5 and u_6 are both adjacent to v_7 or both adjacent to v_8 in G_r, say u_4 and u_5 are adjacent to v_7, and each of u_4, u_5 and u_6 is adjacent to v_{10} in G_r. However then, $v_7, v_{10} \in N_R(u_4) \cap N_R(u_5)$ and so there is a red C_4 in G_r, which is a contradiction.

Case 3. $\deg_{G_r} u_1 = 5$. Since $\deg_{G_r} u_2 \geq \lceil \frac{37-5}{10} \rceil = 4$, it follows that $\deg_{G_r} u_2 = 4, 5$. We consider these two subcases.

Subcase 3.1. $\deg_{G_r} u_2 = 5$. Thus $\deg_{G_r} u_3 = 3$ by (2) and (4). Since $m_r \geq 37$, it follows that $\deg_{G_r} u_i = 3$ for $3 \leq i \leq 11$. We may assume, without loss generality, that $N_R(u_1) = \{v_1, v_2, v_3, v_4, v_5\}$, $N_R(u_2) = \{v_5, v_6, v_7, v_8, v_9\}$ and $N_R(u_3) = \{v_9, v_{10}, v_1\}$. Since $\deg_{G_r} u_i = 3$ for $i = 4, 5, 6, 7$, each of u_4, u_5, u_6, u_7 is adjacent to exactly one vertex in $N_R(u_i)$ in G_r for $i = 1, 2, 3$ but not to any of v_1, v_5, v_9 in G_r. Hence $u_i v_{10} \in E(G_r)$ for $i = 4, 5, 6, 7$. Furthermore, at least two vertices in $\{u_4, u_5, u_6, u_7\}$ are both adjacent to one of $v_6, v_7, v_8 \in N_R(u_2)$ in G_r, say u_4 and u_5 are adjacent to v_7 in G_r. However then, $v_7, v_{10} \in N_R(u_4) \cap N_R(u_5)$ and so there is a red C_4 in G_r, which is a contradiction.

Subcase 3.2. $\deg_{G_r} u_2 = 4$. Since $\deg_{G_r} u_3 \geq \lceil \frac{37-9}{9} \rceil = 4$, it follows that $\deg_{G_r} u_3 = 4$. We may assume that $N_R(u_1) = \{v_1, v_2, v_3, v_4, v_5\}$, $N_R(u_2) = \{v_5, v_6, v_7, v_8\}$ and $N_R(u_3) = \{v_8, v_9, v_{10}, v_1\}$. If $\deg_{G_r} u_4 = 4$, then $\sum_{i=1}^{4} \deg_{G_r} u_i = 17$, a contradiction. Thus, $\deg_{G_r} u_4 = 3$ and so $\deg_{G_r} u_i = 3$ for $4 \leq i \leq 11$, as $m_r \geq 37$. Hence, each of u_4, u_5, u_6, u_7, u_8 is adjacent to exactly one vertex in $N_R(u_i)$ for each $i \in \{1, 2, 3\}$ but not adjacent to any vertex in $\{v_1, v_5, v_8\}$ in G_r. Therefore, each of five vertices u_4, u_5, u_6, u_7, u_8 is adjacent to exactly one of $v_6, v_7 \in N_R(u_2)$ and exactly one of $v_9, v_{10} \in N_R(u_3)$ in G_r. Since there are only four such possibilities, namely

$$A_1 = \{v_6, v_9\},\ A_2 = \{v_6, v_{10}\},\ A_3 = \{v_7, v_9\},\ A_4 = \{v_7, v_{10}\},$$

there is $t \in \{1, 2, 3, 4\}$ such that $A_t \in N_R(u_i) \cap N_R(u_j)$ where $i, j \in \{4, 5, 6, 7, 8\}$ and $i \neq j$, which is a contradiction.

Case 4. $\deg_{G_r} u_1 = 4$. Since $\deg_{G_r} u_2 \geq \lceil \frac{37-4}{10} \rceil = 4$, $\deg_{G_r} u_3 \geq \lceil \frac{37-8}{9} \rceil = 4$ and $\deg_{G_r} u_4 \geq \lceil \frac{37-12}{8} \rceil = 4$, it follows that $\deg_{G_r} u_i = 4$

for $1 \le i \le 4$. First, suppose that either $N_R(u_i) \cap N_R(u_j) = \emptyset$ for some pair $ij \in \{1,2,3\}$ and $i \ne j$ or $N_R(u_1) \cap N_R(u_1) \cap N_R(u_3) \ne \emptyset$. Then $N_R(u_1) \cup N_R(u_1) \cup N_R(u_3) = V$. Since $\deg_{G_r} u_4 \ge 4$, it follows that u_4 must be adjacent to two vertices in one of the sets $N_R(u_1)$, $N_R(u_2)$, $N_R(u_3)$, resulting in a red C_4, which is a contradiction. Next, suppose that $|N_R(u_i) \cap N_R(u_j)| = 1$ for every pair $ij \in \{1,2,3\}$ and $i \ne j$ and $N_R(u_1) \cap N_R(u_1) \cap N_R(u_3) = \emptyset$. We may assume that $N_R(u_1) = \{v_1, v_2, v_3, v_4\}$, $N_R(u_2) = \{v_4, v_5, v_6, v_7\}$ and $N_R(u_3) = \{v_7, v_8, v_9, v_1\}$. Furthermore, u_4 is adjacent to exactly one vertex in each of the sets $\{v_2, v_3\}$, $\{v_5, v_6\}$, $\{v_8, v_9\}$ and is adjacent to v_{10} in G_r. We may assume, without loss of generality, that $N_R(u_4) = \{v_2, v_5, v_8, v_{10}\}$. Since $\deg_{G_r} u_5 \ge \lceil \frac{37-16}{7} \rceil = 3$, it follows that $\deg_{G_r} u_5 = 3, 4$. We consider these two subcases.

Subcase 4.1. $\deg_{G_r} u_5 = 4$. Since (a) u_5 is adjacent to exactly one vertex in each of $\{v_2, v_3\}$, $\{v_5, v_6\}$, $\{v_8, v_9\}$ and is adjacent to v_{10} and (b) $N_R(u_4) = \{v_2, v_5, v_8, v_{10}\}$, it follows that $N_R(u_5) = \{v_3, v_6, v_9, v_{10}\}$. Hence, each vertex $v \in V$ belongs to exactly two of the five sets $N_R(u_1), N_R(u_2), \ldots, N_R(u_5)$. Next, we consider u_6. Since $\deg_{G_r} u_6 \ge \lceil \frac{37-20}{6} \rceil = 3$, it follows that $N_R(u_6)$ contains at least three vertices. Each of these three vertices, however, must belong to two of the red neighborhoods of u_1, u_2, u_3, u_4 and u_5. Therefore, at least one of u_1, u_2, u_3, u_4 and u_5 must share two red neighbors with u_6, which is impossible.

Subcase 4.2. $\deg_{G_r} u_5 = 3$. Since $m_r \ge 37$, it follows that $\deg_{G_r} u_i = 3$ for $5 \le i \le 11$. We now consider the possible 3-element sets for $N_R(u_i)$ for $5 \le i \le 11$. Let $S \in \{N_R(u_5), N_R(u_6), \ldots, N_R(u_{11})\}$ and let i be the smallest integer $i \in \{1, 2, \ldots, 10\}$ such that $v_i \in S$. Hence $1 \le i \le 7$.

(1) $i = 1$. Since $v_1 \in N_R(u_1) \cap N_R(u_3)$, $v_5, v_6 \in N_R(u_2)$, $v_5, v_{10} \in N_R(u_4)$ and $u_5 \notin S$, it follows that $v_2, v_3, v_4, v_7, v_8, v_9 \notin S$ and so $S = \{v_1, v_6, v_{10}\}$.

(2) $i = 2$. Since $v_2 \in N_R(u_1) \cap N_R(u_4)$, $v_6, v_7 \in N_R(u_2)$, $v_7, v_9 \in N_R(u_3)$ and $u_7 \notin S$, it follows that $v_3, v_4, v_5, v_8, v_{10} \notin S$ and so $S = \{v_2, v_6, v_9\}$.

(3) $i = 3$. Since $v_3 \in N_R(u_1)$, it follows that $v_4 \notin S$.

(3.1) If $v_5 \in S$, then $v_4, v_6, v_7, v_8, v_{10} \notin S$ because $v_5 \in N_R(u_2) \cap N_R(u_4)$. Hence $S = \{v_3, v_5, v_9\}$.

(3.2) If $v_5 \notin S$ but $v_6 \in S$, then $v_4, v_5, v_7 \notin S$ because $v_6 \in N_R(u_2)$. Hence S is one of the three sets $\{v_3, v_6, v_8\}$, $\{v_3, v_6, v_9\}$ and $\{v_3, v_6, v_{10}\}$.

(3.3) If $v_5, v_6 \notin S$ but $v_7 \in S$, then $v_8, v_9 \notin S$ because $v_7 \in N_R(u_3)$. Hence $S = \{v_3, v_7, v_{10}\}$.

(3.4) If $v_5, v_6, v_7 \notin S$, then $u_8 \notin S$ because $v_8, v_9 \in N_R(u_3)$ and $v_8, v_{10} \in N_R(u_4)$. Hence $S = \{v_3, v_9, v_{10}\}$.

(4) $i = 4$. Since $v_4 \in N_R(u_2)$, it follows that $v_5, v_6, v_7 \notin S$. Furthermore, because $v_8, v_9 \in N_R(u_3)$ and $v_8, v_{10} \in N_R(u_4)$, it follows that $u_8 \notin S$ and so $S = \{v_4, v_9, v_{10}\}$.

(5) $i = 5$. Since $v_5 \in N_R(u_2) \cap N_R(u_4)$, it follows that $v_6, v_7, v_8, v_{10} \notin S$. However then, $|S| < 3$, which is impossible.

(6) $i = 6$. Since $v_6 \in N_R(u_2)$, it follows that $v_7 \notin S$. Because $v_8, v_9 \in N_R(u_3)$ and $v_8, v_{10} \in N_R(u_4)$, it follows that $u_8 \notin S$ and so $S = \{v_6, v_9, v_{10}\}$.

(7) $i = 7$. Hence, S must contain at least one of v_8 and v_9, say $v_8 \in S$. However then, $|S \cap N_R(u_3)| = 2$, which is impossible.

In summary, there are only six possibilities for the seven sets $N_R(u_i)$ for $5 \le i \le 11$:

$$\begin{array}{ll} S_1: & \{v_1, v_6, v_{10}\} \\ S_2: & \{v_2, v_6, v_9\} \\ S_3: & \{v_3, v_5, v_9\} \\ S_4: & \{v_3, v_7, v_{10}\} \\ S_5: & \{v_3, v_6, v_8\}, \{v_3, v_6, v_9\}, \{v_3, v_6, v_{10}\} \\ S_6: & \{v_3, v_9, v_{10}\}, \{v_4, v_9, v_{10}\}, \{v_6, v_9, v_{10}\} \end{array}$$

Therefore, there are two sets $N_R(u_i)$ and $N_R(u_j)$, where $i, j \in \{5, 6, \ldots, 11\}$ and $i \ne j$, such that $N_R(u_i)$ and $N_R(u_j)$ are chosen to be the same set. This produces a red C_4, which is a contradiction.

By Theorems 2.1 and 2.2, it follows that $20 \le R_2(C_4, C_4, C_4) \le 21$.

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