Uniquely Bipancyclic Graphs

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Abstract

A *bipancyclic* graph on v vertices is a bipartite graph that contains, as subgraphs, cycles of length n for every even integer n such that $4 \leq n \leq v$. Such a graph is *uniquely bipancyclic* if it contains exactly one subgraph of each permissible length.

In this paper we find all uniquely bipancyclic graphs on 30 or fewer vertices.

1 Introduction

For definitions and theorems involving graph theory, the reader is referred to standard texts on the subject, such as [12]. In this paper all graphs will be finite, simple and undirected.

A graph with v vertices is called *pancyclic* if it contains cycles of every length from 3 to v. Pancyclic graphs were first defined by Bondy in 1971 ([3]). A pancyclic graph with exactly one cycle of every possible order is called *uniquely pancyclic* ([11, 6]).

Recall that a *bipartite graph* G is a graph whose vertices can be partitioned into two sets, V_1 and V_2, where no edge joins two members of the same set. Say there are n_1 vertices in V_1 and n_2 vertices in V_2; we shall say G is bipartite *of type* (n_1, n_2). G is a *complete bipartite graph* K_{n_1,n_2} if every vertex in V_1 is joined to every vertex in V_2, otherwise G will be a proper subgraph of K_{n_1,n_2}. Clearly the largest cycle in a type (n_1, n_2) bipartite graph will contain at most $2 \times min(n_1, n_2)$ edges, so the graph can be Hamiltonian only if $n_1 = n_2$.

Obviously a bipartite graph can only contain cycles of even length, so such a graph cannot be pancyclic — no cycle of length 3 is possible. To get

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around this problem, a graph with v vertices is defined to be *bipancyclic* if it is bipartite and contains cycles of every even length from 4 to v. For further discussion of bipancyclic graphs, see [1, 2, 7, 8, 10].

In this paper we look at the problem of uniquely bipancyclic graphs, that is bipartite graphs that contain exactly one cycle of each length from 4 up to the number of vertices. If such a graph contains c cycles, their lengths are 4, 6, ..., $2c+2$, so the graph has $2c+2$ edges; moreover the sum of the lengths of the cycles (total number of edges in the cycles, with multiple appearances in different cycles counted multiply) is $4+6+\ldots+(2c+2) = c^2+3c$. We shall denote this as $s(c)$.

This topic arose in conversation with Saad El-Zanati. I wish to express my gratitude for his help and support.

2 Representing the graphs and counting cycles

Suppose we have a bipancyclic graph on v vertices. It must contain a Hamilton cycle, so the graph could be represented as a cycle of length v together with some other edges which we shall call *chords*.

In every case, we represent our graph as a circle with the chords as straight lines. The segments of the outer circle may contain a number of vertices, but the chords only have vertices at their ends. If vertex labels are needed, we assume the vertices are $x_1, x_2, \ldots, x_v$ in clockwise order.

If a v-vertex graph contains a chord xy, there will be two cycles containing it, one obtained by going from x to y clockwise and then along yx, the other by going anticlockwise. The numbers of vertices in these two cycles will total $v+2$, because the edges on the cycle will be counted once each and the chord counted twice. We shall say a chord is *of type* (p,q) if the two cycles it generates have lengths p and q. If there are two chords, there may be one or two cycles that contain both of them; in the latter case, the total is $v+4$. Any collection of chords generates either one or two cycles that contain them all; if there are c chords and they generate two cycles, then the two lengths will add to $v+2c$.

In this paper we shall examine all cases of three or fewer chords.

3 Graphs with fewer than two chords

If there are no chords, the graph contains only one (Hamilton) cycle, of length v, and the only bipancyclic case is $v = 4$. If there is one chord, there are two further (one-chord) cycles. So, if the graph has only one chord, there are exactly three cycles. These cycles are illustrated in Figure 1 (there are three drawings of the same graph, with the cycles shown in bold).

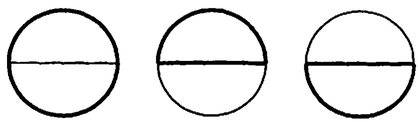

Figure 1: Cycles in the case of one chord

If the graph is uniquely bipancyclic the lengths of the cycles must be 4, 6 and 8. This can be achieved by inserting a chord x_1x_4 into a cycle of length 8; any other example will obviously be isomorphic to this.

One can construct a bipancyclic graph on six vertices, again by inserting chord x_1x_4, but the graph will contain two cycles $x_1x_2x_3x_4$ and $x_1x_6x_5x_4$ of length 4, so it is not uniquely bipancyclic.

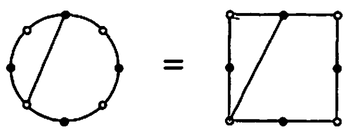

Figure 2: The uniquely bipancyclic graph of order 8

As we shall see in the next section, any uniquely bipancyclic graph with more than one chord will have at least six cycles and therefore at least 14 vertices, so there are no uniquely bipancyclic graphs on 6, 10 or 12 vertices.

4 Two chords

There are three possible patterns for two chords: case A, where the chords share an endpoint; case B, where they do not cross in the standard diagram, and case C, where they cross. The three types are illustrated in Figure 3.

In addition to the cycles that contain no chord or one chord, there well be new cycles that contain two chords. Let us count cycles in the three

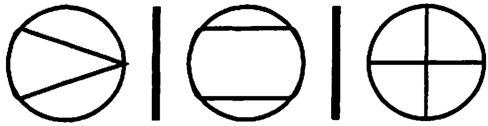

Figure 3: The possible cases with two chords

cases:
A. There is one new cycle, so together with the Hamilton cycle and the four one-chord cycles (two per chord) there are six cycles in total;
B. There is one new cycle, for six in total;
C. There are two new cycles, for seven in total.

Types A and B produce 6 cycles (including the Hamiltonian) so a uniquely bipancyclic graph of type A or B would have 14 vertices; one of type C would have 16.

In the 14-vertex case, the total number of edges in the two cycles containing a given chord (but not the other one) will total 16, so four cycles each containing exactly one chord will contain a total of 32 edges, and the Hamiltonian cycle contains 14, so if the cycle containing both chords has z edges then

$$14 + 32 + z = 4 + 6 + 8 + 10 + 12 + 14 = 54$$

and $z = 8$.

In the 16-vertex case, the total number of edges in the two cycles containing a given chord (but not the other one) will total 18, the two cycles that contain both chords will have a total of 20 edges, and the Hamiltonian cycle contains 16. So the seven cycles have a total of $16+18+18+20 = 72$ edges. But $4 + 6 + 8 + 10 + 12 + 14 + 16 = 70$. So there is no uniquely bipancyclic graph on 16 vertices.

Type A The four cycles containing one edge each must contain 4, 6, 10 and 12 edges. So one chord must be of type (4,12) and the other of type (6,10). Up to isomorphosm, we can assume the common endpoint of the chords to be x_1 and the first chord to be (x_1, x_4). The second must be (x_1, x_6) or (x_1, x_{10}). Only the second choice causes the cycle containing both chords to be of length 8. So there is exactly one solution.

Without loss of generality the chords are (x_1, x_i) and (x_j, x_k) where, in order for the cycle containing both chords to have length 8, $(j-i)+(15-k) = 6$. (The eight edges include the two chords.) The first chord is of type $(i, 16-i)$ and the second is of type $(k-j+1, 15-j+k)$. One of these pairs must be (4,12), so let us take $i = 4$. Then $(j - i) + (15 - k) = 6$ becomes $j + 11 - k = 6$ or $k = j + 5$. Then type $(k - j + 1, 15 - j + k)$ becomes type (6,10), as required. There are five possibilities: $j = 5, 6, 7, 8$ or 9. Cases $j = 5$ and $j = 9$ are easily seen to be isomorphic, as are cases $j = 6$ and $j = 8$. So there are three solutions, giving a total of four nonisomorphic uniquely bipancyclic graphs of order 14.

The four uniquely bipancyclic graphs are shown in Figure 4.

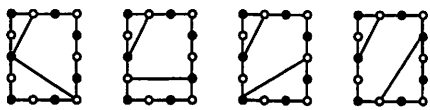

Figure 4: The possible cases with two chords

5 Three chords

If a graph has three chords, we classify by looking at the three pairs of chords. We refer to the configuration by the string of three letters corresponding to the three types of chord interaction. For example, type AAB is a graph in which two of the pairs of chords are type A (they do not cross, and have no common endpoint) and one pair is type B (they have a common endpoint). There are 14 types of graph: AAAi, AAAii, AABi, AABii, AAC, ABBi, ABBii, ABC, ACC, BBBi, BBBii, BBC, BCC, CCC. (There are two types AAA, two types AAB, two types ABB (one where the three chords form a "C" pattern and one where they form a "Z"), and two types BBB (one where all three chords have a common endpoint and one where they form a triangle).) They are illustrated in Figure 5, below, which is taken from [4], where they were used to look for minimal pancyclic graphs.

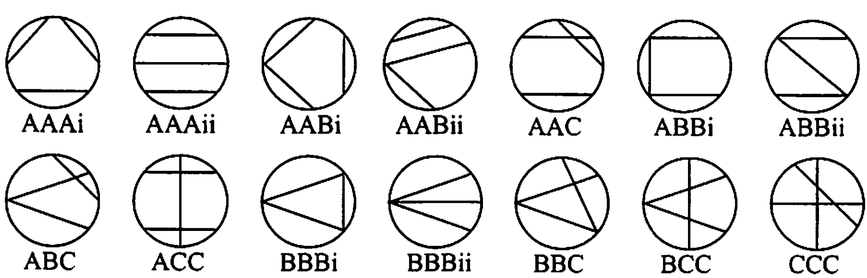

Figure 5: Cases of three chords

The following table, also from that paper, counts the number c of cycles in a graph, in each of the 14 cases. C(n) means the number of cycles involving exactly n chords. We have added rows showing the number v of vertices required for a uniquely bipancyclic graph with this chord structure, and the total $s(c)$ of the edges in the cycles.

	AAAi	AAAii	AABi	AABii	AAC	ABBi	ABBii
C(0)	1	1	1	1	1	1	1
C(1)	6	6	6	6	6	6	6
C(2)	3	3	3	3	4	3	3
C(3)	1	0	1	0	1	1	0
c	11	10	11	10	12	11	10
v	24	22	24	22	26	24	22
$s(c)$	154	130	154	130	180	154	130

	ABC	ACC	BBBi	BBBii	BBC	BCC	CCC
C(0)	1	1	1	1	1	1	1
C(1)	6	6	6	6	6	6	6
C(2)	4	5	3	3	4	5	6
C(3)	1	2	1	0	1	1	2
c	12	14	11	10	12	13	15
v	26	30	24	22	26	28	32
$s(c)$	180	238	154	130	180	208	270

Whenever a diagram is shown in the discussion of a chord pattern, lower-case letters in the figure are the numbers of edges between the points where chord and basic cycle meet; we refer to these sets of edges as *arcs* and the number of edges is the *arc length.* Capitals are chord names. We identify cycles by the chords that they contain: for example, "an XY cycle" will mean one that contains chords X and Y and no others.

Given a chord X, the lengths of the two X cycles will add to $2v + 2$, because they each contain the edge X and every edge of the Hamilton cycle appears once. Similarly, if chords X and Y are of type C, the lengths of the two XY cycles adds to $2v + 4$. If there are two cycles containing chords X, Y and Z, their lengths total $2v + 6$.

Type AAAi

This graph has $v = 24$ vertices. There will need to be 11 cycles, totalling 154 edges. The Hamiltonian cycle and the six one-chord cycles have a total of $v + 3(v + 2) = 102$ edges. Therefore the remaining three cycles have a total of 52 edges.

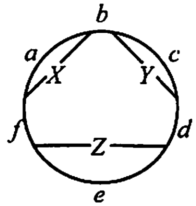

Figure 6: Chord type AAAi

The cycle containing both chords X and Y has $b+d+e+f+2$ edges, the YZ cycle has $a+b+d+f+2$ edges, the XZ cycle has $b+c+d+f+2$ edges, and the XYZ cycle has $b+d+f+3$ edges. So the four cycles together total $3(b+d+f)+(a+b+c+d+e+f)+9$ edges. Since $(a+b+c+d+e+f = 24$, we have $3(b+d+f) = 19$, so $b+d+f$ is not an integer, a contradiction. So there is no uniquely bipancyclic graph with chord pattern AAAi.

Type AAAii

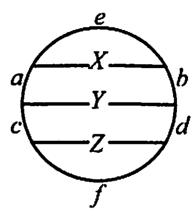

Figure 7: Chord type AAAii

This graph has $v = 22$ vertices. There will need to be 10 cycles, totalling 130 edges. The Hamiltonian cycle and the six one-chord cycles have a total of $v + 3(v + 2) = 94$ edges. Therefore the remaining three cycles have a total of 36 edges.

The cycle containing both chords X and Y has $a+b+2$ edges, the YZ cycle has $c+d+2$ edges, and the XZ cycle has $a+b+c+d+2$ edges. So the three cycles together total $2(a+b+c+d)+6$ edges. This equals 36, so $a+b+c+d = 15$, and $e+f = 7$. But there are cycles of lengths $e+1$ and $f+1$, so e and f must both be odd, which is a contradiction. So there is no uniquely bipancyclic graph — and in fact no bipancyclic graph — with chord pattern AAAii.

Type AABi

The graph will have $v = 24$ vertices; $a+b+c+d+e = 24$. There are

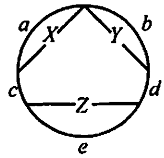

Figure 8: Chord type AABi

11 cycles, of lengths $24, 1+a, 1+b, 1+e, 25-a, 25-b, 25-e, c+d+e+2, b+c+d+2, a+c+d+2$ and $c+d+3$, and for a uniquely bipancyclic graph these must all be different. As the cycles have even length a, b, e are all odd numbers and $c+d$ is also odd. Without loss of generality we can assume $a < b < e$. Then the only way to have a cycle of length 4 is if $a = 3$. To form a 6-cycle, we must have $d+e = 3$ or $b = 5$. In the former case, we must have $b+1 \geq 8$, so $b \geq 7$. If $b = 7$ then $b+1 = 8 = a+c+d$, a repeated cycle length. If $b > 7$ we would have $e \leq b$, a contradiction. So $d+e = 3$ is impossible, and $b = 5$. There are three cases: $e = 7, c+d = 9$, and $25-e = 18 = c+d+e+2$; $e = 9, c+d = 7$, and $e+1 = 10 = a+c+d$; or $e = 11, c+d = 5$, and $e+1 = 12 = b+c+d+2$. Therefore there is no uniquely bipancyclic graph with chord pattern AABi.

Type AABii

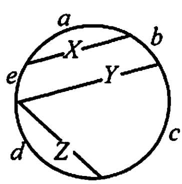

Figure 9: Chord type AABii

This pattern involves 10 cycles (including the Hamiltonian). So, if there is a uniquely bipancyclic graph with this pattern, it has chords of lengths 4, 6, 8, 10, 12, 14, 16, 18, 20, 22. So the number of vertices is 22, and $a+b+c+d+e = 22$. Each of a, b, c, d, e is greater than 0.

Here are the cycle lengths (left column names them; second column shows all chords in the cycle.)

$C1$	--	22		
$C2$	X	$a+1$		
$C3$	X	$b+c+d+e+1$	$=$	$21-a$
$C4$	Y	$c+d+1$		
$C5$	Y	$a+b+e+1$	$=$	$21-c-d$
$C6$	Z	$d+1$		
$C7$	Z	$a+b+c+e+1$	$=$	$21-d$
$C8$	XY	$b+e+2$	$=$	$24-a-c-d$
$C9$	XZ	$b+c+e+2$	$=$	$24-a-d$
$C10$	YZ	$c+2$		

Since all cycles must be even, we must have

a is odd (from $C2$)

d is odd (from $C6$)

$c+d$ is odd (from $C4$) so c is even (also follows from $C10$)

$b+e$ is even (from $C8$)

Also, since all cycles are length 4 or greater,

$a \geq 3$ (from $C2$)

$d \geq 3$ (from $C6$)

and since no two cycles are the same length,

$a \neq 3$ (from $C2$ and $C6$).

Now there must be a cycle of length 4. Candidates are

$C2$, implying $a=3$

$C6$, implying $d=3$

$C8$, implying $b=e=1$

$C10$, implying $c=2$

Two other cases,

$C4$, implying $c=2, d=1$

$C5$, implying $a=b=d=1$

are ruled out by the facts that a and d are each at least 3.

C2: This does not work, because both $C7$ and $C9$ would be length $21-d$.

C6: This does not work, because both $C3$ and $C9$ would be length $21-a$.

C8: We assume $b=e=1$ Also $a \geq 5$ and $d \geq 5$ (both are odd, ≥ 3, and if either equalled 3 we have two cycles length 4). Similarly $c \geq 4$. Cycle lengths are

$$\begin{array}{lllll} C1: 22, & C2: a+1, & C3: 21-a & C4: c+d+1, & C5: 21-c-d \\ C6: d+1, & C7: 21-d, & C8: 4, & C9: c+4, & C10: c+2. \end{array}$$

As $a+c+d = 20$, we can get upper bounds on a, c and d by using the given lower bounds. For example, $d \geq 5$ and $c \geq 4$ imply $a \leq 11$, so $5 \leq a \leq 11$; similarly $5 \leq d \leq 11$ and $4 \leq c \leq 10$. (Actually, as $a \neq d$, $a + d \geq 12$ and $c \leq 8$.

So how do we get a cycle of length 20? The only case not immediately eliminated is $C4$, but that would need $c + d = 19$, so $a = 1$ ...impossible.

C10: We assume $c = 2$. Again $a \geq 5$ and $d \geq 5$ (both are odd, ≥ 3, and if either equalled 3 we have two cycles length 4). Also $b + e \geq 4$. Cycle lengths are

$$\begin{array}{lllll} C1: 22, & C2: a+1, & C3: 21-a & C4: d+3, & C5: 19-d \\ C6: d+1, & C7: 21-d, & C8: 22-a-d, & C9: 24-a-d, & C10: 4. \end{array}$$

The only way to get a cycle of length 20 is if $d = 17$ or 19. But $a+b+c+e \geq 9$, so $d \leq 13$. So there is no uniquely bipancyclic graph with chord pattern AABii.

Type AAC

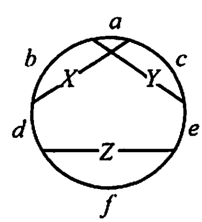

Figure 10: Chord type AAC

A uniquely bipancyclic graph will contain 12 cycles, $v = 26$ and the chords total 180 edges. The cycles with 0 and 1 chord and two XY cycles total 140 edges; XZ has $2 + b + c + e$, YZ has $2 + b + d + e$, and XYZ has $3 + a + d + e$. So $7 + a + b + c + 3d + 3e = 40$. Using the fact that $a + b + c + d + e + f = 26$, we have $33 + 2(d + e) - f = 40$, or $2(d + e) - f = 7$. Therefore $2(d + e + f) = 7 + 3f$, and $2(a + b + c) = 2(a + b + c + d + e + f) - 2(d + e + f) = 52 - 7 - 3f = 45 - 3f$. Up to isomorphism we can assume $b \geq c$.

The cycle lengths are

$$\begin{array}{llll} a+b+1 & 27-a-b & a+c+1 & 27-a-c \\ f+1 & 27-f & 2+b+c & 28-b-c \\ 2+c+d+e & 2+b+d+e & 3+a+d+e & 26 \end{array}$$

So b and c cannot be equal, or there would be two equal cycles. Also b and c are of the same parity, opposite to that of a. Moreover, none of the 1-chord cycle lengths $a+b+1, a+c+1$ and $f+1$ can equal 14, as the other cycle associated with the same chord would also be of length 14.

The only possible lengths for a 4-cycle are $a+b+1, a+c+1, f+1$ and $2+b+c$. But $a+b+1=4$ implies $a+c+1<4$, which is impossible, and $2+b+c=4$ implies $b=c=1$, also impossible.

Suppose $a+c+1=4$. Then $2(a+b+c)=45-3f$ becomes $2b+6=45-3f$ or $2b=39-3f$. So b is a multiple of 3.If $b=15$ then $f=3$, which gives another 4-cycle. So $b=3,6,9$ or 12. Clearly $\{a,c\}=\{1,2\}$, but if $c=1$ then $a+b+1=2+b+c$, so $c=2, a=1$, and therefore b is even. There remain two cases:

$b=6, a=1, c=2, f=9, d+e=8$, in which $f+1=2+b+c=10$;

$b=12, a=1, c=2, f=5, d+e=6$, in which $a+b+1=27-a-b=14$.

So there are no examples with $a+c+1=4$.

Suppose $f+1=4$. Then $d+e=5$ and $a+b+c=18$. a must be even and b and c odd. If we avoid cases where $a+b+1$ or $a+c+1$ equals 14, there remain ten cases, all with $f=3$ and $d+e=5$:

$a=2, b=9, c=7$, in which $27-a-c=2+b+c=18$;

$a=2, b=13, c=3$, in which $27-a-b=28-b-c=12$;

$a=4, b=11, c=3$, in which $a+b+1=2+b+c=16$;

$a=4, b=13, c=1$, and no problem arises;

$a=6, b=9, c=3$, in which $a+b+1=28-b-c=16$;

$a=6, b=11, c=1$, in which $a+c+1=2+c+d+e=8$;

$a=8, b=7, c=3$, in which $27-a-b=1+a+c=12$;

$a=8, b=9, c=1$, in which $27-a-b=1+a+c=10$;

$a=10, b=7, c=1$, in which $27-a-b=2+b+c=10$;

$a=14, b=3, c=1$, in which $27-f=28-b-c=24$.

So there is one solution. This gives rise to four uniquely bipancyclic graphs with chord pattern AAC, according as $(d,e)=(1,4),(2,3),(3,2)$ or $(4,1)$. An example is shown in Figure 11.

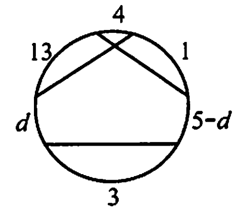

Figure 11: Uniquely bipancyclic graphs of type AAC: $d =$ 1,2,3,4.

Type ABBi

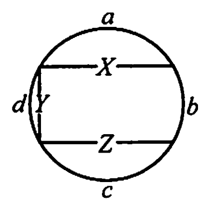

Figure 12: Chord type ABBi

$v = 24$ and the chords total 154 edges. A uniquely bipancyclic graph will contain 11 cycles. The Hamiltonian cycle and the six one-chord cycles have a total of $v + 3(v + 2) = 102$ edges. Therefore the remaining four cycles have a total of 52 edges. The XY, XZ, YZ and XYZ cycles have $(b + c + 2)$, $(b + d + 2)$, $(a + b + 2)$ and $(b + 3)$ edges respectively, for a total of $(a + b + c + d) + 3b + 9$, which equals $3b + 33$ as $a + b + c + d = v$. So $3b = 19$, which is not a multiple of 3 — a contradiction. So there is no uniquely bipancyclic graph with chord pattern ABBi.

Type ABBii

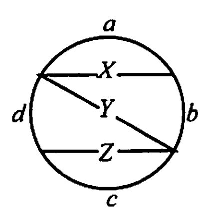

Figure 13: Chord type ABBii

$v = 22$ and the chords total 130 edges. A uniquely bipancyclic graph will contain 10 cycles. The Hamiltonian cycle and the six one-chord cycles have a total of 94 edges. Therefore the remaining four cycles have a total of 36 edges. The XY, XZ and YZ cycles have $(b+2)$, $(b+d+2)$ and $(d+2)$ edges respectively, for a total of $2(b+d)+6$, so $b+d=15$, and the XZ cycle has 17 vertices, an odd number.. So there is no uniquely bipancyclic graph with chord pattern ABBii.

Type ABC

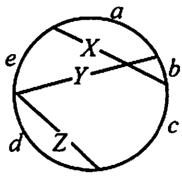

Figure 14: Chord type ABC

A uniquely bipancyclic graph will contain 12 cycles, $v = 26$ and the chords total 180 edges. The cycles with 0 and 1 chord and two XY cycles total 140 edges; the XZ cycle havs $2+c+e$, YZ's has $2+b+c$, and XYZ's has $3+a+c$. So $7+a+b+3c+e=40$. Using the fact that $a+b+c+d+e=26$, we have $33+2c-d=40$, or $2c-d=7$.

The cycle lengths are (Hamilton) 26; (X): $a+b+1$, $c+d+e+1$; (Y): $a+e+1$, $b+c+d+1$; (Z): $d+1$, $a+b+c+e+1$; (XY): $a+c+d+2$, $b+e+2$; (XZ): $c+e+2$; (YZ): $b+c+2$; (XYZ): $a+c+3$. From these we see that $a \neq b+1$ (or the cycle lengths $a+e+1$ and $b+e+2$ would be equal), $a \neq e+1$ (or the cycle lengths $a+b+1$ and $b+e+2$ would be equal), and $d \neq 1$ (or there would be a cycle of length 2); since d is odd, $d \geq 3$ and therefore $c \geq 5$. The only possible cycles of length 4 have lengths $a+b+1$ (which would imply $a=1, b=2$), $a+e+1$ (which would imply $a=1, e=2$), $b+e+2$ (which would imply $b=e=1$), and $d+1$ (so $d=3$).

Case $a=1, b=2$: then $c+d+e=23$; from $2c-d=7$, we get $3c+e=30$. So e is a multiple of 3. As $a+e+1=e+2$ is even, e is even, so $e=6, 12, 18$ or 24. But $c+d \geq 8$, so the only possibilities are $e=6$ and $e=12$. If $e=6$ then $c=8$, so $d=9$, so $d+1=10=b+e+2$, and there are two cycles of length 10. If $e=12$ then $c=6, d=5$, and $a+e+1=14=b+c+d+1$, and there are two cycles of length 14. Neither case is uniquely bicyclic.

Case $a=1, e=2$: $b+c+d=23$, and similarly to the above we deduce $b+3c=30$, b is even, so 6 divides b. The only possibilities are $b=6, c=8, d=9$, whence $d+1=10=b+e+2$, and there are two cycles of length 10, and $b=12, c=6, d=5$, so $a+b+1=14=c+d+e+1$, and there are two cycles of length 14. Neither case is uniquely bicyclic. .

Case $b = e = 1$ is impossible because the cycles of length $a + b + 1$ and $a + e + 1$ would be of the same length.

Case $d = 3$: $c = 5$. Clearly a is even and b and e are odd. Cycle lengths are $26, a + b + 1, e + 9, a + e + 1, b + 9, 4, 24$ (and $a + b + e = 18$), $a + 10, b + e + 2, e + 7, b + 7, a + 8$. Clearly b and e differ by at least 4 (otherwise $\{e+7, b+7, e+9, b+9\}$ must contain a duplication) so $b+e \geq 6$ and therefore $a \leq 12$. If $a = 2$ we have cycles of length 10 and 12, so neither b nor e can equal 3 or 5; the only solution for $b + e = 16$ is that they equal 7 and 9, so they do not differ by at least 4, a contradiction. If $a = 4$ the only possibilities for a cycle of length 6 are the $a + b + 1$ and $a + e + 1$ cycles, with $b = 1$ and $e = 1$ respectively. Both of these work, yielding two uniquely bipancyclic graphs on 26 vertices, shown in Figure 15.

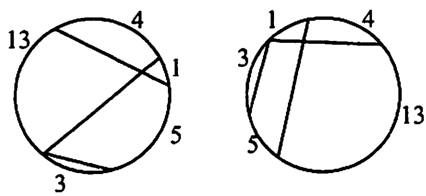

Figure 15: Uniquely bipancyclic graphs of type ABC

If $a = 6$, then $a + b + 1 = b + 7$; if $a = 8$, then $a + b + 1 = b + 9$; so both of these cases yields a duplication. If $a = 10$ there are cycles of length $e + 11$ and $b + 11$, so b and e cannot differ by 4; the only possibility is $\{b, e\} = \{1, 7\}$. Then $b + e + 2 = 10$, and so does either $e + 9$ or $b + 9$. If $a = 10$ then $\{b, e\} = \{1, 7\}$. Then $b + e + 2 = 8$, and so does either $e + 7$ or $b + 7$. So there are no further uniquely bipancyclic graphs of type ABC.

Type ACC

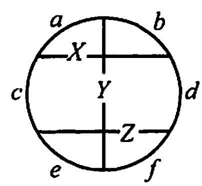

Figure 16: Chord type ACC

A uniquely bipancyclic graph will contain 14 cycles, $v = 30$ and the chords total 238 edges. The cycles with 0 and 1 chord, the two XY cycles, the two YZ cycles and the two XYZ cycles total 230 edges; the XZ cycle has $2+c+d$, so $2+c+d=8$, $c+d=6$. Without loss of generality, we can assume $c \le d$, so $c \le 3 \le d$. Each arc length is at least 1, so the only possible cycles of length 4 are the X cycle of length $(1+a+b)$, the Y cycle of length $(1+a+c+e)$ and the Z cycle of length $(1+e+f)$; all others include at least five pieces (chords and arcs) or the arc of length d and at least two other pieces.

If $(1+a+c+e)=4$, then $a=c=e=1$ and $d=5$. One X cycle has length $1+c+d+e+f=8+f$ and one XY cycle has length $2+a+d+f=8+f$ — a duplication. So there is no uniquely bipancyclic example with $(1+a+c+e)=4$.

If $(1+a+b)=4$, then either $a=1, b=2$, and the YZ cycle of length $2+b+d+e$ is the same length as the XYZ cycle of length $3+a+d+e$; or else $a=2, b=1$, and the YZ cycle of length $2+a+c+f$ is the same length as the XYZ cycle of length $3+b+c+f$ — a duplication in either case. So there is no uniquely bipancyclic example with $(1+a+b)=4$.

If $(1+e+f)=4$ then either $e=1, f=2$, and the XY cycle of length $2+a+d+f$ is the same length as the XYZ cycle of length $3+a+d+e$; or else $e=2, f=1$, and the XY cycle of length $2+b+c+e$ is the same length as the XYZ cycle of length $3+b+c+f$ — a duplication in either case. So there is no uniquely bipancyclic example with $(1+e+f)=4$.

So in no case is there a uniquely bipancyclic graph of chord type ACC.

Type BBBi

This is impossible because it would contain a cycle of length 3.

Type BBBii

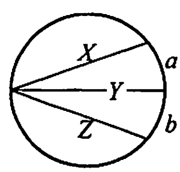

Figure 17: Chord type BBBii

In this case, there are 10 cycles and 22 vertices, so the sum of the numbers of edges in the cycles is $4+6+\ldots+22=130$. There is one cycle of length 22 (the Hamilton circuit), six whose edges add to $3v+6=72$ (the one-chord cycles), and cycles of lengths $a+2$, $b+2$ and $a+b+2$, so the total number of edges in the cycles is $2a+2b+6+22+72=2(a+b)+100$. So $a+b=15$. But $a+2$ and $b+2$ must be even (because the XY and YZ chords are of equal lengths), so a and b must be even — a contradiction.

So there is no uniquely bicyclic graph of type BBBii.

Type BBC

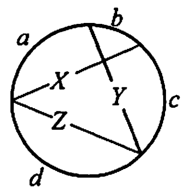

Figure 18: Chord type BBC

Chord type BBC has 12 cycles, 26 vertices and a total of 180 edges among the chords.The Hamilton cycle has 26 edges, the six one-chord cycles have 84 edges in total, and the two XY cycles total 30 edges. The XZ, YZ and XYZ cycles have $c+2$, $a+2$ and $b+3$ edges respectively, so $a+b+c+7-180-26-84-30=40$. So $a+b+c=33$, which is impossible as there are only 26 vertices. Therefore Chord pattern BBC is impossible.

Type BCC

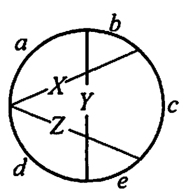

Figure 19: Chord type BCC

A uniquely bipancyclic graph will contain 13 cycles, $v=28$ and the chords total 208 edges. The cycles with 0 and 1 chord and the XY and YZ cycles total 182 edges; the XZ cycle has $2+c$ and the XYZ cycle has $3+b+e$. So $5+b+c+e=208-182=26$, $b+c+e=21$, and $a+d=7$. Without loss of generality we can assume $a<d$, so $a\leq 3$.

If we use the substitutions $(7-a)$ for d and $(21-b-c)$ for e, the lengths of the non-hamiltonian cycles are

$a+b+1$	$29-a-b$	$a+b+c+1$	$29-a-b-c$
8	22	$9+b-a$	$23+a-b$
$23+a-b-c$	$9-a+b+c$	$2+c$	$24-c$

So $a+b$ is odd and c is even. Clearly c cannot equal 2,(there would be two cycles of length 22), whence $b + c \leq 17$; $a + b + c$ cannot equal 21 (two two 22 cycles); and $a - b$ cannot equal ± 1 (two 8s or two 22s).

How can we achieve a cycle of length 4? The only possibilities are: $a + b = 3$, which would imply $a - b = \pm 1$; $a + b + c = 25$ (impossible as $b + c \leq 17$); $a - b = 5$ (impossible as $a \leq 3$); and $b + c = a + 19$ (again, impossible because $b + c \leq 17$). Therefore chord pattern BCC is impossible.

Type CCC

In a type CCC graph, there are six cycles with one chord, totalling $3(v + 2)$ edges, six with two chords, totalling $3(v + 4)$, two with three chords, contributing $v + 6$ edges, and the Hamilton cycle. So the chord lengths add to

$$v + 3(v + 2) + 3(v + 4) + v + 6 = 8v + 24 = 280,$$

as $v = 32$. But the sum must be 270. So no solution is possible.

6 Summary; Further Results

A bipancyclic graph with four chords will contain at least 15 cycles – two containing each chord, at least one for each pair of chords, plus the Hamiltonian cycle. So a uniquely bipancyclic graph with four chords will contain at least 32 vertices. Therefore we have found all uniquely bipancyclic graphs on 30 or fewer vertices.

There are uniquely bipancyclic graphs of orders, 4, 8, 14 and 26 – examples of all of them are shown in Figure 20 – and no other orders smaller than 32.

We have examined the case of 32 vertices, and shown that no examples exist [9]. This work was done using a combination of hand work and computer programs. Subsequently, Peterson *et al* [5] used computers to extend to 56 vertices, and showed that there are six non-isomorphic uniquely bipancyclic graphs on 44 vertices, and no other cases in the range $32 \leq v \leq 56$.

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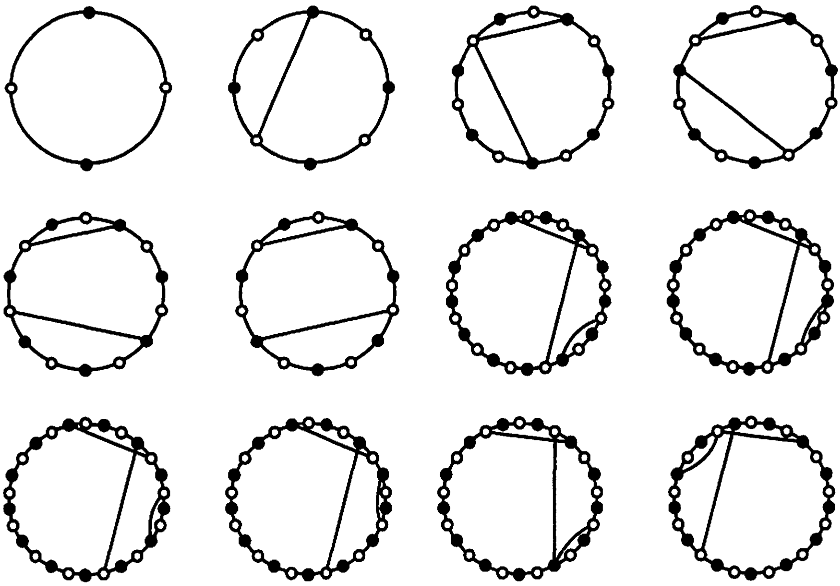

Figure 20: The uniquely bipancyclic graphs of order less than 32

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