New Proof of Adjacency Lemmas of Edge Critical Graphs

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Abstract

A graph G with maximum degree Δ and edge chromatic number $\chi'(G) > \Delta$ is *edge-Δ-critical* if $\chi'(G-e) = \Delta$ for each $e \in E(G)$. In this article we provide new proof of adjacency Lemmas on edge critical graphs such that Vizing's adjacency lemma becomes a corollary of our results.

1 Introduction

In this paper, we consider a finite simple graph G with maximum degree $\Delta(\geq 2)$, edge set E and edge chromatic number $\chi'(G)$. Vizing's Theorem [8] states that the edge chromatic number of a simple graph G is either Δ or $\Delta+1$. A graph G is *class one* if $\chi_e(G) = \Delta$ and is *class two* otherwise. A class two graph G is *critical* if $\chi_e(G-e) < \chi_e(G)$ for each edge e of G. A critical graph G is *edge- Δ-critical* if it has maximum degree Δ.

Vizing [8] conjectured that if G is an edge-Δ-critical graph of edge set E, then $|E| \geq \frac{1}{2}(|V|(\Delta-1)+3)$.

The conjecture has been verified for $\Delta \leq 6$ (see [4],[3],[5],[7]). For $\Delta > 6$, the conjecture is still open. D. Woodall [10] gave a good result on average degree, denote it by q, of edge-Δ-critical graph G where $q \geq \frac{2}{3}(\Delta+1)$. However any nontrivial improvement to his result will require a new adjacency lemma. We introduce a vertex-rotation method for studying adjacency properties for edge-coloring problems, with it we conclude a generalized version of Vizing's Adjacency Lemma which is stronger than those known to the author.

2 Adjacency Lemmas

Let ϕ be the Δ-edge coloring of $G - xw$, $\phi(v)$ be the set of colors of the edges adjacent to the vertex v under edge coloring ϕ. A vertex v sees color j if v is adjacent to an edge colored by j. Denote by $P_{j,k}(v)_\phi$ the (j,k)-bicolored path starting at v under edge coloring ϕ, or by $P_{j,k}(v)$ if there is no confusion.

Through this paper, without loss of generality, under coloring ϕ, edges incident with x in $G - xw$ are colored by $1, 2, ...d - 1$, while those incident with w are colored by $\Delta - k + 2, ...\Delta$ where $d = d(x), k = d(w)$.

Let C_1 be the set of colors present at only one of x, w and C_2 be the set of colors present at both. Further let C_{11} be the set of colors present only at x, and C_{12} be the set of colors present only at w. We may assume that $C_1 = C_{11} \cup C_{12} = \{1, \cdots, \Delta - k + 1\} \cup \{d, d + 1, \cdots, \Delta\}$ and $C_2 = \{\Delta - k + 2, \cdots, d - 1\}$, where $C_2 = \emptyset$ if $d + k = \Delta + 2$, $|C_1| = 2\Delta - d - k + 2, |C_2| = d + k - \Delta - 2$.

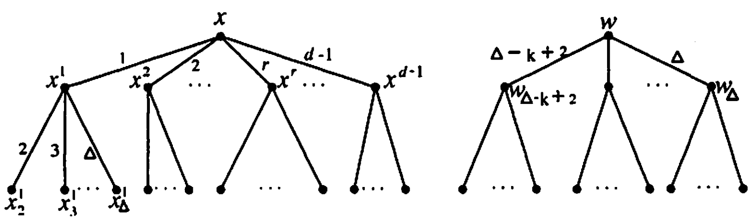

Figure 1: Δ-edge coloring ϕ of G-xw exhibited at $N(x) \cup N(w)$

In order to give improved adjacency properties on the i-vertex, we provide some Facts which are given in [7] and [6].

Fact 1. *For each neighbor w_j of w in $G - xw$ where $\phi(ww_j) = j$ present only at w, then w_j must see each color in C_1.*

Fact 1 will be often used in the discussion through this paper without notifying.

Fact 2. *For each neighbor x^i of x where $\phi(xx^i) = i$ present only at x, then x^i must see each color in C_1. Note that x has at least $\Delta - k + 1$ such x^i.*

Fact 3. *For $w_b \in N(w)$ where $b \in C_2$, if one of such $w_b \in N(w)$ misses a color in C_1, then we could assume that it misses color 1. Note that we can only assure there is one such vertex w_b.*

Fact 4 *Let x and w be adjacent in Δ-critical graph G with $d(x) = d, d(w) = k$. $G - xw$ has a Δ-edge coloring ϕ. Let $xx^a y$ be a path in $G - xw$ where $\phi(xx^a) = a \in C_{11}$ and $y \neq w$ such that $\phi(x^a y) \in C_1$. Then y must see each color in C_1, that is, $d(y) \geq 2\Delta - d - k + 2$.* Note that there are $2\Delta - d - k + 1$ such $y's$, and some of them may be adjacent to vertices in $N(x)$.

Fact 5 $|\phi(x) \cap \phi(w)| = d + k - \Delta - 2$ where ϕ is a Δ-coloring of $G - xw$ defined as above.

Denote $d + k - \Delta - 2$ by L. As $d(w) = k$, we have $d - 1 \geq |\phi(x) \cap \phi(w)| \geq L$. Suppose that $|\phi(x) \cap \phi(w)| > L$. Then $|\phi(x) \cup \phi(w)| = |\phi(x)| + |\phi(w)| - |\phi(x) \cap \phi(w)| < d - 1 + k - 1 - L = \Delta$, therefore, $C \setminus \{\phi(x) \cup \phi(w)\} \neq \emptyset$ since $|C| = \Delta$. Let $\alpha \in C \setminus \{\phi(x) \cup \phi(w)\}$. Now ϕ could be extended to a Δ-coloring of G by coloring xw with α. A contradiction. Hence $|\phi(x) \cap \phi(w)| = L$.

For $w \in V(G)$, we let $C_w = \{\alpha \in C | \alpha \textit{ is missing at } w\}$

The following lemma (proved in Andersen[1]) provides a foundational construction on property of edge coloring of edge critical graph.

Lemma 2.1. *[1] Let xw be an edge of G, and let ϕ be a coloring of $G - xw$. Then $C_w \cap C_x = \emptyset$. Furthermore, for any pair of distinct vertices (w, y) at $N(x)$, $C_w \cap C_y = \emptyset$.*

Proof. If $C_w \cap C_x \neq \emptyset$, let $\alpha \in C_w \cap C_x$. We color edge xw by α which gives a Δ-coloring of G, a contradiction. Suppose now that $C_w \cap C_y \neq \emptyset$ for two distinct vertices $w, y \in N(x)$. Let $\beta \in C_w \cap C_y$. Note that $C_x \neq \emptyset$ by the assumption that G is critical, let α be any color missed by x. $\alpha \neq \beta$ by beginning part of the proof. Consider the subgraph $G(\alpha, \beta)$ of G induced by the edges of G colored by α or β. Notice that x has degree one in $G(\alpha, \beta)$ since it is missing α, and the same holds for w and y since they are missing β. Notice that, all x, w, y cannot be in the same component of $G(\alpha, \beta)$. We assume that x and w are on two different component of $G(\alpha, \beta)$. Then, swapping colors (α, β) in one component containing x, and under current coloring, x and w are both missing color α, which contradicts the fact that $C_w \cap C_x = \emptyset$. This proves that $C_w \cap C_y = \emptyset$ for any pair (w, y) of distinct vertices in $N(x)$. □

Corollary 2.2. *For a Δ-edge coloring ϕ of G-xw with $d(x) = d, d(w) = k$ (see Figure 1), then $C_{w_i} \cap C_{w_j} = \emptyset$ where $i, j \in C_{12}$.*

Proof. It suffices to notice that the $C_{w_i} \cap C_1 = \emptyset$, $C_{w_j} \cap C_1 = \emptyset$ $i, j \in C_{12}$, and if there is $\alpha \in C_{w_i} \cap C_{w_j} \neq \emptyset (\alpha \in C_2)$, then not all w_i, w_j, w_α are in the same component of $G(1, \alpha)$. By argument in previous lemma, we have the desired result. □

Lemma 2.3. *For a Δ-edge coloring ϕ of G-xw with $d(x) = d, d(w) = k$ (see Figure 1), If there is a vertex $w_\beta \in N(w)(\beta \in C_{12})$ misses a color $r_1(\in C_2)$, then (i) $C_{w_{r_1}} \cap C_1 = \emptyset$, and (ii) there is a vertex $w_{r_1^*} \in N(w)(r_1^* \in C_2)$ with $C_{w_{r_1^*}} \cap \{C_1 \cup C_2\} = \emptyset$, that is $d(w_{r_1^*}) = \Delta$.*

Proof. (i) (1) We claim that $C_{w_{r_1}} \cap C_{11} = \emptyset$.

Suppose that $C_{w_{r_1}} \cap C_{11} \neq \emptyset$. Let $\alpha \in C_{w_{r_1}} \cap C_{11}$. Vertices w_r and w_β must be in the same component of $G(\alpha, r_1)$. Otherwise exchange α and r_1 in the component containing w_β, under current coloring, vertices w and w_β are both missing color α which contradicts to the fact that $C_w \cap C_\beta = \emptyset$. So w_{r_1} and w are in one component of $G(\alpha, r_1)$ which means that w_{r_1} sees α, a contradiction.

(2) We claim that $w_{r_1} \cap C_{12} = \emptyset$.

Otherwise let $j \in C_{w_{r_1}} \cap C_{12}$. Notice that x has degree one in $G(\alpha, j)$ since it misses j and same hold for w_{r_1} since it is missing j. vertex w_j has degree 2 in $G(\alpha, j)$ since it sees both α and j. Suppose that x and w_{r_1} are in same component of $G(\alpha, j)$, we exchange colors (α, j) in a component containing x and w_{r_1}, under current coloring, x and w are both missing α, which contradicts the fact that $C_w \cap C_x = \emptyset$. So x and w_{r_1} are in different component of $G(\alpha, j)$, swapping colors (α, j) in the component containing w_{r_1} which implies w_{r_1} missing α which contradicts the result of (1).

From previous two claims, we have that $C_{w_{r_1}} \cap C_1 = \emptyset$.

(ii) If $C_{w_{r_1}} \cap C_2 = \emptyset$, then $d(w_{r_1}) = \Delta$, we have desired result. So we assume that $C_{w_{r_1}} \cap C_2 \neq \emptyset$. Let $r_2 \in C_{w_{r_1}} \cap C_2$. We consider vertex $w_{r_2} \in N(w)$ where $\phi(ww_{r_2}) = r_2$.

(3) We claim that $C_{w_{r_2}} \cap C_1 = C_{w_{r_2}} \cap \{C_{11} \cup C_{12}\} = \emptyset$.

First suppose that $C_{w_{r_2}} \cap C_{11} \neq \emptyset$, let $\alpha \in C_{w_{r_2}} \cap C_{11}$. We re-color edge ww_{r_1} and ww_{r_2} by r_2 and α respectively. Now color r_1 present only at x, so by Claim D, vertex w_β sees r_1 which contradicts our assumption that w_β is missing r_1. So vertex w_{r_2} sees each color present at x only. Now suppose that $C_{w_{r_2}} \cap C_{12} \neq \emptyset$, let $\eta \in C_{w_{r_2}} \cap C_{12}$. By Fact 1, vertices x and w_{r_2} are not in one component of $G(\eta, 1)$. We exchange colors $(\eta, 1)$ in a component containing w_{r_2} which doesn't effect colors of edges incident with x and w. Under current coloring, w_{r_2} misses color 1 which contradicts the fact that $C_{w_{r_2}} \cap C_{11} = \emptyset$.

(4) If $C_{w_{r_2}} \cap C_2 = \emptyset$, then $d(w_{r_2}) = \Delta$, we have desired result. Hence we assume that $C_{w_{r_2}} \cap C_2 \neq \emptyset$, let $r_3 \in C_{w_{r_2}} \cap C_2$. Consider vertex w_{r_3}

where $\phi(ww_{r_3}) = r_3$. By using the same argument as that for vertex w_{r_2}, we have $C_{w_{r_3}} \cap C_1 = \emptyset$. If $C_{w_{r_3}} \cap C_2 = \emptyset$, then $d(w_{r_3}) = \Delta$, our result holds. So we assume that w_{r_3} misses a color r_4. By repeating above discussion up to $|C_2|$ steps, we obtain a vertex sequence $[w_{r_1}, w_{r_2}, \cdots, w_{r_s}]$ such that either (a) $d(w_{r_s}) = \Delta$ or (b) ww_{r_i} misses color $r_{i+1}(i = 1, \cdots, s)$ and ww_{r_s} misses r_1 (Note that, by the same argument used in Lemma 2.1, $C_{w_{r_i}} \cap C_{w_{r_j}} = \emptyset$, hence, vertex w_{r_s} must miss color r_1, not any color $r_i(i > 1)$. Since otherwise, $C_{w_{r_s}} \cap C_{w_{r_{i-1}}} = r_i \neq \emptyset$).

Now we claim that (a) must be true. Otherwise (b) holds. Notice that path $P_{\Delta,r_1}(w_\beta)$ must pass through w and ends at x. Otherwise, we swap colors (r_1, Δ) along a path starts at x which causes vertex w_β seeing r_1(since under current coloring, $r_1 \in C_{12}$ and $C_{w_\beta} \cap C_{12} = \emptyset$), a contradiction.

We re-color ww_{r_i} by $r_{i+1}(i = 1, \cdots, s)$ and ww_{r_s} by r_1 respectively, and denote current coloring by ϕ^*. Under ϕ^*, exchange colors β, r_1 along path starting at x and denote updated coloring by ϕ^{**}, under ϕ^{**}, color r_1 present at x only, so by (i), w_β must see r_1, a contradiction.

Hence $d(w_{r_s}) = \Delta$. This terminates the proof.

□

Lemma 2.4. *For a Δ-coloring ϕ of $G-xw$ defined earlier, let $Q = \{w_n; n \in C_{12}(n \geq d), C_{w_n} \cap C_2 \neq \emptyset\}$ and $S = \{w_s; s \in C_2$ and $d(w_s) = \Delta\}$. Then we have following: $|Q| \leq |S|$.*

Proof. Let $w_n \in Q$ miss a color $r \in C_2$, by Lemma 2.3 (i), each $w_j \in N(w) \setminus \{w_n\}$ where $(j \geq d)$ must see color r. For each such $w_n \in Q$, through previous lemma, there exists a Δ-vertex $w_{s_n} \in S$.

Next, we claim that $|Q| \leq |S|$.

It suffices to show that if there are two different vertices $w_n, w_{n'} \in Q$, w_n and $w_{n'}$ miss color r and r' in C_2 respectively, then there must exist two different Δ-vertices w_{s_r} and $w_{s_{r'}}$ in S. By Lemma 2.1, we have that $r \neq r'$. By previous lemma, for each of $w_n, w_{n'}$, there exists a corresponding vertex-sequence V_1 and V_2 for $w_n, w_{n'}$ respectively. Let V_1 be $w_{r_1}, \cdots, w_{r_s}$ with $r_1 = r$; and V_2 be $w_{r'_1}, \cdots, w_{r'_{s'}}$ with $r'_1 = r'$, where $d(w_{r_s}) = d(w_{r'_{s'}}) = \Delta$. Each vertex in $V_i(i = 1, 2)$ sees each color in C_1. We claim that $w_{r_s} \neq w_{r'_{s'}}$. Otherwise, suppose that $w_{r_s} = w_{r'_{s'}}$, Through our assumption, $V_1 \cap V_2 \neq \emptyset$. We choose a vertex $w_{r_{i+1}}(= w_{r'_{j'+1}}) \in V_1 \cap V_2$ such that $w_{r_t} \neq w_{r'_{t'}}$ for all $t \leq i, t' \leq j'$, but $w_{r_{i+1}} = w_{r'_{j'+1}}$. For the sake of convenience, denote $w_{r_i}, w_{r'_{j'}}$ by w_p, w_q respectively. Each of w_p, w_q sees colors in C_1, and misses same color r^* where $r^* = r_{i+1} = r'_{j'+1}$ due to the fact that $w_{r_{i+1}} = w_{r'_{j'+1}}$.

We consider two paths $P_{1,r^*}(w_p), P_{1,r^*}(w_q)$ where $r^* \in C_2$. Note that at least one of those two paths will not pass through w, without loss of generality, let $P_{1,r^*}(w_p)$ doesn't pass through w. Thus we swap colors $(1, r^*)$ along $P_{1,r^*}(w_p)$ which doesn't affect colors incident with x and w. Under current coloring, w_p misses color 1, a contradiction. This contradiction leads us to that $|Q| \leq |S|$.

□

Note that $|C_x| = |C_{12}| = |w_j, w_j \in N(w), j \geq d| = \Delta - d(x) + 1$, and $|C_{11}|+|C_{12}| \geq 2$, through previous two lemmas and Fact 5, Vizing's Lemma becomes our corollary.

Corollary 2.5 (Vizing [9]). *If xw is an edge of a Δ-critical graph G, then w has at least $(\Delta - d(x) + 1)$ Δ-neighbors. Any vertex of G has at least two Δ-neighbors.*

By using this vertex-rotation method, with more sophistically discussion, we also generalized adjacency lemma obtained by R. Luo and Y. Zhao [7].

Lemma 2.6. *[6] For a Δ-edge coloring ϕ of G-xw (see Figure 1), $d(x) = d, d(w) = k$ and $|C_2| = d + k - \Delta - 2$. If the number of $(\leq \Delta - \frac{|C_2|}{2})$-neighbors of w is $|C_2| - 1$ or $|C_2|$(a $u \in N(w)$ is called a $(\leq s)$-neighbor of w if $d(u) \leq s$), then there are $\Delta - k + 1 + \lfloor \frac{1}{2}|C_2| \rfloor$ neighbors x^α of x satisfying: $x^\alpha \neq w$; x^α is adjacent to at least $2\Delta - d - k + 1 + \lfloor \frac{1}{2}|C_2| \rfloor$ vertices y different from x with degree at least $2\Delta - d - k + 2 + \lfloor \frac{1}{2}|C_2| \rfloor$.*

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