THE STRONG ADMISSIBILITY OF FINITE GROUPS: AN UPDATE

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ABSTRACT. For a finite group G, a bijection $\theta\colon G \to G$ is a *strong complete mapping* if the mappings $g \mapsto g\theta(g)$ and $g \mapsto g^{-1}\theta(g)$ are both bijections. A group is *strongly admissible* if it admits strong complete mappings. Strong complete mappings have several combinatorial applications. There exists a latin square orthogonal to both the multiplication table of a finite group G and its normal multiplication table if and only if G is strongly admissible. The problem of characterizing strongly admissible groups is far from settled. In this paper we will update progress towards its resolution. In particular, we will present several infinite classes of strongly admissible dihedral and quaternion groups and determine all strongly admissible groups of order at most 31.

1. INTRODUCTION

For a group G, a bijection $\theta\colon G \to G$ is a *complete mapping* of G if the mapping $x \mapsto x\theta(x)$ is a bijection, an *orthomorphism* of G if the mapping $x \mapsto x^{-1}\theta(x)$ is a bijection, and a *strong complete mapping* of G if it is both a complete mapping and an orthomorphism of G. Note that a bijection $\theta\colon G \to G$ is a complete mapping of G if and only if the mapping $x \mapsto x\theta(x)$ is an orthomorphism of G, and an orthomorphism of G if and only if the mapping $x \mapsto x^{-1}\theta(x)$ is a complete mapping of G. We say that a group is *admissible* if it admits complete mappings (equivalently if it admits orthomorphisms), and *strongly admissible* if it admits strong complete mappings. A strong complete mapping is *normalized* if it fixes the identity. If θ is a strong complete mapping of G, then the mapping $\theta'\colon x \mapsto x\theta(x)\theta(1)^{-1}$ is a normalized strong complete mapping of G. Thus a group is strongly admissible if and only if it admits normalized strong complete mappings. Here are two simple examples of strong complete mappings. If $\gcd(|G|, 6) = 1$, then $\theta\colon x \mapsto x^2$ is a strong complete mapping as $x \mapsto x^{-1}\theta(x) = x$, $x \mapsto \theta(x) = x^2$, and $x \mapsto x\theta(x) = x^3$ are all bijections. If $q \geq 4$ and $a \neq 0, \pm 1$, then $\theta\colon x \mapsto ax$ is a strong complete mapping of $GF(q)^+$ as $x \mapsto \theta(x) - x = (a-1)x$, $x \mapsto \theta(x) = ax$, and $x \mapsto \theta(x) + x = (a+1)x$ are all bijections.

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Admissible, finite groups have been characterized. In 1955 Hall and Paige [8] proved that finite groups with nontrivial, cyclic Sylow 2-subgroups are not admissible. They conjectured the converse. In 2009 Wilcox [15] proved that any minimal counterexample to the Hall-Paige conjecture must be the Tits group or a sporadic simple group and in 2009 Evans [5] proved that the only possible minimal counterexample is J_4. Bray [3] showed that J_4 is not a minimal counterexample, thus completing the proof of the Hall-Paige conjecture.

In this paper we will update the 2013 survey by Evans [7] of work done on the strong admissibility of finite groups. As strongly admissible groups are also admissible, Hall and Paige's theorem establishes that some finite groups cannot be strongly admissible.

Theorem 1. *If the Sylow* 2 *subgroup of a finite group G is nontrivial and cyclic, then G is not strongly admissible.*

In 1990 Evans [4] showed that the structure of the Sylow 3-subgroup also plays a role in determining the strong admissibility of finite groups.

Theorem 2. *If a finite group G has a nontrivial, cyclic Sylow* 3*-subgroup that is a homomorphic image of G, then G is not strongly admissible.*

For finite abelian groups strong admissibility is completely determined by the structure of the Sylow 2-subgroups and the Sylow 3-subgroups: this was proved by Evans [6] in 2012.

Theorem 3. *A finite abelian group with a trivial or noncyclic Sylow* 2*-subgroup and a trivial or noncyclic Sylow* 3*-subgroup, is strongly admissible.*

Very little is known about the strong admissibility of nonabelian groups. If the order of the group is relatively prime to 6, then the group is strongly admissible, and a strong complete mapping of the dihedral group of order 12 was found by Shieh, Hsiang, and Hsu [14]. We will add to the list of known strongly admissible groups by proving that, if $\gcd(m, 6) = 1$, then the dihedral groups of orders $4m$, $12m$, $16m$, and $24m$ are strongly admissible, and the quaternion groups of orders $16m$ and $24m$ are strongly admissible.

It should be noted that, for nonabelian groups, there is no proof in the literature that if H is a normal subgroup of G and H and G/H are both strongly admissible, then G is strongly admissible: this result was erroneously cited by Evans [6, 7].

In 2013 Evans [7] conjectured that a finite group is strongly admissible if its Sylow 2-subgroup is trivial or noncyclic and its Sylow 3-subgroup is also trivial or noncyclic. In the same paper he also made the stronger conjecture that a finite group is strongly admissible if its Sylow 2-subgroup is trivial or noncyclic and its Sylow 3-subgroup is trivial, noncyclic, or nontrivial, cyclic, and not a homomorphic image. We will prove both conjectures false. Embarassingly, our counterexamples are the smallest possible counterexamples: the dihedral group of order 8 and the quaternion group of order 8. We will also determine the strong admissibility of all groups of order at most 31: this yields no further counterexamples to these conjectures.

Strong complete mappings have been used in constructing group sequencings [1], Knut Vic designs [9, 10], strong starters [11], solutions to the toroidal n-queens problem [12], and check digit systems [13].

2. Strong complete mappings and latin squares

A *latin square* of order n is an $n \times n$ array in which each symbol from a symbol set of order n appears exactly once in each row and once in each column. Throughout this section G will denote a group of order n, all latin squares will be of order n, and the rows and columns of latin squares will be indexed by the elements of G. There are two latin squares of particular interest: the *Cayley table* of G, which we will denote by M, which has (g,h)th entry equal to gh; and the *normal multiplication table* of G, which we will denote by N, which has (g,h)th entry equal to gh^{-1}.

Two latin squares of the same order are *orthogonal* if, when superimposed, each ordered pair of symbols appears exactly once. Orthomorphisms can be used to construct latin squares orthogonal to M. If $\theta\colon G \to G$ is a bijection and M_θ is the latin square with (g,h)th entry $g\theta(h)$, then M_θ is orthogonal to M if and only if θ is an orthomorphism of G. In fact, there exists a latin square orthogonal to M if and only if G is admissible. The strong admissibility of G determines the existence of latin squares orthogonal to both M and N?

Theorem 4. *There exists a latin square orthogonal to both M and N if and only if G is strongly admissible.*

Proof. Let L be a latin square orthogonal to both M and N, pick a, an entry of L, and define $\theta\colon G \to G$ by $\theta(h) = g^{-1}$ if the (g,h)th entry of L is a. Let (g_i, h_i), $i = 1, \ldots, n$ be the cells of L with entry a. As L is orthogonal to M, the corresponding cells of M form a transversal of M. Thus $\theta(h_i)^{-1}h_i = (h_i^{-1}\theta(h_i))^{-1}$, $i = 1, \ldots, n$ are all distinct and so θ is an orthomorphism of G. As L is orthogonal to N, the corresponding cells of N form a transversal of N. Thus $\theta(h_i)^{-1}h_i^{-1} = (h_i\theta(h_i))^{-1}$, $i = 1, \ldots, n$ are all distinct and so θ is a complete mapping of G. It follows that θ is a strong complete mapping of G.

Let θ be a strong complete mapping of G and let M_θ be the latin square with (g,h)th entry $g\theta(h)$. M_θ is a latin square orthogonal to both M and N. □

For a latin square L a *transversal* of L is a set of cells, one from each row, one from each column, each symbol of L appearing exactly once. For a latin square L the *kth generalized left diagonal* of L with respect to G consists of the (kg, g) cells of L as g runs through the elements of G, and the *kth generalized right diagonal* of L with respect to G consists of the $(g, g^{-1}k)$ cells of L as g runs through the elements of G. A latin square with rows and columns indexed by the elements of a group G is a *generalized Knut Vic design* over G if each of its generalized left and right diagonals with respect to G is a transversal. A generalized Knut Vic design over $\mathbb{Z}_n$ is called a *Knut Vic design.*

Theorem 5. *There exists a generalized Knut Vic design over G if and only if G is strongly admissible.*

Proof. Let θ be a strong complete mapping of G and define M_θ to be the latin square with (g, h)th entry $g\theta(h)$. The entries on the kth generalized left diagonal are $k(g\theta(g))$, $g \in G$. It follows that each generalized left diagonal of M_θ is a transversal of M_θ. The entries on the kth generalized right diagonal are $g\theta(g^{-1}k) = k((g^{-1}k)^{-1}\theta(g^{-1}k)))$, $g \in G$. It follows that each generalized right diagonal of M_θ is also a transversal of M_θ. Hence M_θ is a generalized Knut Vic design over G.

Next, suppose that L is a generalized Knut Vic design over G. As every cell in the kth generalized right diagonal of M has entry k, L is orthogonal to M, and as every cell in the kth generalized left diagonal of N has entry k, L is orthogonal to N. Hence, by Theorem 4, G is strongly admissible. □

Theorems 4 and 5 both characterize generalized Knut Vic designs.

Corollary 1. *A latin square L is a generalized Knut Vic design over G if and only if L is orthogonal to both M and N.*

3. Quotient group constructions

In the literature there is no general quotient group construction, that is no proof that, if H is normal in G and H and G/H are both strongly admissible, then G is strongly admissible. Such a result was incorrectly cited in 2012 and 2013 by Evans [6, 7]. We will give some quotient group constructions in this section. The first of these, for abelian groups, was proved by Horton [11] and Evans [4], independently in 1990.

Theorem 6 (Evans and Horton, 1990). *If G is an abelian group, and H is a subgroup of G, and H and G/H are both strongly admissible, then G is strongly admissible.*

The characterization of strongly admissible abelian groups renders Theorem 6 moot. For general groups, it is easy to establish the strong admissibility of direct products of strongly admissible groups.

Theorem 7. *If G and H are both strongly admissible, then $G \times H$ is strongly admissible.*

Proof. Let θ be a strong complete mapping of G and ϕ a strong complete mapping of H. Then $\theta \times \phi$: $(g,h) \mapsto (\theta(g), \phi(h))$ is a strong complete mapping of $G \times H$. □

Let H be a normal subgroup of G and D a system of distinct coset representatives for H in G. We will say that D is *strongly admissible* if there exist bijections θ, δ, η: $D \to D$ for which $d\theta(d)H = \eta(d)H$ for all $d \in D$ and $d^{-1}\theta(d)H = \delta(d)H$ for all $d \in D$. We will call θ a *strong complete mapping* of D. Note that D being strongly admissible is equivalent to G/H being strongly admissible.

Theorem 8. *Let H be a normal subgroup of G and D a system of distinct coset representatives for H in G. If D is strongly admissible and there exists a strong complete mapping ϕ of H for which the mapping $h \mapsto h^{-1}d^{-1}\phi(h)d$ is a bijection for all $d \in D$, then G is strongly admissible.*

Proof. Let θ be a strong complete mapping of D and set $\theta'(dh) = \phi(h)\theta(d)$. θ' is a bijection $G \to G$ as are the mappings

$$dh \mapsto (dh)\theta'(dh) = (dh\phi(h)d^{-1})(d\theta(d))$$

and

$$dh \mapsto (dh)^{-1}\theta'(dh) = (h^{-1}(d^{-1}\phi(h)d))(d^{-1}\theta(d)).$$

It follows that θ' is a strong complete mapping of G. □

Note that, in the statement of Theorem 8, as ϕ is a bijection, the mapping $h \mapsto h^{-1}d^{-1}\phi(h)d$ is a bijection for $d \in D$ if and only if the mapping $h \mapsto d^{-1}\phi(h)d$ is an orthomorphism of H. An immediate corollary to Theorem 8.

Corollary 2. *If $Z(G)$ and $G/Z(G)$ are both strongly admissible, then G is strongly admissible.*

There are many variants of Theorem 8 including variants in which H is not a normal subgroup of G and D is a dual system of coset representatives for H in G. Alas, there is, as yet, no general result.

4. Dihedral and quaternion groups

We will use $D_{4k} = \langle a, b \mid a^{2k} = b^2 = 1, ab = ba^{-1} \rangle$ to denote the dihedral group of order $4k$ and $Q_{4k} = \langle a, b \mid a^{2k} = 1, b^2 = a^k, bab^{-1} = a^{-1} \rangle$ to denote the quaternion group of order $4k$. Note that D_{4k} is often denoted D_{2k}. In this section we will prove that D_8 and Q_8 are not strongly admissible and that, if $\gcd(m, 6) = 1$, then D_{4m}, D_{12m}, D_{16m}, D_{24m}, Q_{16m}, and Q_{24m} are strongly admissible.

Let us begin by noting that, as a direct consequence of Theorem 8, we obtain the following quotient group construction for dihedral and quaternion groups.

Lemma 1. *Let $G = D_{2k}$ or Q_{2k}, $H \cong \mathbb{Z}_m$ a subgroup of $\langle a \rangle$, where $m|k$ and $\gcd(m,6) = 1$. If G/H is strongly admissible, then G is strongly admissible.*

Proof. Let D be a system of distinct coset representatives for H in G and define $\phi\colon H \to H$ by $\phi(h) = h^2$. ϕ is a strong complete mapping of H and for $d \in D$, either $d^{-1}\phi(h)d = h^2$ for all $h \in H$ or $d^{-1}\phi(h)d = h^{-2}$ for all $h \in H$. In either case the mapping $h \mapsto d^{-1}\phi(h)d$ is an orthomorphism of H. □

Any mapping $\theta\colon D_{4k} \to D_{4k}$ or $\theta\colon Q_{4k} \to Q_{4k}$ can be expressed as

$$\theta(x) = \begin{cases} a^{\alpha(i)} & \text{if } x = a^i, i \in A, \\ ba^{\beta(i)} & \text{if } x = a^i, i \in \bar{A}, \\ a^{\gamma(i)} & \text{if } x = ba^i, i \in B, \\ ba^{\delta(i)} & \text{if } x = ba^i, i \in \bar{B}, \end{cases}$$

for some partitions $\{A, \bar{A}\}$ and $\{B, \bar{B}\}$ of $\mathbb{Z}_{2k}$ and some mappings $\alpha\colon A \to \mathbb{Z}_{2k}$, $\beta\colon \bar{A} \to \mathbb{Z}_{2k}$, $\gamma\colon B \to \mathbb{Z}_{2k}$, and $\delta\colon \bar{B} \to \mathbb{Z}_{2k}$. We will call α the *aa-mapping* for θ, β the *ab-mapping* for θ, γ the *ba-mapping* for θ, and δ the *bb-mapping* for θ. We will call A the *aa-set* for θ, $\bar{A}$ the *ab-set* for θ, B the *ba-set* for θ, and $\bar{B}$ the *bb-set* for θ. A characterization of the *aa*-sets, *ab*-sets, *ba*-sets, *bb*-sets, *aa*-mappings, *ab*-mappings, *ba*-mappings, and *bb*-mappings that correspond to strong complete mappings of dihedral groups or strong complete mappings of quaternion groups is given next.

Theorem 9. *Let A be the aa-set, $\bar{A}$ the ab-set, B the ba-set, $\bar{B}$ the bb-set, α the aa-mapping, β the ab-mapping, γ the ba-mapping, and δ the bb-mapping for $\theta\colon D_{4k} \to D_{4k}$ or $\theta\colon Q_{4k} \to Q_{4k}$.*

(1) *θ is a strong complete mapping of D_{4k} if and only if $|A| = |\bar{A}| = |B| = |\bar{B}| = k$; α, β, γ, and δ are 1-1 mappings; and the following hold.*
   (a) $\{\alpha(i) + i \mid i \in A\} = \{\alpha(i) - i \mid i \in A\}$.
   (b) $\{\beta(i) + i \mid i \in \bar{A}\} = \{\beta(i) - i \mid i \in \bar{A}\}$.
   (c) $\{\alpha(i) \mid i \in A\}$ *and* $\{\gamma(i) \mid i \in B\}$ *partition* $\mathbb{Z}_{2k}$.
   (d) $\{\beta(i) \mid i \in \bar{A}\}$ *and* $\{\delta(i) \mid i \in \bar{B}\}$ *partition* $\mathbb{Z}_{2k}$.
   (e) $\{\alpha(i) \pm i \mid i \in A\}$ *and* $\{\delta(i) - i \mid i \in \bar{B}\}$ *partition* $\mathbb{Z}_{2k}$.
   (f) $\{\beta(i) \pm i \mid i \in \bar{A}\}$ *and* $\{\gamma(i) + i \mid i \in B\}$ *partition* $\mathbb{Z}_{2k}$.

(2) *θ is a strong complete mapping of Q_{4k} if and only if $|A| = |\bar{A}| = |B| = |\bar{B}| = k$; α, β, γ, and δ are 1-1 mappings; and the following hold.*

(a) $\{\alpha(i)+i \mid i \in A\} = \{\alpha(i)-i+k \mid i \in A\}$.
(b) $\{\beta(i)+i \mid i \in \bar{A}\} = \{\beta(i)-i+k \mid i \in \bar{A}\}$.
(c) $\{\alpha(i) \mid i \in A\}$ *and* $\{\gamma(i) \mid i \in B\}$ *partition* $\mathbb{Z}_{2k}$.
(d) $\{\beta(i) \mid i \in \bar{A}\}$ *and* $\{\delta(i) \mid i \in \bar{B}\}$ *partition* $\mathbb{Z}_{2k}$.
(e) $\{\alpha(i)-i \mid i \in A\}$ *and* $\{\delta(i)-i \mid i \in \bar{B}\}$ *partition* $\mathbb{Z}_{2k}$.
(f) $\{\beta(i)-i \mid i \in \bar{A}\}$ *and* $\{\gamma(i)+i \mid i \in B\}$ *partition* $\mathbb{Z}_{2k}$.

Proof.

(1) Let us assume that θ is a strong complete mapping of D_{4k}. As θ is a bijection, each of the mappings α, β, γ, and δ is 1-1. As

$$x\theta(x) = \begin{cases} a^{\alpha(i)+i} & \text{if } x = a^i, i \in A, \\ ba^{\beta(i)-i} & \text{if } x = a^i, i \in \bar{A}, \\ ba^{\gamma(i)+i} & \text{if } x = ba^i, i \in B, \\ a^{\delta(i)-i} & \text{if } x = ba^i, i \in \bar{B}, \end{cases}$$

and

$$x^{-1}\theta(x) = \begin{cases} a^{\alpha(i)-i} & \text{if } x = a^i, i \in A, \\ ba^{\beta(i)+i} & \text{if } x = a^i, i \in \bar{A}, \\ ba^{\gamma(i)+i} & \text{if } x = ba^i, i \in B, \\ a^{\delta(i)-i} & \text{if } x = ba^i, i \in \bar{B}, \end{cases}$$

each mapping $i \mapsto \alpha(i)+i$, $i \in A$; $i \mapsto \alpha(i)-i$, $i \in A$; $i \mapsto \beta(i)+i$, $i \in \bar{A}$; $i \mapsto \beta(i)-i$, $i \in \bar{A}$; $i \mapsto \gamma(i)+i$, $i \in B$; and $i \mapsto \delta(i)-i$, $i \in \bar{B}$ is 1-1. It is easy to see that (1a) through (1f) hold. From $|A|+|\bar{A}| = 2k$, $|B|+|\bar{B}| = 2k$, $|A|+|B| = 2k$, $|\bar{A}|+|\bar{B}| = 2k$, $|A|+|\bar{B}| = 2k$, and $|\bar{A}|+|B| = 2k$, we deduce that $|A| = |\bar{A}| = |B| = |\bar{B}| = k$.

The converse is routine.

(2) Similar to the proof of (1).

□

Theorem 9 yields a simple proof that D_8 and Q_8 are not strongly admissible.

Theorem 10.

(1) D_8 *is not strongly admissible.*
(2) Q_8 *is not strongly admissible.*

Proof.

(1) Let θ be a strong complete mapping of D_8. Let A be the *aa*-set, $\bar{A}$ the *ab*-set, B the *ba*-set, $\bar{B}$ the *bb*-set, α the *aa*-mapping, β the *ab*-mapping, γ the *ba*-mapping, and δ the *bb*-mapping for θ.

We may assume, without loss of generality, that θ is normalized and, hence, that $0 \in A$ and $\alpha(0) = 0$. Thus $A = \{0, h\}$ for some

$h \in A \setminus \{0\}$ and by Theorem 9(1a) $\alpha(h) + h = \alpha(h) - h$. It follows that $h = 2$ and so $B = \{1, 3\}$. But, if $\beta(1) = c$ and $\beta(3) = d$, then $\{c - 1, d - 3\} = \{c + 1, d + 3\}$ and, as $c - 1 \neq c + 1$, $c + 1 = d - 3$ from which it follows that $c = d$, a contradiction from which the result follows.

(2) Let θ be a strong complete mapping of Q_8. Let A be the *aa*-set, $\bar{A}$ the *ab*-set, B the *ba*-set, $\bar{B}$ the *bb*-set, α the *aa*-mapping, β the *ab*-mapping, γ the *ba*-mapping, and δ the *bb*-mapping for θ.

We may assume, without loss of generality, that θ is normalized and, hence, that $0 \in A$ and $\alpha(0) = 0$. Thus $A = \{0, h\}$ for some $h \in A \setminus \{0\}$ and by Theorem 9(2a) $\{0, \alpha(h) + h\} = \{2, \alpha(h) - h + 2\}$. As $0 \neq 2$, $\alpha(h) + h = \alpha(h) - h = 2$. It follows that $h = 2$ and $\alpha(h) = 0$, or $h = 0$ and $\alpha(h) = 2$; a contradiction from which the result follows.

□

Let us illustrate how Theorem 9 can be used by constructing a strong complete mapping of D_{12}. We begin by choosing $A = \{0, 1, 2\}$, and thus $\bar{A} = \{3, 4, 5\}$. Next we choose α and β as follows.

g	0	1	2
$\alpha(g)$	0	2	5
$\alpha(g) + g$	0	3	1
$\alpha(g) - g$	0	1	3

and

g	3	4	5
$\beta(g)$	0	2	5
$\beta(g) + g$	3	0	4
$\beta(g) - g$	3	4	0

Hence

$$\{\gamma(g) \mid g \in A\} = \{1, 3, 4\} \quad \text{and} \quad \{\gamma(g) + g \mid g \in A\} = \{1, 2, 5\}$$

and

$$\{\delta(g) \mid g \in \bar{A}\} = \{1, 3, 4\} \quad \text{and} \quad \{\delta(g) - g \mid g \in \bar{A}\} = \{2, 4, 5\}.$$

Next form a bordered partial latin square as follows.

		$\gamma(g)$			$\delta(g) - g$		
		1	3	4	2	4	5
$\gamma(g) + g$	1	**0**	4	3			
	2	1	5	**4**			
	5	4	**2**	1			
$\delta(g)$	1				5	**3**	2
	3				**1**	5	4
	4				2	0	**5**

Here the entries are obtained by subtracting column headings from row headings. A transversal of this partial latin square is shown in bold. From this transversal we can determine B, $\bar{B}$, γ, and δ. We find that $B = \{0,2,4\}$, $\bar{B} = \{1,3,5\}$,

g	0	2	4
$\gamma(g)$	1	3	4
$\gamma(g)+g$	1	5	2

and

g	1	3	5
$\delta(g)$	3	1	4
$\delta(g)-g$	2	4	5

.

The corresponding strong complete mapping of D_{12} is shown in Figure 1.

x	1	a	a^2	a^3	a^4	a^5	b	ba^2	ba^4	ba	ba^3	ba^5
$\theta(x)$	1	a^2	a^5	b	ba^2	ba^5	a	a^3	a^4	ba^3	ba	ba^4
$x\theta(x)$	1	a^3	a	ba^3	ba^4	b	ba	ba^5	ba^2	a^2	a^4	a^5
$x^{-1}\theta(x)$	1	a	a^3	ba^3	b	ba^4	ba	ba^5	ba^2	a^2	a^4	a^5

FIGURE 1. A strong complete mapping of D_{12}.

Theorem 11. *If* $\gcd(m,6) = 1$, *then* D_{4m}, D_{12m}, D_{16m}, *and* D_{24m} *are strongly admissible.*

Proof. As D_4 is the elementary abelian group of order four, it is strongly admissible by Theorem 3. Hence, by Lemma 1, if $\gcd(m,6) = 1$, then D_{4m} is strongly admissible.

As we have already shown D_{12} to be strongly admissible, it follows from Lemma 1 that, if $\gcd(m,6) = 1$, then D_{12m} is strongly admissible.

For D_{16}, let $A = \{0,1,2,5\}$, $\bar{A} = \{3,4,6,7\}$, $B = \{0,2,3,7\}$, and $\bar{B} = \{1,4,5,6\}$, and let $\alpha, \beta, \gamma, \delta \colon \mathbb{Z}_8 \to \mathbb{Z}_8$ be given by

x	0	1	2	5
$\alpha(x)$	0	6	3	4
$\alpha(x)+x$	0	7	5	1
$\alpha(x)-x$	0	5	1	7

,

x	3	4	6	7
$\beta(x)$	4	0	3	6
$\beta(x)+x$	7	4	1	5
$\beta(x)-x$	1	4	5	7

,

x	0	2	3	7
$\gamma(x)$	2	1	5	7
$\gamma(x)+x$	2	3	0	6

, and

x	1	4	5	6
$\delta(x)$	5	2	7	1
$\delta(x)-x$	4	6	2	3

.

It is routine to show that A, $\bar{A}$, B, $\bar{B}$, α, β, γ, δ satisfy the conditions of Theorem 9. Thus D_{16} is strongly admissible and, by Lemma 1, if $\gcd(m,6) = 1$, then D_{16m} is strongly admissible.

For D_{24}, let $A = \{0,1,3,7,8,11\}$, $\bar{A} = \{2,4,5,6,9,10\}$, $B = \{2,3,4,6,10,11\}$, and $\bar{B} = \{0,1,5,7,8,9\}$, and let $\alpha, \beta, \gamma, \delta \colon \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ be given by

x	0	1	3	7	8	11
$\alpha(x)$	0	6	10	11	9	5
$\alpha(x)+x$	0	7	1	6	5	4
$\alpha(x)-x$	0	5	7	4	1	6

,

x	2	4	5	6	9	10
$\beta(x)$	1	5	8	6	2	7
$\beta(x)+x$	3	9	1	0	11	5
$\beta(x)-x$	11	1	3	0	5	9

,

x	2	3	4	6	10	11
$\gamma(x)$	2	7	4	1	8	3
$\gamma(x)+x$	4	10	8	7	6	2

, and

x	0	1	5	7	8	9
$\delta(x)$	11	10	3	9	4	0
$\delta(x)-x$	11	9	10	2	8	3

.

It is routine to show that A, $\bar{A}$, B, $\bar{B}$, α, β, γ, δ satisfy the conditions of Theorem 9. Thus D_{24} is strongly admissible and, by Lemma 1, if $\gcd(m,6) = 1$, then D_{24m} is strongly admissible. □

Theorem 12. *If* $\gcd(m,6) = 1$, *then* Q_{16m}, *and* Q_{24m} *are strongly admissible.*

Proof. For Q_{16}, let $A = \{0,1,5,6\}$, $\bar{A} = \{2,3,4,7\}$, $B = \{2,5,6,7\}$, and $\bar{B} = \{0,1,3,4\}$, and let $\alpha, \beta, \gamma, \delta \colon \mathbb{Z}_8 \to \mathbb{Z}_8$ be given by

x	0	1	5	6
$\alpha(x)$	0	5	7	3
$\alpha(x)+x$	0	6	4	1
$\alpha(x)-x$	0	4	2	5

,

x	2	3	4	7
$\beta(x)$	4	6	1	0
$\beta(x)+x$	6	1	5	7
$\beta(x)-x$	2	3	5	1

,

x	2	5	6	7
$\gamma(x)$	4	2	6	1
$\gamma(x)+x$	6	7	4	0

, and

x	0	1	3	4
$\delta(x)$	3	7	2	5
$\delta(x)-x$	3	6	7	1

.

It is routine to show that A, $\bar{A}$, B, $\bar{B}$, α, β, γ, δ satisfy the conditions of Theorem 9. Thus Q_{16} is strongly admissible and, by Lemma 1, if $\gcd(m,6) = 1$, then Q_{16m} is strongly admissible.

For Q_{24}, let $A = \{0,2,5,8,10,11\}$, $\bar{A} = \{1,3,4,6,7,9\}$, $B = \{3,4,6,8,10,11\}$, and $\bar{B} = \{0,1,2,5,7,9\}$, and let $\alpha, \beta, \gamma, \delta \colon \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ be given by

x	0	2	5	8	10	11
$\alpha(x)$	0	5	1	9	4	10
$\alpha(x)+x$	0	7	6	5	2	9
$\alpha(x)-x$	0	3	8	1	6	11

,

x	1	3	4	6	7	9
$\beta(x)$	3	0	2	6	1	5
$\beta(x)+x$	4	3	6	0	8	2
$\beta(x)-x$	2	9	10	0	6	8

,

x	3	4	6	8	10	11
$\gamma(x)$	8	3	11	7	6	2
$\gamma(x)+x$	11	7	5	3	4	1

, and

x	0	1	2	5	7	9
$\delta(x)$	4	8	11	10	9	7
$\delta(x)-x$	4	7	9	5	2	10

.

It is routine to show that A, $\bar{A}$, B, $\bar{B}$, α, β, γ, δ satisfy the conditions of Theorem 9. Thus Q_{24} is strongly admissible and, by Lemma 1, if $\gcd(m, 6) = 1$, then Q_{24m} is strongly admissible. □

5. Data for small groups

The admissibility of a finite group G is completely determined by the structure of its Sylow 2-subgroup S. If S is nontrivial and cyclic, then G is not admissible. If S is trivial or noncyclic, then G is admissible. For strong admissibilty we also need to take into account the structure of the Sylow 3-subgroup. The strong admissibility of a finite abelian group G is completely determined by the structures of its Sylow 2-subgroup S and its Sylow 3-subgroup T. By Theorems 1 and 2, if either of S or T is nontrivial and cyclic, then G is not strongly admissible: otherwise, by Theorem 3, G is strongly admissible. The situation is not as straightforward for nonabelian groups. We have seen strongly admissible groups, the smallest D_{12}, with nontrivial, cyclic Sylow 3-subgroups, and two groups, D_8 and Q_8, that are not strongly admissible despite having noncyclic Sylow 2-subgroups and trivial Sylow 3-subgroups. In this section we will determine all strongly admissible groups of order 31 or less. As the strongly admissible, finite, abelian groups have been characterized, we will only consider nonabelian groups.

The smallest nonabelian group, $D_6 \cong S_3$, of order 6, has a cylic Sylow 2-subgroup and so is not strongly admissible by Theorem 1. The two nonabelian groups of order 8, D_8 and Q_8, are not strongly admissible by Theorem 10. The nonabelian group of order 10, D_{10}, has a cyclic Sylow 2-subgroup and so is not strongly admissible by Theorem 1.

There are three nonabelian groups of order 12, D_{12}, Q_{12}, and A_4. By Theorem 11, D_{12} is strongly admissible. Q_{12} has a cyclic Sylow 2-subgroup and so is not strongly admissible by Theorem 1, and A_4 has $\mathbb{Z}_3$ as a homomorphic image and so is not strongly admissible by Theorem 2. The nonabelian group of order 14, D_{14}, has a cyclic Sylow 2-subgroup and so is not strongly admissible by Theorem 1.

There are nine nonabelian groups of order 16. D_{16} and Q_{16} were shown to be strongly admissible in Theorems 11 and 12. For the remaining seven groups a search using magma [2] found strong complete mappings for all seven groups.

There are three nonabelian groups of order 20. D_{20} is strongly admissible by Theorem 11. Q_{20} and $HOL(\mathbb{Z}_5)$ have cyclic Sylow 2-subgroups and so are not strongly admissible by Theorem 1. The nonabelian group of order 22, D_{22}, has a cyclic Sylow 2-subgroup and so is not strongly admissible by Theorem 1.

There are twelve nonabelian groups of order 24. D_{24} and Q_{24} are strongly admissible by Theorems 11 and 12. Of the other ten groups, one has a cyclic Sylow 2-subgroup and so is not strongly admissible by Theorem 1, and four have $\mathbb{Z}_3$ as a homomorphic image and hence are not strongly admissible by Theorem 2. For the remaining five groups a search using magma [2] found strong complete mappings for all five groups.

The nonabelian group of order 26, D_{26}, has a cyclic Sylow 2-subgroup and so is not strongly admissible by Theorem 1. There are two nonabelian groups of order 27. For these two groups a search using magma [2] found strong complete mappings for both groups.

There are two nonabelian groups of order 28. D_{28} is strongly admissible by Theorem 11, and Q_{28} has a cyclic Sylow 2-subgroup and so is not strongly admissible by Theorem 1. There are three nonabelian groups of order 30, but, as each of these groups has a cyclic Sylow 2-subgroup, all three are not strongly admissible by Theorem 1.

Data for small groups along with the constructions of Theorems 7 and 8 yield a number of infinite classes of strongly admissible groups.

There is much work to do to characterize strongly admissible, finite groups. We have shown that the conjectures made in 2013 by Evans [7] are false, D_8 and Q_8 being counterexamples. But, we have also shown that these are the only counterexamples of order at most 31. The question presents itself: are these the only counterexamples to these conjectures?

REFERENCES

[1] B. A. Anderson, Sequencings and houses, Congr. Numer. **43** (1984), 23–43.

[2] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput., **24** (1997), 235-265.

[3] J. N. Bray, personal communication.

[4] A. B. Evans, On strong complete mappings, Congr. Numer. **70** (1990), 241–248.

[5] A. B. Evans, The admissibility of sporadic simple groups, J. Algebra **321** (2009), 105–116.

[6] A. B. Evans, The existence of strong complete mappings, Electronic J. Combin. **19** (2012), # P34.

[7] A. B. Evans, The existence of strong complete mappings of finite groups: a survey. Discrete Math. **313** (2013), 1191–1196.

[8] M. Hall and L. J. Paige, Complete mappings of finite groups, Pacific J. Math. **5** (1955), 541–549.

[9] A. Hedayat, A complete solution to the existence and non-existence of Knut Vic designs and orthogonal Knut Vic designs, J. Combin. Theory Ser. A **22** (1977), 331–337.

[10] A. Hedayat and W. T. Federer, On the non-existence of Knut Vic designs for all even orders, Ann. Statist. **3** (1975), 445–447.

[11] J. D. Horton, Orthogonal starters in finite abelian groups, Discrete Math. **79** (1990), 265–278.

[12] I. Rivin, I. Vardi, and P. Zimmermann, The n-queens problem, Amer. Math. Monthly **101** (1994), 629–639.

[13] R. Shaheen and A. Winterhof, Permutations of finite fields for check digit systems, Des. Codes Cryptogr. **57** (2010), 361–371.
[14] Y.-P. Shieh, J. Hsiang, and D. F. Hsu, On the existence problems of complete mappings, preprint.
[15] S. Wilcox, Reduction of the Hall-Paige conjecture to sporadic simple groups, J. Algebra **321** (2009), 1407–1428.

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