Linear Operators on Graphs which Preserve the Dot-Product Dimension*†

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Abstract

Let $\mathcal{G}_n$ be the set of all simple loopless undirected graphs on n vertices. Let T be a linear mapping, $T : \mathcal{G}_n \to \mathcal{G}_n$ for which the dot product dimension of $T(G)$ is the same as the dot product dimension for G for any $G \in \mathcal{G}_n$. We show that T is necessarily a vertex permutation. Similar results are obtained for mappings preserving sets of graphs with specified dot product dimension.

1 Introduction

Let $\mathcal{G}_n$ denote the set of all simple loopless undirected graphs on n vertices. In this paper we will investigate transformations of $\mathcal{G}_n$ which preserve the dot product dimension of graphs, both sets of graphs of dot product dimension $= 1$ and graphs of dot product dimension $= 2$, and those that strongly preserve the set of graphs of dot product dimension $= 1$.

In section 2 we give necessary definitions and notation; in section 3 we will investigate preservers of dot product dimension, etc.

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2 Preliminaries

We will assume that the reader is familiar with the basic concepts of graph theory and matrix theory. See [3, 4, 5, 11] for basic definitions. Let G be a graph. We use the notation $\mathcal{V}(G)$, or just $\mathcal{V}$, to denote the set of all vertices, and $\mathcal{E}(G)$, or just $\mathcal{E}$, to denote the set of all edges of G, and we write $G = (\mathcal{V}, \mathcal{E})$. We call a graph on n vertices an *edge graph* if the cardinality of the edge set is one, that is if the edge set of a graph is $\{ab\}$ where a and b are vertices of G and ab is the edge joining vertex a to vertex b, then the graph is an edge graph and is denoted $E_{a,b}$. A *star graph*, or simply a *star*, is a graph all of whose edges are incident with a single vertex. If this vertex is the vertex a and there are $n-1$ edges in the graph we call it a *full star graph*, or simply a *full star*, and is denoted S_a. If the vertex set of G is $\mathcal{V} = \{v_1, v_2, \cdots, v_n\}$ we shorten the notation to $\mathcal{V} = \{1, 2, \cdots, n\}$ and use the notation S_i to be the full star centered at v_i and $E_{i,j}$ to denote the edge $\{v_i, v_j\}$.

Let G and H be graphs with the same vertex set, that is, $\mathcal{V}(G) = \mathcal{V}(H)$. We say that G *dominates* H, written $G \sqsupseteq H$, if the edge set of H is a subset of the edge set of G. Similarly, if H and K are two $m \times n$ matrices we say H dominates K, written $H \sqsupseteq K$ if $k_{i,j} \neq 0$ implies that $h_{i,j} \neq 0$. If $H \sqsubseteq G$ we let $G \setminus H$ be the graph whose vertex set is the vertex set of G and the edge set is those edges in G which are not in H.

Let X denote a set. We let $|X|$ denote the cardinality of the set X. Let G be a graph. For convenience, we let $|G|$ denote the number of edges in G, i.e., $|G| = |\mathcal{E}(G)|$.

2.1 The Dot-Product Dimension of a Graph

A *dot product representation of* $G = (\mathcal{V}, \mathcal{E})$ is a function $f : \mathcal{V} \to \mathbb{R}^k$ for some k such that for any $x, y \in \mathcal{V}$ $xy \in \mathcal{E}$ if and only if $f(x) \cdot f(y) \geq t$. These representations were introduced independently by Reiterman et al. [10] and Fiduccia et al. [6]. The definition given by Reiterman et al. allowed t to be any real number, while Fiduccia et al limited to $t > 0$. Fiduccia et al. also showed that for $t > 0$ the vectors of the representation can be scaled such that $t = 1$. We will use this latter restriction on t.

To understand dot product representations of graphs, consider the graph, H (Figure 1). We can turn this into a dot product representation by as-

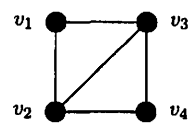

Figure 1: H: An Example of a Dot Product Representation of Graph.

signing each vertex a vector such that the dot product of adjacent vertices is greater than or equal to 1 and the dot product of non adjacent vertices is less than 1. A brief examination of the dot products of the following vectors shows that the following assignment also produces the graph in Figure 1:

$$f(v_1) = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \; f(v_2) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \; f(v_3) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \; f(v_4) = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix},$$

a dot product representation of dimension 2.

It has been proven that such a representation exists for all undirected, simple graphs. These representations have been used in a variety of applications. Those applications include work on social networks[12] and ecology[1].

Since every graph has a dot product representation, a minimization problem exists: to determine the minimum $k \in \mathbb{N}$ such that a representation exists for a graph G. This minimum is called the *dot product dimension of* G, written $\rho(G)$. The dot product dimension of a graph has been shown to be NP-hard [8], but there has been some work done on the dot product dimension for specific classes of graphs. The following are some known results about dot product dimension (See [9][10][6][7]):

- $\rho(G) \leq 1$ if and only if no induced subgraph of G is either a $3K_2$, P_4, or C_4
- $\rho(C_n) = 2$ if $n \geq 4$

- $\rho(P_n) = 2$ if $n \geq 4$
- If G is an interval graph, then $\rho(G) \leq 2$
- If G is a tree, then $\rho(G) \leq 3$
- If $G = K_{n,m}$, then $\rho(G) = \min\{n, m\}$.
- Let $G = (\mathcal{V}, \mathcal{E})$, $A \subset \mathcal{V}$, and K_A be the clique on A. Then $\rho(G \cup K_A) \leq \rho(G) + 1$.
- Let $G = (\mathcal{V}, \mathcal{E})$, $a \in \mathcal{V}$, and s_a be the star at a. Then $\rho(G \cup s_a) \leq \rho(G) + 1$.

These last two results give rise to the question of how does the addition or removal of an edge affect the dot product dimension of a graph.

The following lemma will be used throughout without reference:

Lemma 1 *Let E and F be edge graphs on the same vertex set,*

- *If E and F are parallel (not adjacent), $\rho(K \setminus (E \cup F)) = 2$, and $\rho(K \setminus E) = 1$;*
- *If E and F are adjacent, $\rho(K \setminus (E \cup F)) = 1$.*

Proof. The proof is straightforward and is left to the reader. ∎

2.2 Linear Operators on Graphs

A transformation on $\mathcal{G}_n$ is linear if it is additive and $T(O) = O$, that is, a transformation on $\mathcal{G}_n$ is linear if the image of the union of two graphs is the union of the images of the two graphs and the image of the edgeless graph is edgeless.

Let $W \subseteq \mathcal{G}_n$. A linear transformation $T : \mathcal{G}_n \to \mathcal{G}_n$ is said to *preserve* the set W if $G \in W$ implies that $T(G) \in W$. We say that T *strongly* preserves the set W if, $G \in W$ if and only if $T(G) \in W$. We similarly define preservers of sets of matrices. Consider a mapping $\varphi : \mathcal{G}_n \to \mathbb{Z}_+$ where $\mathbb{Z}_+$ is the set of nonnegative integers. Let $W_i = \{G \in \mathcal{G}_n | \varphi(G) = i\}$. Then we say that T preserves φ if T preserves all the sets W_i. Further, we say that T (strongly) preserves $\varphi = i$ if T (strongly) preserves W_i.

The investigation of preservers of sets and functions has been an active area of research in the past few years. The study of linear preservers began with Fröbenius in 1896 and for most of a century, all of the problems considered were preservers of sets and functions of matrices over fields or rings. In 1984, Beasley and Pullman [2] came out with the first article on linear preservers of sets of matrices over a semiring, specifically over $\mathbb{B}$, the binary Boolean semiring. Since the symmetric matrices over the binary Boolean semiring is equivalent to the set of undirected graphs, this began the investigation of linear preservers of functions and subsets of graphs.

Let W be any subset of $\mathcal{G}_n$ and let T be any transformation on $\mathcal{G}_n$ whose image is a subset of W. Then T preserves the set W. In the investigation of preservers of sets of $\mathcal{G}_n$, an additional condition has to be added to T to have any hope of characterizing T. This condition is usually that T is bijective (or equivalently surjective or injective), that T strongly preserves the set or that T preserves two or more (usually disjoint) sets. Of these conditions, the condition that T be bijective is the most restrictive, and that T preserve two sets is the least. To illustrate the need for additional conditions, suppose $W \subseteq \mathcal{G}_n$ and $W \neq \mathcal{G}_n$. Then, if the image of T is a subset of W, T is not bijective, T does not strongly preserve W and T cannot preserve two disjoint sets unless they are both in W, however, T preserves W. If T preserves a function, like the dot product dimension of graphs, then clearly T preserves two disjoint sets and also T strongly preserves the set of matrices of dot product dimension $= k$ for any fixed k.

3 Preservers of Dot Product Dimension

When talking about edges in a graph, the term edge will mean an edge, but, when talking about a graph, an edge will refer to an edge graph. No confusion should arise. Let $\mathcal{E}(\mathcal{G}_n)$ denote the set of all edge graphs in $\mathcal{G}_n$.

Lemma 2 *Let $n \geq 4$ and $\mathcal{Z} \subseteq \mathcal{E}(\mathcal{G}_n)$. If $\mathcal{Z}$ contains no pair of non adjacent edges, then $|\mathcal{Z}| \leq n-1$.*

Proof. Suppose that $|\mathcal{Z}| \geq 2$. Then, since any two elements of $\mathcal{Z}$ are adjacent, we can assume that $E_{1,2}, E_{1,3} \in \mathcal{Z}$. Now suppose that $E_{r,s} \in \mathcal{Z}$ for some $1 < r < s \leq n$. Then, we must have that $r = 2$ and $s = 3$, for otherwise $E_{r,s}$ would be non adjacent to one of $E_{1,2}$ or $E_{1,3}$. But then, $|\mathcal{Z}| = 3 < n-1$ since $E_{1,b}$ is not adjacent to $E_{2,3}$ if $b \geq 4$.

The other possibility is that all elements of $\mathcal{Z}$ are of the form $E_{1,b}$ and hence $|\mathcal{Z}| \leq n-1$. ■

Lemma 3 *Let $n \geq 4$, and $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator that preserves the dot product dimension of graphs, or preserves dot product dimensions 1 and 2, or strongly preserves dot product dimension 1. If $X \in \mathcal{G}_n$ and $T(K \setminus X) = T(K)$, then $|X| \leq n-1$.*

Proof. Suppose that X dominates two non adjacent edges E and F. Then $T(K \setminus (E+F)) \sqsupseteq T(K \setminus X) = T(K)$ so that $T(K \setminus (E+F)) = T(K)$. But $K \setminus (E+F)$ has dot product dimension 2, so that $T(K \setminus (E+F))$ does not have dot product dimension 1. But $T(K)$ has dot product dimension 1, a contradiction. Thus, X does not dominate a pair of non adjacent edges. By Lemma 2, $|X| \leq n-1$. ■

Lemma 4 *Let $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator. If $|T(E)| \geq n$ for some edge graph E, then there exists a graph Z with $|Z| \geq n$ such that $T(K \setminus Z) = T(K)$.*

Proof. Suppose with out loss of generality that $|T(E_{1,2})| \geq n$. Then $T(K) = T(E_{1,2}) + \sum_{1 \leq r < s, 3 \leq s \leq n} T(E_{r,s})$. Let $r_1 = 1, s_1 = 2$ and for $k \geq 2$, when possible, choose E_{r_k,s_k} so that $|\sum_{i=1}^{k-1} T(E_{r_i,s_i}) + T(E_{r_k,s_k})| > |\sum_{i=1}^{k-1} T(E_{r_i,s_i})|$. This can be possible for k at most $|T(K)| - n$, say for $k = \ell$. Let $Y = \sum_{i=1}^{\ell} E_{r_i,s_i}$ and $Z = K \setminus Y$. Then $T(Y) = T(K)$ and hence $T(K \setminus Z) = T(K)$. Further $|Z| = |K| - \ell \geq |K| - (|K| - n) = n$. ■

An immediate consequence of Lemmas 3 and 4 is the following.

Corollary 4.1 *Let $n \geq 4$, and $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator that preserves the dot product dimension of graphs, or preserves dot product dimensions 1 and 2, or strongly preserves dot product dimension 1. Then, $max_{i<j}|T(E_{i,j})| < n$.*

3.1 $n \geq 6$.

Lemma 5 *Let $n \geq 6$ and $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator that preserves the dot product dimension of graphs, or preserves dot product dimensions 1 and 2, or strongly preserves dot product dimension 1. Then T maps edge graphs to edge graphs.*

Proof. Suppose that $T(E) = O$. Then, $\rho(E) = 1$ while $\rho(O) = 0$, a contradiction. Therefore, T is nonsingular.

Suppose $T(E)$ dominates more than one edge. Then, $T(K) = T(K \setminus E)$ for some edge E. Suppose, without loss of generality, that $T(K) = T(K \setminus E_{1,2})$. Consider $T(K \setminus (E_{1,2} + E_{r,s}))$ for $3 \leq r < s \leq n$. Since $K \setminus (E_{1,2} + E_{r,s})$ has dot product dimension 2, $T(K \setminus (E_{1,2} + E_{r,s}))$ cannot have dot product dimension 1. But, $K \setminus E_{r,s}$ has dot product dimension 1, and hence , $T(K \setminus E_{r,s})$ has dot product dimension 1. Thus, there is some edge graph $F_{(r,s)}$ such that $T(E_{1,2})$ dominates $F_{(r,s)}$, $T(E_{r,s})$ dominates $F_{(r,s)}$, and $T(E_{u,v})$ dominates $F_{(r,s)}$ implies that $(u, v) = (1, 2)$ or $(u, v) = (r, s)$. But then, for each $3 \leq r < s \leq n$, there is a distinct edge graph dominated by $T(E_{1,2})$. Thus, $|T(E_{1,2})| \geq \frac{(n-2)(n-3)}{2} \geq n$, since $n \geq 6$. We now have a contradiction by Corollary 4.1. ∎

Lemma 6 *Let $n \geq 6$ and $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator that preserves the dot product dimension of graphs, or preserves dot product dimensions 1 and 2, or strongly preserves dot product dimension 1. Then Tis a bijection on the set of all edge graphs.*

Proof. By Lemma 5 the image of an edge graph is an edge graph. Suppose that there are two edge graphs, E and F, such that $T(E) = T(F)$. Then, we can find edge graphs G and H such that $E \cup F \cup G \cup H$ is a C_4 so that $\rho(E \cup F \cup G \cup H) = 2$, and hence, $\rho(T(E \cup F \cup G \cup H)) \neq 1$, But $\rho(E \cup G \cup H) = 1$, so that $\rho(T(E \cup G \cup H)) = 1$. Since $T(E) = T(F)$, $T(E \cup F \cup G \cup H) = T(E \cup G \cup H)$, a contradiction. Thus, T is bijective on the set of edge graphs. ∎

Lemma 7 *Let $n \geq 6$ and $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator that preserves the dot product dimension of graphs, or preserves dot product dimensions 1 and 2, or strongly preserves dot product dimension 1. Then, T maps star graphs to star graphs.*

Proof. Suppose that the image of a star graph is not a star. Then, some pair of adjacent edges is mapped to a pair of parallel edges. Say E and F are adjacent and $T(E)$ and $T(F)$ are parallel. Then, $1 = \rho((K \setminus (E+F)) = \rho(T(K \setminus (E + F))$, but $\rho(T(K \setminus (E + F)) = \rho(K \setminus (T(E) + T(F)) \neq 1$, a contradiction. Thus T maps stars to stars. ∎

Theorem 8 *Let $n \geq 6$ and $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator that preserves the dot product dimension of graphs, or preserves dot product dimensions 1 and 2, or strongly preserves dot product dimension 1. Then, T is a vertex permutation.*

Proof. Since T maps stars to stars by Lemma 7, Define a mapping $\sigma : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ by $\sigma(i) = j$ if the image of the star centered at vertex i is mapped to the star centered at j. First, since T is bijective by Lemma 6, full stars are mapped to full stars and hence, σ is well defined, and since T is bijective, so is σ. That is σ is a permutation, and hence, T is a vertex permutation corresponding to σ. ∎

3.2 $n \leq 5$.

If $n \leq 3$, any nonsingular mapping preserves the dot product dimension, since any non empty graph has dot product dimension equal 1.

Lemma 9 *Let $n = 4$ and $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator. Then, the following are equivalent:*

- *T preserves the dot product dimension of graphs.*
- *T preserves dot product dimensions 1 and 2.*
- *T strongly preserves dot product dimension 1.*

Proof. The equivalences all follow from the fact that for $n = 4$, every graph in G_4 has dot product dimension 0, 1 or 2. ■

Lemma 10 *Let $n = 4$ and $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator that preserves the dot product dimension of graphs, or preserves dot product dimensions 1 and 2, or strongly preserves dot product dimension 1. Then T maps edge graphs to edge graphs.*

Proof. Suppose that $T(E) = O$. Then, $\rho(E) = 1$ while $\rho(O) = 0$, a contradiction. Therefore, T is nonsingular.

First observe that for $n = 4$, every graph with dot product dimension =2 has either has 3 edges and is a P_4, or has 4 edges, and is a C_4. Suppose that $|T(E)| > 1$ for some edge graph E. Then, there are two edge graphs, F, G, such that $E \cup F \cup G$ is a 4-path graph which has dot product dimension = 2. So $\rho(T(E \cup F \cup G)) = 2$. But then, there one of F, G, say with out loss of generality F, such that $T(E \cup F))$ is a 4-path or a 4 cycle since $|T(E)| > 1$. This is a contradiction since $\rho(T(E \cup F)) = 1$. Thus, T maps edge graphs to edge graphs. ■

Lemma 11 *Let $n = 4$ and $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator that preserves the dot product dimension of graphs, or preserves dot product dimensions 1* *and 2, or strongly preserves dot product dimension 1. Then T is a bijection on the set of all edge graphs.*

Proof. The proof is parallel to the proof of Lemma 6. ∎

Lemma 12 *Let $n = 4$ and $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator that preserves the dot product dimension of graphs, or preserves dot product dimensions 1 and 2, or strongly preserves dot product dimension 1. Then, T maps star graphs to star graphs.*

Proof. The proof is parallel to the proof of Lemma 7. ∎

Theorem 13 *Let $n = 4$ and $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator that preserves the dot product dimension of graphs, or preserves dot product dimensions 1 and 2, or strongly preserves dot product dimension 1. Then, T is a vertex permutation.*

Proof. The proof is parallel to the proof of Theorem 8. ∎

The case $n = 5$ is the only remaining case. For $n = 5$, at this point the authors can only conjecture that if T preserves dot product dimension of graphs then T is a vertex permutation (see below).

4 Summary

In the previous section we have shown

Theorem 14 *If $n = 4$ or $n \geq 6$ and $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator that preserves the dot product dimension of graphs, or preserves dot product dimensions 1 and 2, or strongly preserves dot product dimension 1. Then, T is a vertex permutation.*

Let $\mathcal{S}_n^{(0)}(\mathbb{B})$ denote the set of all $n \times n$ $(0,1)$-matrices with all diagonal entries equal 0. The arithmetic used is Boolean: (1+1=1), other wise as for real numbers. Further, let the dot product dimension of a $(0,1)$-matrix, A, be the dot product dimension of the graph whose adjacency matrix is A. The matrix theoretic equivalent of Theorem 14 is :

Theorem 15 *If $n = 4$ or $n \geq 6$ and $T\mathcal{S}_n^{(0)}(\mathbb{B}) \to \mathcal{S}_n^{(0)}(\mathbb{B})$ be a linear operator that preserves the dot product dimension of matrices, or preserves dot product dimensions 1 and 2, or strongly preserves dot product dimension 1. Then, there is a permutation matrix, P such that $T(X) = PXP^t$ for all $X \in \mathcal{S}_n^{(0)}(\mathbb{B})$.*

The case for $n = 5$ is still unresolved, however the authors conjecture:

Conjecture 16 *If $n = 5$ and $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator that preserves the dot product dimension of graphs, or preserves dot product dimensions 1 and 2, or strongly preserves dot product dimension 1. Then, T is a vertex permutation.*

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