On local metric dimension of $(n-3)$-regular graph

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Abstract

A set of vertices W *locally resolves* a graph G if every two adjacent vertices is uniquely determined by its coordinate of distances to the vertices in W. The minimum cardinality of a local resolving set of G is called the *local metric dimension* of G. A graph G is called k-regular graph if every vertex of G is adjacent to k other vertices of G. In this paper, we determine the local metric dimension of $(n-3)$-regular graph G of order n where $n \geq 5$.

Keywords: local basis, local metric dimension, local resolving set, $(n-3)$-regular graph

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1 Introduction

Throughout this paper, all graphs are finite, simple, and connected. The *vertex set* and the *edge set* of graph G are denoted by $V(G)$ and $E(G)$, respectively.

For any two distinct vertices $u, v \in V(G)$, the distance between u and v in G, denoted by $d(u,v)$, is the length of a shortest (u,v)-path in G. Let $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$. For a vertex $v \in V(G)$, a *representation* of v with respect to W is defined as k-tuple $r(v \mid W) = (d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$. W *resolves* a graph G if every two distinct vertices $x, y \in V(G)$ satisfy $r(x \mid W) \neq r(y \mid W)$. A *basis* of G is the minimum resolving set of G. The cardinality of a basis of G is called the *metric dimension* of G, denoted by $\beta(G)$.

Metric dimension problems were first studied by Harary and Melter [7], and independently by Slater [12, 13]. Slater [12] considered the minimum resolving set of a graph as the location of the placement of a minimum number of sonar/loran detecting devices in a network. So, the position of every vertex in the network can be uniquely described by its distances to the devices in the set.

Trivially, if G is a graph of order n, then $\beta(G) \leq n$, by taking all vertices of G to be a resolving set. However, we may obtain a resolving set whose cardinality is less than n. All graphs of order $n \geq 2$ with metric dimension 1, $n-1$, or $n-2$ have been characterized by Chartrand *et al.* [4].

Metric dimension problem is a difficult problem. Garey and Johnson [6] have showed that determining the metric dimension of any graph is an NP-problem. However, some results for certain class of graphs have been obtained, which can be seen in [1, 2, 3, 4, 5, 7, 8, 10, 11, 14].

Now, we consider the local version of metric dimension. In this problem, two distinct vertices may have the same representation with respect to an ordered subset W of $V(G)$. If $r(x \mid W) \neq r(y \mid W)$ for every two adjacent vertices $x, y \in V(G)$, then W is called a *local resolving set* of G. The *local basis* of G is a local resolving set of G with minimum number of vertices, and the *local metric dimension* of G refers to its cardinality, denoted by $lmd(G)$. Note that, for a nontrivial connected graph G of order n, since every resolving set is also a local resolving set, then

$$1 \leq lmd(G) \leq \beta(G) \leq n-1.$$

The local metric dimension problems were first studied by Okamoto *et al.* [9]. They provided some bounds of local metric dimension for a connected graph. They also have obtained some characterizations as follows.

Theorem 1 *[9]Let G be a connected graph of order $n \geq 2$. Then*

1. *$lmd(G) = 1$ if and only if G is bipartite.*
2. *$lmd(G) = n-1$ if and only if $G = K_n$.*
3. *$lmd(G) = n-2$ if and only if $\omega(G) = n-2$ where $\omega(G)$ is the order of the biggest clique in G.*

In this paper, we consider the local metric dimension of a regular graph. A graph G is called k-regular graph if every vertex of G is adjacent to k other vertices of G. Since every vertex of G is adjacent to the same number of vertices of G, every vertex of G has the same probability to be considered as the member of a resolving set of G. The local metric dimension of regular graph was firstly studied by Okamoto *et al.* [9]. They obtained the local metric dimension of $(n-1)$-regular graph and $(n-2)$-regular graph of order $n \geq 3$. In this paper, we will determine the local metric dimension of $(n-3)$-regular graph of order $n \geq 5$.

2 Main Results

Let G be an $(n-3)$-regular graph of order $n \geq 5$. In order to find $lmd(G)$, we must consider that G have a subgraph which is isomorphic to $K_m \backslash E(C_m)$ where $m \in \{3, 4, \ldots, n\}$. Let $G' \subseteq G$ such that $G' = K_m \backslash E(C_m)$ where $m \in \{3, 4, ..., n\}$. Note that every two vertices $v \in V(G')$ and $w \in V(G) \backslash V(G')$ satisfy $vw \in E(G)$.

Lemma 1 *Let G be an $(n-3)$-regular graph of order $n \geq 5$. Let $G' \subseteq G$ such that $G' = K_m \backslash E(C_m)$ where $m \in \{4, ..., n\}$. If W is a local resolving set of G, then W includes a local basis of G'.*

Proof. Let B be a local basis of G'. Suppose that there exists a vertex $z \in B$ such that $z \notin W$. Let $B' \subseteq B \backslash \{z\}$ such that $B' \subseteq W$. Since every vertices $u \in V(G')$ and $v \in V(G) \backslash V(G')$ satisfy $uv \in E(G)$, all vertices $w \in W \backslash B'$ do not locally resolve two distinct vertices in G'. Therefore, we obtain two possibilities of representation with respect to B' below.

1. There exist two adjacent vertices $x, y \in V(G')$ such that $r(x \mid B') = r(y \mid B')$ which implies $r(x \mid W) = r(y \mid W)$.
2. There exists a vertex $x \in V(G')$ which is adjacent to z such that $r(x \mid B') = r(z \mid B')$ which implies $r(x \mid W) = r(z \mid W)$.

From both possibilities above, we have a contradiction. ■

By Lemma 1, every two distinct vertices in G' must be locally resolved by a vertex in G'. So, firstly we need to find the local resolving set of G'. In the other hand, we will determine the local metric dimension of G'. Note that G' is a graph which is isomorphic to a complete graph minus a Hamiltonian cycle.

2.1 Complete Graph minus a Hamiltonian Cycle

Let G be an $(n-3)$-regular graph of order $n \geq 5$. Let $G' \subseteq G$ such that $G' = K_m \backslash E(C_m)$ where $m \in \{3, 4, ..., n\}$. By Lemma 1, a local basis of G' must be contributed to a local basis of G. Therefore, we will determine the local metric dimension of G' for each $m \in \{3, 4, \ldots, n\}$.

In Lemma 2 and Remark 1, we provide the local metric dimension of G' for $m \in \{3, 4, 5\}$.

Lemma 2 *Let G be an $(n-3)$-regular graph of order $n \geq 5$. Let $G' \subseteq G$ such that $G' = K_m \backslash E(C_m)$ where $m \in \{3, 4, 5\}$.*

1. *For $m = 3$, there exists a local resolving set W of G such that $W \cap V(G') = \emptyset$.*
2. *For $m \in \{4, 5\}$, G' contributes at least 2 vertices in a local resolving set W of G.*

Proof. For $m \in \{3, 4, 5\}$, let $V(G') = \{x_1, x_2, \ldots, x_m\}$ and $C_m = x_1 x_2 \ldots x_m x_1$. We distinguish two cases.

Case 1: $m = 3$

We define $W = V(G) \setminus V(G')$. Since x_1, x_2, and x_3 are not adjacent each other in G', trivially W locally resolves G'.

Case 2: $m \in \{4, 5\}$

First, we show that $lmd(G') \leq 2$ by constructing a local resolving set W' of G' with 2 vertices. We define $W' = \{x_1, x_2\}$. For $m \in \{4, 5\}$, $x_i x_{i+1} \notin E(G')$ where $1 \leq i \leq m-1$ and for $m = 5$, $x_3 x_5 \in E(G')$ but $x_1 x_3 \in E(G')$ and $x_1 x_5 \notin E(G')$. Since every two adjacent vertices u, v of G' satisfy $r(u \mid W') \neq r(v \mid W')$, W' is a local resolving set of G'.

Now, suppose that $lmd(G') \geq 1$. Since there exists two adjacent vertices in G', we assume that $lmd(G') = 1$. Let B be a local basis of G' and $B = \{u\}$. Note that there exists two distinct vertices $v, w \in V(G')$ such that $uv, uw \notin E(G')$ but $vw \in E(G')$. It follows that $r(v \mid B) = r(w \mid B)$, a contradiction. ■

Remark 1 *Lemma 2 says that the local metric dimension of a subgraph G' above is given by*

$$lmd(G') = \begin{cases} 0, & \textit{for } m = 3 \\ 2, & \textit{for } m \in \{4, 5\}. \end{cases}$$

To determine the local metric dimension of $G' = K_m \backslash E(C_m)$ for $m \geq 6$, we use the idea of a gap between two vertices.

For $m \geq 6$, let S be a set of two or more vertices of G'. Let $v, w \in S$ and $P(v,w)$ be a shortest (v,w)-path in C_m. Note that all edges of $P(v,w)$ are not element of $E(G')$. We define a *gap* between v and w as the set of vertices in $P(v,w)\backslash \{v,w\}$ such that every vertex $z \in P(v,w)\backslash \{v,w\}$ satisfies $z \notin S$. Then the vertices v and w are called the *end points* of a gap. The two gaps which have at least one common end point, will be referred to as *neighboring gaps*. Consequently, if $|S| = r$, then S has r gaps, some of gaps may be empty.

Now, let W be a basis of G'. We make the following three observations.

Observation 1 *Every gap of W contains at most four vertices.*

Proof. Suppose that there exists a gap of W containing at least five vertices a_1, a_2, a_3, a_4, a_5 of G' where $a_j a_{j+1} \notin E(G')$ with $1 \leq j \leq 4$. However, $a_2 a_4 \in E(G')$ and for every $u \in W$, $d(u, a_2) = d(u, a_4)$ which implies $r(a_2 \mid W) = r(a_4 \mid W)$, a contradiction. ■

Observation 2 *At most one gap of W contains at least three vertices.*

Proof. Suppose that there are two different gaps A_1 and A_2 such that $a_1, a_2, a_3 \in V(A_1)$ and $b_1, b_2, b_3 \in V(A_2)$ where $a_j a_{j+1}, b_j b_{j+1} \notin E(G')$ for $1 \leq j \leq 2$. Since a_2 and b_2 are adjacent each other and adjacent to every vertex in W, we obtain $r(a_2 \mid W) = r(b_2 \mid W)$, a contradiction. ■

Observation 3 *If a gap A of W contains k vertices where $2 \leq k \leq 4$, then any neighboring gaps of A contain at most one vertex.*

Proof. Suppose that there are $k+3$ vertices $a_1, a_2, \ldots, a_{k+3}$ of G' where $a_j a_{j+1} \notin E(G')$ with $1 \leq j \leq k+2$, and a_{k+1} is the only vertex of W. We obtain that $a_k a_{k+1} \in E(G')$ but $r(a_k \mid W) = r(a_{k+2} \mid W)$, a contradiction. ■

Now, we consider any set of vertices S of G' satisfying Observations 1-3 above, and let $u \in V(G') \backslash S$. There are four possibilities of u with respect to gaps formed by S.

1. *u belongs to a gap of size one in S.*

   Let a, b be two distinct end points of this gap. Then the vertex u have a distance 2 to both a and b, and it is the only vertex which has this distance property. Therefore, for all $x \in V(G')$ and $x \neq u$, we have $r(x \mid S) \neq r(u \mid S)$.

2. *u belongs to a gap of size two in S.*

   Let us consider the vertices a_1, u, a_2, a_3 of G' with $a_1, a_3 \in S$. Then $d(u, a_1) = 2$, $d(u, a_3) = 1$, and for every $v \in S \backslash \{a_1, a_3\}$, $d(u, v) = 1$. By Observation 3, the vertex u is the only one which has all of these distance properties. Therefore, for all $x \in V(G')$ and $x \neq u$, we have $r(x \mid S) \neq r(u \mid S)$.

3. *u belongs to a gap of size three in S.*

   Let us consider the vertices a_1, a_2, a_3, a_4, a_5 of G' with only $a_1, a_5 \in S$. If $u = a_2$, then $d(u, a_1) = 2$, and if $u = a_3$, then $d(u, a_1) = 1$. For every $v \in S \backslash \{a_1\}$, we have $d(u, v) = 1$. By Observations 1-3, the vertex u is the only one which has all of these distance properties. Therefore, for all $x \in V(G')$ and $x \neq u$, we have $r(x \mid S) \neq r(u \mid S)$.

4. *u belongs to a gap of size four in S.*

   Let us consider the vertices $a_1, a_2, a_3, a_4, a_5, a_6$ of G' with only $a_1, a_6 \in S$. We distinguish two cases.

   (a) $u = a_2$

   Then $d(u, a_1) = 2$ and for every $v \in S \backslash \{a_1\}$, $d(u, v) = 1$. By Observation 3, the vertex u is the only one which has all of these distance properties. Therefore, for all $x \in V(G')$ and $x \neq u$, we have $r(x \mid S) \neq r(u \mid S)$.

   (b) $u = a_3$

   Note that for every $v \in S$, $d(u, v) = 1 = d(a_4, v)$ which implies $r(u \mid S) = r(a_4 \mid S)$. However, u and a_4 are not adjacent in G'. By Observation 3, the vertex u and a_4 are the only ones which have all of these distance properties. Therefore, for all $x \in V(G') \backslash \{u, a_4\}$, we have $r(x \mid S) \neq r(u \mid S)$.

Consequently, any set S satisfying Observations 1-3 locally resolves $V(G')$.

Theorem 2 *For $m \geq 6$, let G' be a connected graph with $G' = K_m \backslash E(C_m)$. Then $lmd(G') = \lceil \frac{2m-4}{5} \rceil$.*

Proof. Let $V(G') = \{x_1, x_2, \ldots, x_m\}$ and $C_m = x_1 x_2 \ldots x_m x_1$. We distinguish two cases.

Case 1: $lmd(G') \leq \lceil \frac{2m-4}{5} \rceil$

We show that $lmd(G') \leq \lceil \frac{2m-4}{5} \rceil$ by constructing a local resolving set W with $\lceil \frac{2m-4}{5} \rceil$ vertices. We consider the integer $k \geq 1$. We obtain six cases as follows.

(a) $m = 6$

Thus, $\lceil \frac{2m-4}{5} \rceil = 2$. We define $W = \{x_1, x_6\}$. Since W contains 2 vertices and satisfies Observations 1-3, then W is a local resolving set.

(b) $m \geq 7$ and $m = 0(\mod 5)$

Let $m = 5k$ with $k \geq 2$. Thus, $\lceil \frac{2m-4}{5} \rceil = 2k$. We define $W = \{x_1, x_6, x_{m-2}, x_m\} \cup \{x_{5j+3}, x_{5j+6} \mid 1 \leq j \leq k-2 \text{ and } k \geq 3\}$. Since W contains $2k$ vertices and satisfies Observations 1-3, then W is a local resolving set.

(c) $m \geq 7$ and $m = 1(\mod 5)$

Let $m = 5k+1$ with $k \geq 2$. Thus, $\lceil \frac{2m-4}{5} \rceil = 2k$. We define $W = \{x_1, x_6, x_{m-3}, x_{m-1}\} \cup \{x_{5j+3}, x_{5j+6} \mid 1 \leq j \leq k-2 \text{ and } k \geq 3\}$. Since W contains $2k$ vertices and satisfies Observations 1-3, then W is a local resolving set.

(d) $m \geq 7$ and $m = 2(\mod 5)$

Let $m = 5k+2$ with $k \geq 1$. Thus, $\lceil \frac{2m-4}{5} \rceil = 2k$. We define $W = \{x_1, x_6\} \cup \{x_{5j+3}, x_{5j+6} \mid 1 \leq j \leq k-1 \text{ and } k \geq 2\}$. Since W contains $2k$ vertices and satisfies Observations 1-3, then W is a local resolving set.

(e) $m \geq 7$ and $m = 3(\mod 5)$

Let $m = 5k+3$ with $k \geq 1$. Thus, $\lceil \frac{2m-4}{5} \rceil = 2k+1$. We define $W = \{x_1, x_6, x_m\} \cup \{x_{5j+3}, x_{5j+6} \mid 1 \leq j \leq k-1 \text{ and } k \geq 2\}$. Since W contains $2k+1$ vertices and satisfies Observations 1-3, then W is a local resolving set.

(f) $m \geq 7$ and $m = 4(\mod 5)$

Let $m = 5k+4$ with $k \geq 1$. Thus, $\lceil \frac{2m-4}{5} \rceil = 2k+1$. We define $W = \{x_1, x_6, x_{m-1}\} \cup \{x_{5j+3}, x_{5j+6} \mid 1 \leq j \leq k-1 \text{ and } k \geq 2\}$. Since W contains $2k+1$ vertices and satisfies Observations 1-3, then W is a local resolving set.

Case 2: $lmd(G') \geq \lceil \frac{2m-4}{5} \rceil$

Let S be a local basis of G'. We consider two cases as follows.

1. $lmd(G')$ is even.

Let $|S| = 2l$ for some integer $l \geq 2$. By Observation 3, there are at most l gaps containing more than one vertex. By Observations 1-3, all of them contain 2 vertices, except possibly one gap contains 3 or 4 vertices. Then, the number of vertices belonging to the gaps of S is at most $3l+2$. Hence $m-2l \leq 3l+2$, which implies $|S| = 2l \geq \lceil \frac{2m-4}{5} \rceil$.

2. $lmd\,(G')$ is odd.

   Let $|S| = 2l+1$ for some integer $l \geq 2$. By Observation 3, there are at most l gaps containing more than one vertex. By Observations 1-3, all of them contain 2 vertices, except possibly one gap contains 3 or 4 vertices. Then, the number of vertices belonging to the gaps of S is at most $3l+3$. Hence $m-2l-1 \leq 3l+3$, which implies $|S| = 2l+1 \geq \lceil \frac{2m-8}{5} + 1 \rceil \geq \lceil \frac{2m-4}{5} \rceil$.

■

2.2 $(n-3)$-regular graph

For $n \geq 5$, we consider certain cycles contained in a complete graph K_n. For $r \geq 1$, let $R_1, R_2, \ldots, R_r$ be r disjoint cycles contained in K_n such that $V(R_1) \cup V(R_2) \cup \ldots \cup V(R_r) = V(K_n)$. Then $K_n \backslash (E(R_1) \cup E(R_2) \cup \ldots \cup E(R_r))$ is an $(n-3)$-regular graph.

Let $G = K_n \backslash (E(R_1) \cup E(R_2) \cup \ldots \cup E(R_r))$ and $m_i = |V(R_i)|$. For every $i \in \{1, 2, \ldots, r\}$, let $G_i = K_{m_i} \backslash E(R_i)$. So, $G_i = K_{m_i} \backslash E(C_{m_i})$ and $G = G_1 + G_2 + \ldots + G_r$.

Let W be a local basis of G. By considering Lemmas 1 and 2, and Remark 1, we can say that $lmd(G) \geq lmd(G_1) + lmd(G_2) + \ldots + lmd(G_r)$. However, for $r \geq 2$, a local basis W of G must satisfy two conditions deduced from Observations 1 and 2, respectively.

(a) Every gap in W contains at most four vertices.

(b) At most one gap in W contains at least three vertices.

Let $G = G_1 + G_2 + \ldots + G_r$ where $3 \leq |V(G_1)| \leq |V(G_2)| \leq \ldots \leq |V(G_r)|$. Let B_i be a local basis of G_i where $1 \leq i \leq r$. If G contains $t \geq 1$ subgraphs G_i of G with $|V(G_i)| = 3$, by Lemmas 1 and 2, $B_{t+1} \cup B_{t+2} \cup \ldots \cup B_r \subseteq W$. Otherwise, $B_1 \cup B_2 \cup \ldots \cup B_r \subseteq W$. However, there may be exists some i such that a local resolving set of G_i which is contained in W, is not B_i. For example, if there is $j \in \{1, 2, \ldots, r\}$ and $j \neq i$ with B_j is contained in W such that B_i and B_j have a gap containing at least three vertices, then by (b), we cannot use B_j as a local resolving set of G_j in G. We must add at least one more vertex on B_j such that the new local resolving set of G_j satisfies (a)-(b). So, we need to know the gaps property of the local basis of $G_i = K_{m_i} \backslash E(C_{m_i})$ for $m_i \geq 4$, which can be seen in Lemmas 3 and 4.

Lemma 3 *Let G be an $(n-3)$-regular graph of order $n \geq 5$. Let $G' \subseteq G$ such that $G' = K_m \backslash E(C_m)$ where $m \in \{3, 4, ..., n\}$. If $m \in \{4, 5\}$ or $m = 0(\mod 5)$ for $m \geq 6$, then there exists a local basis of G' where every gap contains at most two vertices.*

Proof. Let $V(G') = \{x_1, x_2, \ldots, x_m\}$ and $C_m = x_1 x_2 \ldots x_m x_1$. We distinguish three cases as follows.

1. $m = 4$

   We define $W = \{x_1, x_2\}$.

2. $m = 5$

   We define $W = \{x_1, x_3\}$.

3. $m \geq 6$ and $m = 0(\mod 5)$

   Let $m = 5k$ with the integer $k \geq 2$. Thus, $\lceil \frac{2m-4}{5} \rceil = 2k$. We define $W = \{x_{5j+2}, x_{5j+4} \mid 0 \leq j \leq k-1\}$.

Note that every gap of W from all three cases above contains at most two vertices. Since $|W| = lmd(G')$ and every two adjacent vertices $u, v \in V(G') \backslash W$ satisfies $r(u \mid W) \neq r(v \mid W)$, then W is a local basis of G'. ∎

Lemma 4 *Let G be an $(n-3)$-regular graph of order $n \geq 5$. Let $G' \subseteq G$ such that $G' = K_m \backslash E(C_m)$ where $m \in \{3, 4, ..., n\}$. If $m \geq 6$ and $m \neq 0(\mod 5)$, then a local basis of G' has a gap containing at least three vertices.*

Proof. Suppose that every gap of a local basis of G' contains at most two vertices. Then we have two following cases.

Case 1: $m = 1$ or $2(\mod 5)$

Let $m = 5k+1$ or $m = 5k+2$ with the integer $k \geq 1$. By Observation 3, there are at most $\frac{lmd(G')}{2}$ gaps which contain two vertices each and $\frac{lmd(G')}{2}$ gaps which contain one vertex each. Then, $|V(G')| \leq \frac{5}{2} lmd(G')$. By Theorem 2, we have $|V(G')| \leq 5k$. Since there are $5k+1$ or $5k+2$ vertices, we obtain a contradiction.

Case 2: $m = 3$ or $4(\mod 5)$

Let $m = 5k+3$ or $m = 5k+4$ with the integer $k \geq 1$. By Observation 3, there are at most $\frac{lmd(G')-1}{2}$ gaps which contain two vertices each and $\frac{lmd(G')+1}{2}$ gaps which contain one vertex each. Then, $|V(G')| \leq \frac{5}{2}lmd(G') - \frac{1}{2}$. By Theorem 2, we have $|V(G')| \leq 5k+2$. Since there are $5k+3$ or $5k+4$ vertices, we obtain a contradiction. ■

For $G_i = K_{m_i} \setminus E(C_{m_i})$ with $m_i = 3$, since by Lemma 2 there exists a local basis W of G satisfying $V(G_i) \cap W = \emptyset$ and $|V(G_i)| = 3$, we can say that G_i has a gap containing three vertices.

Remark 2 *Let G be an $(n-3)$-regular graph of order $n \geq 5$. Let $G' \subseteq G$ such that $G' = K_m \backslash E(C_m)$ where $m \in \{3, 4, ..., n\}$. If $m = 3$, then we can say that G' has a gap containing three vertices.*

The following theorem provides the local metric dimension of $(n-3)$-regular graphs.

Theorem 3 *For $n \geq 5$ and $r \geq 1$, let $R_1, R_2, \ldots, R_r$ be r disjoint cycles contained in K_n such that $V(R_1) \cup V(R_2) \cup \ldots \cup V(R_r) = V(K_n)$. For $i \in \{1, 2, \ldots, r\}$, let $G = K_n \backslash (E(R_1) \cup E(R_2) \cup \ldots \cup E(R_r))$, $m_i = |V(R_i)|$, and $G_i = K_{m_i} \backslash E(R_i)$. If k is the number of disjoint cycles of order $m_i \geq 6$ such that $m_i \not\equiv 0 (\mod 5)$ and t is the number of disjoint cycles of order $m_i = 3$, then*

$$lmd(G) = \begin{cases} 1, & \text{if } n = 5, \\ \sum_{i=1}^{r} lmd(G_i), & \text{if } t = 0 \text{ and } n \geq 6 \text{ and } (k \in \{0,1\} \text{ or } r = \\ \sum_{i=1}^{r} lmd(G_i) + k - 1, & \text{if } t = 0 \text{ and } n \geq 6 \text{ and } k \geq 2, \\ \sum_{i=1}^{r} lmd(G_i) + k + t - 1, & \text{if } 1 \leq t \leq r \text{ and } n \geq 6 \text{ and } k \geq 0. \end{cases}$$

Proof. For $n = 5$, the graph G is isomorphic to the bipartite graph. Okamoto *et al.* [9] proved that the local metric dimension of the cycle of order n is equal to 1. Now, we assume that $n \geq 6$.

The second case for $lmd(G)$ is a direct consequence of Theorem 2, Lemma 3, and conditions (a)-(b) above.

For the two last cases, let $G' = K_{m_i} \backslash E(R_i)$ be a subgraph of G. We consider two possibilities of G'.

- $V(G') = \{a_1, a_2, a_3\}$;

By Remark 2, G' has a gap containing three vertices. We define $W^* = \{a_2\}$. Since a_1 is not adjacent to a_3, trivially W^* is a local resolving set of G'. Note that W^* only has one gap, that is a gap containing two vertices.

- $|V(G')| > 3$ and G' has a gap containing either three vertices a_1, a_2, a_3 where $a_j a_{j+1} \notin E(G')$ with $1 \leq j \leq 2$ or four vertices a_1, a_2, a_3, a_4 where $a_j a_{j+1} \notin E(G')$ with $1 \leq j \leq 3$;

  Let W' be a local basis of G'. By Observation 2, either gap $\{a_1, a_2, a_3\}$ or gap $\{a_1, a_2, a_3, a_4\}$ is the only one containing at least three vertices. Thus $W^* = W' \cup \{a_2\}$ is a local resolving set of G' which all the gaps contain at most two vertices.

So, by using above property, Theorem 2, Lemma 1 - Lemma 4, Remarks 1 and 2, and also (a)-(b), we prove the two last cases. ■

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