On the Edge-Balance Index Sets of n-Wheels

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Abstract

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A labeling f of a graph G is said to be edge-friendly if $|e_f(0)-e_f(1)| \leq 1$, where $e_f(i) = \text{card}\{e \in E(G) : f(e) = i\}$. An edge-friendly labeling $f : E(G) \to \mathbb{Z}_2$ induces a partial vertex labeling $f^+ : V(G) \to A$ defined by $f^+(x) = 0$ if the edges incident to x are labeled 0 more than 1. Similarly, $f^+(x) = 1$ if the edges incident to x are labeled 1 more than 0. $f^+(x)$ is not defined if the edges incident to x are labeled 1 and 0 equally. The edge-balance index set of the graph G, $EBI(G)$, is defined as $\{|v_f(0) - v_f(1)| : \text{the edge labeling f is edge-friendly}\}$, where $v_f(i) = \text{card}\{v \in V(G) : f^+(v) = i\}$.
An n-wheel is a graph consisting of n cycles with each vertex of the cycles connected to one central hub vertex. The edge-balance index sets of n-wheels are presented.

1 Introduction

In [6], Kong and Lee considered a new labeling problem in graph theory. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. An edge labeling $f : E(G) \to \mathbb{Z}_2$ induces a partial vertex labeling $f^+ : V(G) \to A$ defined by

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$f^+(x) = 0$ if the edges incident to x are labeled 0 more than 1. Similarly, $f^+(x) = 1$ if the edges incident to x are labeled 1 more than 0. $f^+(x)$ is not defined if the edges incident to x are labeled 1 and 0 equally. For each $i \in \mathbb{Z}_2$, we let $e_f(i) = \text{card}\{e \in E(G) : f(e) = i\}$ and $v_f(i) = \text{card}\{v \in V(G) : f^+(v) = i\}$. We also let $v_f(\times) = \text{card}\{v \in V(G) : f^+(v) \text{ is undefined}\}$. We will use $v(0)$, $v(1)$, $e(0)$ and $e(1)$ instead of the more complicated $v_f(0)$, $v_f(1)$, $e_f(0)$, $e_f(1)$, when the context is clear.

With these notations, we now introduce the notion of an edge-balanced graph.

Definition 1. *An edge labeling f of a graph G is said to be* **edge-friendly** *if $|e_f(0) - e_f(1)| \leq 1$. A graph G is said to be an* **edge-balanced** *graph if there is an edge-friendly labeling f satisfying $|v_f(0) - v_f(1)| \leq 1$.*

Chen, Lee, et al in [1] proved that all connected simple graphs except the star $K_{1,2k+1}$, where $k \geq 0$ are edge-balanced.

Definition 2. *The* **edge-balance index set** *of the graph G, EBI(G), is defined as $\{|v_f(0) - v_f(1)| :$ the edge labeling f is edge-friendly.$\}$.*

Example 3. *$EBI(nK_2)$ is $\{0\}$ if n is even and $\{2\}$ if n is odd.*

Figure 1: The edge-balance index set of $2K_2$ and $3K_2$

For any $n \geq 1$, we denote the tree with $n+1$ vertices of diameter two by St(n). The star has a center c and n appended edges from c.

Example 4. *The edge-balance index set of the star St(n) is*

$$EBI(St(n)) = \begin{cases} \{0\} & \text{if } n \text{ is even,} \\ \{2\} & \text{if } n \text{ is odd.} \end{cases}$$

The edge-balance index sets can be viewed as the dual of balance index sets. The balance index sets of graphs were considered in [7, 9, 10, 11, 12, 13, 15, 17, 18]. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathbb{Z}_2 = \{0, 1\}$. A labeling $f : V(G) \to \mathbb{Z}_2$ induces a partial edge labeling $f^* : E(G) \to A$ defined by $f^*(vw) = f(v)$, if and only if $f(v) = f(w)$ for each edge $vw \in E(G)$. For $i \in \mathbb{Z}_2$, let $v_f(i) = \text{card}\{v \in V(G) : f(v) = i\}$ and $e_{f^*}(i) = \text{card}\{e \in E(G) : f^*(e) = i\}$. A labeling f of a graph G is said to be ***friendly*** if $|v_f(0) - v_f(1)| \leq 1$. If $|e_f(0) - e_f(1)| \leq 1$ then G is said to be ***balanced***. The ***balance index set*** of the graph G, BI(G), is defined as $\{|e_f(0) - e_f(1)| :$ the vertex labeling f is friendly$\}$.

Edge-balance index sets of various types of graphs were considered in [1, 3, 4, 5, 8, 14, 16]. In particular, edge-balance index sets of wheels were considered in [2].

Definition 5. *For $n \geq 4$, the wheel on n vertices, W_n, is a graph constructed from a cycle C_{n-1} and an extra vertex c that is incident to all vertices on the cycle. The vertex c is called the hub, and the edges connecting the hub to the vertices of the cycle are called the spokes.*

Example 6. *Figure 2 shows the graph of W_7.*

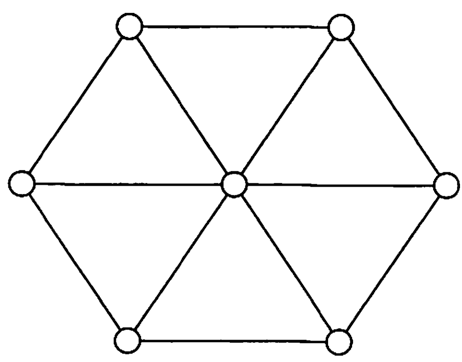

Figure 2: Graph of W_7

In [2] it was shown that the EBI of W_n is

$$EBI(W_n) = \begin{cases} \{0, 2, 4, .., n-2\} & \text{if } n \text{ is even} \\ \{1, 3, 5, .., n-2\} \cup \{0, 2, 4, .., 2\lfloor \frac{n}{4} \rfloor\} & \text{if } n \text{ is odd} \end{cases}$$

2 EBI of n-Wheels

In this paper, we will determine the edge-balance index set of n-wheels.

Definition 7. *For integers $a_1, a_2, \ldots, a_n \geq 3$, an n-wheel $W(a_1, \ldots, a_n)$ is a graph consisting of n cycles of lengths $a_1, a_2, \ldots, a_n$, where every vertex of each cycle is also connected to an additional vertex with an extra edge.*

The additional vertex will be referred to as the hub, and the edges connecting each vertex from a cycle to the hub will be referred to as spokes.

It should be noted that n-wheels also could have been defined as a graph consisting of the wheels $W_{a_1+1}, W_{a_2+1}, \ldots, W_{a_n+1}$, where the hub for each of the wheels is the same vertex. Of course, $W(a_1)$ is just W_{a_1+1}. Figure 3 gives an example of a 2-wheel.

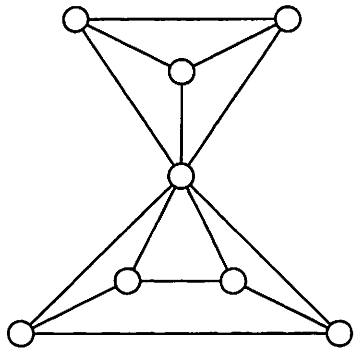

Figure 3: Graph of $W(3,4)$

We will use s for the total number of spokes:

$$s = \sum_{i=1}^{n} a_i$$

The total number of vertices of $W(a_1, \ldots, a_n)$ is then $s+1$, and the total number of edges is $2s$.

In [2] the EBI of wheels was found by splitting the wheel graph into a cycle and a star graph, with the star consisting of the hub and the spokes. While the same strategy could be applied to n-wheels by splitting them into n cycles and a star, we choose to use a different approach here. We will split the n-wheel into s hooks, namely one spoke connected to one edge from a cycle. This approach has the advantage that initially one can investigate each hook irrespective of which cycle it is in, as each wheel part of a hook is only incident to two other hooks.

Since a hook consists of two edges, there are only $2^2 = 4$ different ways to label a hook, as there are only two different ways to label each edge. Figure 4 shows all of the four different labelings.

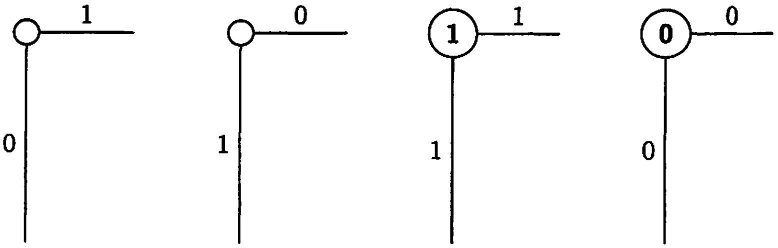

Figure 4: All types of hooks

Since the corner of the hook is a vertex of order three, hooks that have the same label for the spoke and the wheel part induce that same label for the vertex.

We let p be the number of hooks with the spoke labeled 0 and the wheel part labeled 1. We will refer to these hooks as 0-1-hooks. Similarly, we let q be the number of 1-0-hooks.

Since the labeling is edge-friendly, the number of 0-0-hooks and 1-1-hooks is the same. We use r to denote the number of either 0-0-hooks or 1-1-hooks.

We then have

$$p + q + 2r = s.$$

To determine the EBI we will consider two cases:

Case 1: s is odd

In this case, every vertex is labeled. All vertices on the wheel part have order three, and the hub has order s, which is odd as well. There are a total of $s+1$ vertices and, since each vertex is labeled, this implies that the edge-balance index of any labeling is even.

We now show that there is a labeling for every even edge-balance index up to $s-1$. This will complete this case, as there cannot be an edge-friendly labeling for $s+1$.

Suppose $0 \le j \le s-1$ is an even number. Use a labeling with $p = \frac{s+j+1}{2}$, $q = \frac{s-j-1}{2}$ and $r = 0$. Note that $p + q + 2r = s$ and $0 \le q < p \le s$.

Since $q < p$, the hub is labeled 0. Since all hooks are either 0-1-hooks or 1-0-hooks, the label of the vertex at the corner of the hook is determined by the wheel part of the neighboring hook. There are p wheel parts labeled 1 and q wheel parts labeled 0.

Thus the edge-balance index of this labeling is

$$p - q - 1 = \frac{s + j + 1}{2} - \frac{s - j - 1}{2} - 1 = j.$$

Case 2: s is even

If s is even we will look at two different cases, namely, whether the hub is labeled or unlabeled. The hub is unlabeled if $p = q$. Without loss of generality, the other case is $p > q$.

Case 1: $p > q$

In this case, every vertex is labeled. Similar to case 1 above, this implies that the edge-balance index of any labeling is odd, since there are an odd number of vertices. Just as above, we exhibit a labeling for each odd number up to $s - 1$.

Suppose $1 \leq j \leq s-1$ is an odd number. Use a labeling with $p = \frac{s+j+1}{2}$, $q = \frac{s-j-1}{2}$ and $r = 0$. Note that $p + q + 2r = s$ and $0 \leq q < p \leq s$.

Since $q < p$, the hub is labeled 0. Since all hooks are either 0-1-hooks or 1-0-hooks, the label of the vertex at the corner of the hook is determined by the wheel part of the neighboring hook. There are p wheel parts labeled 1 and q wheel parts labeled 0. Thus the edge-balance index of this labeling is

$$p - q - 1 = \frac{s + j + 1}{2} - \frac{s - j - 1}{2} - 1 = j.$$

Case 2: $p = q$

Here, the hub is not labeled. Each vertex on a wheel is labeled, so the edge-balance index has to be even.

We first determine the largest possible edge-balance index. The labels of the vertices of the 0-0-hooks and 1-1-hooks are already determined, and these induce an equal number of vertices labeled 0 and 1. There is no contribution to the edge-balance index from these hooks. The edge-balance index solely depends on the 0-1-hooks and 1-0-hooks. The label of these depends on the label of the wheel part of the previous hook. Since $p = q$, the number of wheel parts labeled 1 is the same as number of wheel parts labeled 0. Thus, there are $\frac{s}{2}$ wheel parts with the same label, and the edge-balance index is at most $\frac{s}{2}$.

Since the edge-balance index has to be even, it is the largest even number less or equal to $\frac{s}{2}$. We can write this as $2\left\lfloor \frac{s}{4} \right\rfloor$.

We now show that for any of these numbers there exists a labeling. Suppose $0 \leq j \leq 2\left\lfloor \frac{s}{4} \right\rfloor$ is an even number. Use a labeling with $p = q = \frac{j}{2}$ and $r = \frac{s-j}{2}$. Note that $p + q + 2r = s$ and $p = q \leq r$.

The labels of the vertices of the 0-0-hooks and 1-1-hooks are already determined by the hooks. The labels for the 0-1-hooks and 1-0-hooks are determined by the label of the wheel part of the neighboring hook. Thus, there are $p + q = j$ vertices that are determined by the previous hook. We now show that it is possible to arrange the hooks in such a manner that all of these vertices will be labeled 1. This implies that the edge-balance index of the labeling is j.

We group hooks together into blocks. Each block starts with a 1-1-hook to ensure that the label of the first block is already determined and will not change by attaching another hook to its vertex. The 1-1-hook will then be attached to as many 0-1-hooks as it is possible to fit into the cycle that this block is going into. This ensures that the vertex of each of these 0-1-hooks will now be labeled 1. The last 0-1-hook of this block will then be attached to a 1-0-hook. The vertex of the 1-0-hook is then also labeled 1.

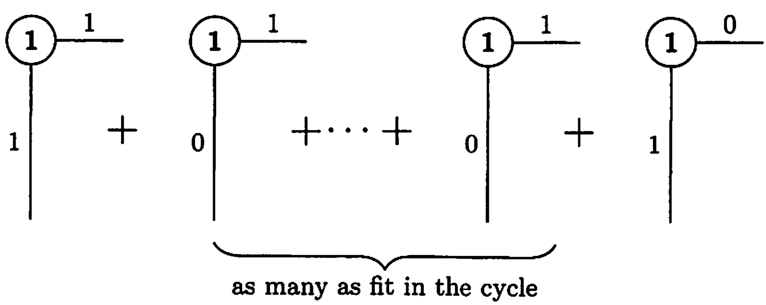

Figure 5: A block of hooks

These blocks will be filled into cycles one by one until all 0-1-hooks are used up in that way. Since each cycle will have at least three hooks in it, there is room for at least one 0-1-hook in each block until they are used up. This means that the 0-1-hooks will be used up either before or at the same time as the 1-0-hooks. Once the 0-1-hooks are used up, the blocks will consist only of two hooks, namely a 1-1-hook and a 1-0-hook. Since $r \geq q$, the 1-0-hook will be used up either at the same time or before the 1-1-hooks. 0-0-hooks will only be used if the cycle cannot accommodate any more blocks. This means that initially, only at most one 0-0-hook will be used in a cycle, since the smallest block will consist of two hooks. This has the consequence that the 1-1-hooks will be used up before the 0-0-hooks, and it is possible to fit all blocks into the cycles in this manner. Once all blocks are placed in the cycles, the remaining hooks will be filled with the remaining 0-0-hooks.

In each block, the vertex of all 0-1-hooks and 1-0-hooks will be labeled 1. Since all the vertices of all 0-1-hooks and 1-0-hooks are labeled 1, the edge-balance index of this labeling is $p+q=j$.

We have proven that

$$\mathrm{EBI}(W(a_1,\ldots,a_n)) = \begin{cases} \{0,2,4,..,s-1\} & \text{if } s \text{ is odd} \\ \{1,3,5,..,s-1\} \cup \{0,2,4,..,2\lfloor \frac{s}{4} \rfloor\} & \text{if } s \text{ is even} \end{cases}$$

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