Constructing the spectrum for packings of the complete graph with trees that have up to five edges

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Abstract

The spectrum problem for decomposition for trees with up to eight edges was introduced and solved in 1978 by Huang and Rosa. Also, the packing problem was settled for all trees with up to six edges by Roditty. For the first time, we consider obtaining all possible leaves in a maximum tree-packing of K_n, which we refer to as the spectrum problem for packings for complete graphs. In particular, we completely solve this problem for trees with at most five edges. The packing designs are used in developing optimal error correcting codes, which have applications in biology, for instance, in DNA sequencing.

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1 Introduction

For graph theoretic terminology, we follow West [17]. We consider our graphs to have no loops nor multiple edges. For graphs G and H, a *G-decomposition* of H is a partition of the edge set of H with graphs all isomorphic to G. The *spectrum problem for decomposition* for a graph G is to determine the necessary and sufficient conditions for n such that the complete graph K_n has a G-decomposition. Graph decompositions were first used by Kirkman in order to attack Kirkman's schoolgirl problem [7]. In 1972, Hell and Rosa considered graph decompositions for graphs which were not cliques for the first time, in their attempt to solve the spectrum problem for decomposition for paths of length three [5].

The spectrum problem for decomposition has been completely solved for paths [16] and stars [18], which are two infinite classes of trees, and also for trees on nine or fewer vertices [6]. Furthermore, Kovar et. al. [8] gave a complete classification of brooms of order $2n$ as a class of trees, that decompose the complete graph K_{2n}.

In cases where a decomposition does not exist, we may instead consider a *packing*. For graphs G and H, a G-packing of the graph H is a collection of subgraphs of H, each isomorphic to G, such that every edge of H is contained in at most one subgraph. Those edges of H which are not included in any of the subgraphs form the *leave graph*. A *maximum G-packing* of H is a packing with the smallest number of edges in the leave graph.

The packing designs are important for finding the optimal numbers in designing experiments where the necessary conditions for the existence of a decomposition do not hold. These designs also have applications in developing optimal error correcting codes that are capable of correcting combinations of deletions and insertions occurring in transmissions [9],[15]. Moreover, the error correcting codes have applications in biology, such as in DNA sequencing [1].

The number of subgraphs in a maximum G-packing of H is called the *G-packing number* of H and the *packing problem* of a graph G is to find the G-packing number of K_n for any positive integer n. The packing and covering problems were solved for trees with up to six edges by Roditty [11, 12, 13, 14].

Since different packings might lead to different leave graphs, we consider if it is feasible to achieve all possible leave graphs, the problem Roditty did not solve. The problem of obtaining all possible leave graphs for G-packings of a complete graph K_n is called the *spectrum problem for packing* for G.

Here, we consider the spectrum problem for trees that have up to five edges. If T is any tree with less than four edges and n any positive integer, the leave graph in a maximum T-packing of K_n has at most one edge and the only possible leave graph will be K_2 [11]. The spectrum problem for packing has been solved by the authors for 4-stars [2] and 5-stars [3]. This was the first time that this natural extension of the packing problem was considered. In this paper, we solve the problem for all trees with up to five edges. We introduce some known results and preliminary lemmas in section 2, then prove our main results in sections 3 and 4 through some recursive constructions.

2 Preliminaries

In 1978 Huang and Rosa [6] solved the spectrum problem for trees with up to eight edges. In particular, they proved the following theorem.

Theorem 1 ([6]). *If n is any positive integer and T is any tree with i edges, where $i = 4$ or 5, then the complete graph K_n has a T-decomposition if and only if $\frac{n(n-1)}{2} \equiv 0 \pmod{i}$.*

In 1986, Roditty solved the packing problem for all trees with four edges.

Theorem 2 ([13]). *If T is a tree with four edges and $n \geq 7$ is any integer, then the T-packing number of K_n, is $\left\lfloor \frac{n(n-1)}{8} \right\rfloor$ and the number of edges in the leave graph of a maximum T-packing of K_n is $\frac{n(n-1)}{2} - 4\left\lfloor \frac{n(n-1)}{8} \right\rfloor$.*

One of our main theorems states that for any tree T with four edges and any integer $n \geq 7$, all possible leave graphs in T-packings of K_n are achievable.

The complete bipartite graph $K_{1,k}$ is called a k-star and is denoted by S_k. Using this terminology, we can say The non-isomorphic trees with four edges are S_4, P_5, and A, where A is a 3-star with an edge attached to one of its leave graphs. The spectrum problem for packing for the graph S_4 was solved in 2013 by the authors [2]. In the next section, we solve the spectrum problem for packing for the graphs A and P_5.

Notation.

1. Let A be the tree with the vertex set $V(A) = \{x_1, x_2, x_3, x_4, x_5\}$ and the edge set $E(A) = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_4, x_5\}\}$. We denote A by $(x_1; x_2, x_3, x_4 - x_5)$.
2. Let x_1, x_2, x_3, x_4, and x_5 be vertices of a path P_5. We denote this path with $(x_1, x_2, x_3, x_4, x_5)$.

The following lemma will be used to prove that all possible leave graphs in T-packings of K_n, where T is any tree with four edges, are achievable.

Lemma 3. *For positive integers m and n, $n \geq 2$, the graph $K_{4m,n}$ has a P_5-decomposition and an A-decomposition.*

Proof. We first consider A. It suffices to show that $K_{4,2}$ and $K_{4,3}$ have A-decompositions. For $K_{4,2}$, label the four vertices of one part with the elements of $\mathbb{Z}_4$ having subscript 1 and the two vertices of the other part with the elements of $\mathbb{Z}_2$ having subscript 2. Then, the following trees form an A-decomposition of the graph $K_{4,2}$. (See Figure 1.)

$$(0_2; 0_1, 1_1, 2_1 - 1_2)$$
$$(1_2; 0_1, 1_1, 3_1 - 0_2)$$

For $K_{4,3}$, label the four vertices of one part with the elements of $\mathbb{Z}_4$ having

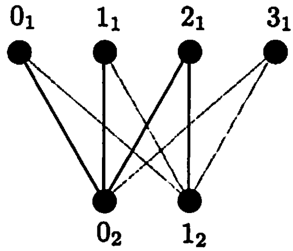

Figure 1: A-decomposition of $K_{4,2}$

subscript 1 and the three vertices of the other part with the elements of $\mathbb{Z}_3$ having subscript 2. Then, the following trees form an A-decomposition of the graph $K_{4,3}$ (See Figure 2).

$$(0_1; 0_2, 1_2, 2_2 - 1_1)$$
$$(2_1; 0_2, 2_2, 1_2 - 1_1)$$
$$(3_1; 1_2, 2_2, 0_2 - 1_1)$$

Now consider P_5.

For $K_{4,2}$, label the four vertices of one part with the elements of $\mathbb{Z}_4$ having subscript 1 and the two vertices of the other part with the elements of $\mathbb{Z}_2$ having subscript 2. The following paths form a P_5-decomposition for the graph $K_{4,2}$.

$$(0_1, 0_2, 1_1, 1_2, 2_1)$$
$$(0_1, 1_2, 3_1, 0_2, 2_1)$$

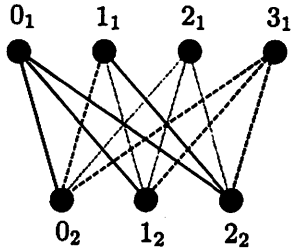

Figure 2: A-decomposition of $K_{4,3}$

For $K_{4,3}$, label the four vertices of one part with the elements of $\mathbb{Z}_4$ having subscript 1 and the three elements of the other part with the elements of $\mathbb{Z}_3$ having subscript 2. The following paths form a P_5-decomposition of $K_{4,3}$.

$$(0_1, 0_2, 1_1, 1_2, 2_1)$$
$$(0_1, 2_2, 2_1, 0_2, 3_1)$$
$$(0_1, 1_2, 3_1, 2_2, 1_1)$$

□

Corollary 4. *If $n \geq 2$ is an integer, T any tree with four edges, and K_n has a T-packing with the leave graph H, then K_{n+8} has a T-packing with the leave graph H.*

Proof. Let $n \geq 1$, T be any tree with four edges, and R be a T-packing of K_n with the leave graph H. Write $K_{n+8} = K_n \vee K_8$. By Theorem 1, the complete graph K_8 has a T-decomposition, S. Moreover, the complete bipartite graph $K_{n,8}$ has a T-decomposition, U, by Lemma 3. Therefore, $R \cup S \cup U$ forms a T-packing of K_{n+8} with the leave graph H. □

All non-isomorphic trees with five edges are shown in Figure 3.

Theorem 5 ([12]). *If T is a tree with five edges and $n \geq 9$ is any integer, then the T-packing number of K_n is $\left\lfloor \frac{n(n-1)}{10} \right\rfloor$ and the number of edges in the leave graph of a maximum T-packing of K_n is $\frac{n(n-1)}{2} - 5\left\lfloor \frac{n(n-1)}{10} \right\rfloor$.*

We will prove that for any tree T with four edges and any integer $n \geq 7$, all possible leave graphs in T-packings of K_n are achievable.

All non-isomorphic trees with five edges are $S_5, B, C, D, E,$ and P_6 (see Figure 3). The spectrum problem for packing for the graph S_5 was solved

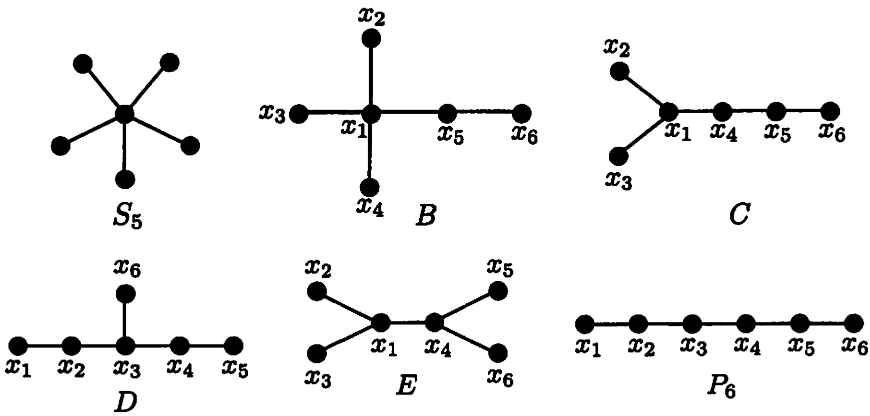

Figure 3: All non-isomorphic trees with five edges

in 2013 by the authors [3]. In this section, we solve the spectrum problem for packing for the remaining trees.

Notation. Consider the vertex labels in Figure 3. We denote the trees $B, C, D, E,$ and P_6 with $(x_1; x_2, x_3, x_4, x_5 - x_6)$, $(x_1; x_2, x_3, x_4 - x_5 - x_6)$, $(x_3; x_6, x_2, x_4 - x_1, x_5)$, $(x_1 - x_2, x_3; x_4 - x_5, x_6)$, and $(x_1, x_2, x_3, x_4, x_5, x_6)$ respectively.

Lemmas 6 and 7 will be used to prove that all possible leave graphs in T-packings of K_n are achievable, where T is any tree with five edges.

Lemma 6. *For positive integers m and n, the graph $K_{5m,n}$ has a B-decomposition and a C-decomposition if $n \geq 2$, a D-decomposition and an E-decomposition if $n \geq 3$, and a P_6-decomposition if $n \geq 4$.*

Proof. Let m and n be positive integers, $n \geq 2$. We first consider B. It suffices to show that $K_{5,2}$ and $K_{5,3}$ have B-decompositions. For $K_{5,2}$, label the five vertices of one part with the elements of $\mathbb{Z}_5$ having subscript 1 and the two vertices of the other part with the elements of $\mathbb{Z}_2$ having subscript 2. The following trees form a B-decomposition of $K_{5,2}$ (see Figure 4).

$$(0_2; 0_1, 1_1, 2_1, 3_1 - 1_2)$$
$$(1_2; 0_1, 1_1, 2_1, 4_1 - 0_2)$$

For $K_{5,3}$, label the five vertices of one part with the elements of $\mathbb{Z}_5$ having subscript 1 and the three vertices of the other part with the elements of $\mathbb{Z}_3$ having subscript 2. The following trees form a B-decomposition of $K_{5,3}$

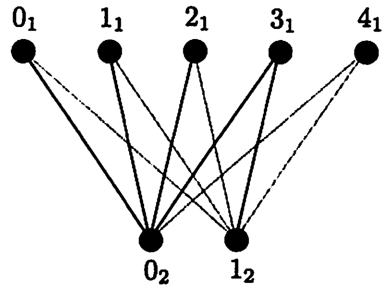

Figure 4: B-decomposition of $K_{5,2}$

(see Figure 5).

$$(0_2; 0_1, 1_1, 2_1, 3_1 - 1_2)$$
$$(1_2; 0_1, 1_1, 4_1, 2_1 - 2_2)$$
$$(2_2; 0_1, 1_1, 3_1, 4_1 - 0_2)$$

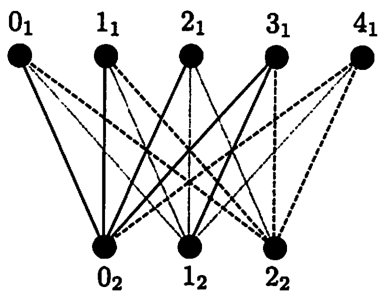

Figure 5: B-decomposition of $K_{5,3}$

Now we consider C. It suffices to show that the graphs $K_{5,2}$ and $K_{5,3}$ have C-decompositions.

For $K_{5,2}$, label the five vertices of one part with the elements of $\mathbb{Z}_5$ having subscript 1 and the two vertices of the other part with the elements of $\mathbb{Z}_2$ having subscript 2. The following trees form a C-decomposition of $K_{5,2}$.

$$(0_2; 0_1, 1_1, 2_1 - 1_2 - 3_1)$$
$$(1_2; 0_1, 1_1, 4_1 - 0_2 - 3_1)$$

For $K_{5,3}$, label the five vertices of one part with the elements of $\mathbb{Z}_5$ having subscript 1 and the three vertices of the other part with the elements of $\mathbb{Z}_3$ having subscript 2. The following trees form a C-decomposition of $K_{5,3}$.

$$(0_2; 0_1, 1_1, 2_1 - 1_2 - 3_1)$$
$$(1_2; 0_1, 1_1, 4_1 - 2_2 - 2_1)$$
$$(2_2; 0_1, 1_1, 3_1 - 0_2 - 4_1)$$

Now let $n \geq 3$. First we consider D. It suffices to prove that the graphs $K_{5,3}, K_{5,4}$, and $K_{5,5}$ have D-decompositions.

For $K_{5,3}$, label the five vertices of one part with the elements of $\mathbb{Z}_5$ having subscript 1 and the three vertices of the other part with the elements of $\mathbb{Z}_3$ having subscript 2. The following graphs form a D-decomposition of $K_{5,3}$.

$$(0_1; 2_2, 0_2, 1_2 - 1_1, 2_1)$$
$$(3_1; 0_2, 1_2, 2_2 - 1_1, 2_1)$$
$$(4_1; 1_2, 0_2, 2_2 - 2_1, 1_1)$$

For $K_{5,4}$, label the five vertices of one part with the elements of $\mathbb{Z}_5$ having subscript 1 and the four vertices of the other part with the elements of $\mathbb{Z}_4$ having subscript 2. The following graphs form a D-decomposition of $K_{5,4}$.

$$(0_2; 0_1, 1_1, 2_1 - 2_2, 1_2)$$
$$(1_2; 0_1, 3_1, 4_1 - 2_2, 3_2)$$
$$(2_2; 0_1, 2_1, 4_1 - 3_2, 0_2)$$
$$(3_2; 0_1, 1_1, 3_1 - 1_2, 0_2)$$

For $K_{5,5}$, label the five vertices of one part with the elements of $\mathbb{Z}_5$ having subscript 1 and the five vertices of the other part with the elements of $\mathbb{Z}_5$ having subscript 2. The following graphs form a D-decomposition of $K_{5,5}$.

$$(3_1; 2_2, 0_2, 1_2 - 0_1, 2_1)$$
$$(4_1; 0_2, 1_2, 2_2 - 0_1, 2_1)$$
$$(0_1; 2_2, 3_2, 4_2 - 3_1, 4_1)$$
$$(1_1; 2_2, 0_2, 3_2 - 2_1, 4_1)$$
$$(4_2; 3_1, 1_1, 2_1 - 1_2, 3_2)$$

Now consider E. It suffices to prove the existence of an E-decomposition of the graphs $K_{5,3}$, $K_{5,4}$, and $K_{5,5}$.

For $K_{5,3}$, label the three vertices of one part with the elements of $\mathbb{Z}_3$ having subscript 1 and the five vertices of the other part with the elements of $\mathbb{Z}_5$ having subscript 2. The following trees form an E-decomposition of the graph $K_{5,3}$.

$$(0_1 - 0_2, 1_2; 2_2 - 1_1, 2_1)$$
$$(1_1 - 0_2, 1_2; 3_2 - 0_1, 2_1)$$
$$(2_1 - 0_2, 1_2; 4_2 - 0_1, 1_1)$$

For $K_{5,4}$, label the four vertices of one part with the elements of $\mathbb{Z}_4$ having subscript 1 and the five vertices of the other part with the elements of $\mathbb{Z}_5$ having subscript 2. The following trees form an E-decomposition of the graph $K_{5,4}$.

$$(0_1 - 1_2, 4_2; 2_2 - 1_1, 2_1)$$
$$(1_1 - 1_2, 3_2; 4_2 - 2_1, 3_1)$$
$$(2_1 - 0_2, 1_2; 3_2 - 0_1, 3_1)$$
$$(3_1 - 1_2, 2_2; 0_2 - 0_1, 1_1)$$

For $K_{5,5}$, label the five vertices of one part with the elements of $\mathbb{Z}_5$ having subscript 1 and the five vertices of the other part with the elements of $\mathbb{Z}_5$ having subscript 2. The following trees form an E-decomposition of the graph $K_{5,5}$. Note that the addition is taken modulo 5.

$$(i_1 - (i+1)_2, (i+2)_2; i_2 - (i+1)_1, (i+2)_1), i \in \mathbb{Z}_5$$

Finally, let $n \geq 4$. Parker proved that there exist P_6-decompositions of $K_{5,4}, K_{5,5}, K_{5,6}$, and $K_{5,7}$ [10]. Therefore, for any $n \geq 4$, the graph $K_{n,5}$ has a P_6-decomposition. □

Let m and n be positive integers. The *disjoint union* of graphs G and H, denoted $G+H$, is the union of graphs G and H with disjoint vertex sets. The *join* of simple graphs G and H, denoted $G \vee H$ is the graph obtained from the disjoint union $G+H$ by adding the edges $\{\{x,y\} | x \in V(G), y \in V(H)\}$. Also for any graph G, mG is the graph consisting of m pairwise disjoint copies of G.

Lemma 7. *If $n \geq 7$ is an integer, T any tree with five edges, and K_n has a T-packing with the leave graph H, then K_{n+5} has a T-packing with the leave graph H. Furthermore, this statement is true if $n = 6$ and T is any of B, C, D, and E, or if $n = 5$ and T is either of B or C.*

Proof. Case 1. $n \geq 5, T = B$

Let R be a B-packing of K_n with the leave graph H. Write $K_{n+5} = K_n \vee K_5$. Label the vertices of K_n with the elements of $\mathbb{Z}_n$ having subscript

1 and the vertices of K_5 with the elements of $\mathbb{Z}_5$ having subscript 2. The set of vertices $\{0_1, 1_1, 2_1\}$, the set of vertices $\{0_2, 1_2, 2_2, 3_2, 4_2\}$, the edges between these two sets, and the edges within the second set form a graph $K_5 \vee 3K_1$. The following trees form a B-decomposition, S, of $K_5 \vee 3K_1$.

$$(i_2; 0_1, 1_1, (i+1)_2, (i+2)_2 - 2_1), \ i \in \mathbb{Z}_5$$

Now, the set of vertices $\{3_1, 4_1, 5_1, \ldots, (n-1)_1\}$, the set of vertices $\{0_2, 1_2, 2_2, 3_2, 4_2\}$, and the edges between these two sets form a complete bipartite graph $K_{5,n-3}$, which has a B-decomposition, T, by Lemma 6. Therefore, $R \cup S \cup T$ forms a B-packing of K_{n+5} with the leave graph H.

Case 2. $n \geq 5, T = C$

Let R be a C-packing of K_n with the leave graph H. Write $K_{n+5} = K_n \vee K_5$. Label the vertices of K_n with the elements of $\mathbb{Z}_n$ having subscript 1 and the vertices of K_5 with the elements of $\mathbb{Z}_5$ having subscript 2. The set of vertices $\{0_1, 1_1, 2_1\}$, the set of vertices $\{0_2, 1_2, 2_2, 3_2, 4_2\}$, the edges between these two sets, and the edges within the second set form a graph $K_5 \vee 3K_1$. The following trees form a C-decomposition, S, of the graph $K_5 \vee 3K_1$.

$$(0_2; 1_2, 2_2, 0_1 - 3_2 - 1_1)$$
$$(1_2; 2_2, 3_2, 1_1 - 4_2 - 2_1)$$
$$(2_2; 3_2, 4_2, 2_1 - 0_2 - 1_1)$$
$$(3_2; 4_2, 0_2, 2_1 - 1_2 - 0_1)$$
$$(4_2; 0_2, 1_2, 0_1 - 2_2 - 1_1)$$

Moreover, the set of vertices $\{3_1, 4_1, 5_1, \ldots, (n-1)_1\}$, the set of vertices $\{0_2, 1_2, 2_2, 3_2, 4_2\}$, and the edges between these two sets form a complete bipartite graph $K_{5,n-3}$, which has a C-decomposition, T, by Lemma 6. Therefore, $R \cup S \cup T$ forms a C-packing with the leave graph H for K_{n+5}.

Case 3. $n \geq 6, T = D$

Let R be a D-packing of K_n with the leave graph H. Write $K_{n+5} = K_n \vee K_5$. Label the vertices of K_n with the elements of $\mathbb{Z}_n$ having subscript 1 and the vertices of K_5 with the elements of $\mathbb{Z}_5$ having subscript 2. The set of vertices $\{0_1, 1_1, 2_1\}$, the set of vertices $\{0_2, 1_2, 2_2, 3_2, 4_2\}$, the edges between these two sets, and the edges within the second set form a graph $K_5 \vee 3K_1$. The following graphs form a D-decomposition, S, of $K_5 \vee 3K_1$.

$$(i_2; 0_1, (i+1)_2, (i+2)_2 - 1_1, 2_1), i \in \mathbb{Z}_5$$

The complete bipartite graph $K_{5,n-3}$ with partite sets $\{3_1, 4_1, 5_1, \ldots, (n-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$, has a D-decomposition, T, by Lemma 6. Therefore, $R \cup S \cup T$ forms a D-packing of K_{n+5} with the leave graph H.

Case 4. $n \geq 6, T = E$

Let R be an E-packing of K_n with the leave graph H. Write $K_{n+5} = K_n \vee K_5$. Label the vertices of K_n with the elements of $\mathbb{Z}_n$ having subscript 1 and the vertices of K_5 with the elements of $\mathbb{Z}_5$ having subscript 2. The set of vertices $\{0_1, 1_1, 2_1\}$, the set of vertices $\{0_2, 1_2, 2_2, 3_2, 4_2\}$, the edges between these sets, and the edges within the second set, form a graph $K_5 \vee 3K_1$. The following trees form an E decomposition of the graph $K_5 \vee 3K_1$.

$$(i_2 - 0_1, (i+2)_2; (i+1)_2 - 1_1, 2_1), i \in \mathbb{Z}_5$$

Since $n \geq 6$, the complete bipartite graph with partite sets $\{3_1, 4_1, 5_1, \ldots, (n-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$ has an E-decomposition, S, by Lemma 6. Therefore, $R \cup S$ forms an E-packing of K_{n+5} with the leave graph H.

Case 5. $n \geq 7, T = P_6$

Let R be a P_6-packing of K_n with the leave graph H. Write $K_{n+5} = K_n \vee K_5$. Label the vertices of K_n with the elements of $\mathbb{Z}_n$ having subscript 1 and the vertices of K_5 with the elements of K_5 having subscript 2. The set of vertices $\{0_1, 1_1, 2_1\}$, the set of vertices $\{0_2, 1_2, 2_2, 3_2, 4_2\}$, the edges between these sets, and the edges within the second set, form a graph $K_5 \vee 3K_1$. The following paths form a P_6-decomposition, S, of $K_5 \vee 3K_1$.

$$(1_1, 1_2, 0_1, 0_2, 2_2, 3_2)$$
$$(0_1, 2_2, 2_1, 1_2, 3_2, 4_2)$$
$$(2_1, 3_2, 1_1, 2_2, 4_2, 0_2)$$
$$(2_1, 0_2, 1_1, 4_2, 1_2, 2_2)$$
$$(2_1, 4_2, 0_1, 3_2, 0_2, 1_2)$$

By Lemma 6, the complete bipartite graph with partite sets $\{3_1, 4_1, 5_1, \ldots, (n-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$ has a P_6-decomposition, T. Consequently, $R \cup S \cup T$ forms a P_6-packing of K_{n+5} with the leave graph H. □

3 Spectrum of packing for trees with four edges

In this section, we will show how to achieve all possible leave graphs in T-packings of K_n, where T is a tree with four edges, using the lemmas stated in the preliminaries section.

Theorem 8. *Let $n \geq 7$ be an integer, T be any tree with four edges, and the leave graph in a maximum T-packing of the complete graph K_n have i*

edges. For any graph H with i edges there exists a maximum T-packing of K_n such that the leave graph is isomorphic to H.

Proof. Let $n \geq 7$ be an integer and T be any tree with four edges. For $n \equiv 0, 1 \pmod 8$, the complete graph K_n has a T-decomposition by Theorem 1. For $n \equiv 2, 7 \pmod 8$, the leave graph is a single edge by Theorem 2. In that case, the only possible leave graph is K_2 and that was obtained by Roditty [13]. We show that for $n \equiv 3, 4, 5, 6 \pmod 8$, we can achieve every possible leave graph. As we stated, the only trees we need to prove the result for are A and P_5. For both cases, we prove the theorem considering congruency classes modulo 8.

Case 1. $n \equiv 3 \pmod 8$, $T = A$

The leave graph has three edges by Theorem 2. Therefore, the possible leave graphs are K_3, S_3, P_4, $3K_2$, and $P_3 + K_2$. By Corollary 4, it suffices to achieve all possible leave graphs for K_{11}.

The leave graph P_4 was obtained by Roditty [13]. In order to obtain the leave graph K_3, write $K_{11} = K_8 \vee K_3$. Label the vertices of K_8 with the elements of $\mathbb{Z}_8$ having subscript 1 and the vertices of K_3 with the elements of $\mathbb{Z}_3$ having subscript 2. By Theorem 1, K_8 has an A-decomposition, R. By Lemma 3, $K_{3,8}$ has an A-decomposition, S. Let S be formed by the following trees.

$$(0_2; 0_1, 1_1, 2_1 - 1_2)$$
$$(1_2; 0_1, 3_1, 1_1 - 2_2)$$
$$(2_2; 0_1, 2_1, 3_1 - 0_2)$$
$$(0_2; 4_1, 5_1, 6_1 - 1_2)$$
$$(1_2; 4_1, 7_1, 5_1 - 2_2)$$
$$(2_2; 4_1, 6_1, 7_1 - 0_2)$$

Therefore, $R \cup S$ forms a maximum A-packing of K_{11} with the leave graph K_3.

In order to obtain the leave graphs $P_3 + K_2$ and $3K_2$, we replace some of the trees in this packing with others. Table 1 shows the required substitutions.

In order to obtain S_3 as the leave graph, label the vertices as before. Let R be the A-decomposition of K_8. The set of vertices $\{0_1, 1_1, 2_1\}$, the set of vertices $\{0_2, 1_2, 2_2\}$, the edges between these two sets, and the edges within the second set, form a graph $K_3 \vee 3K_1$. The following trees form an

New tree(s)	Previous tree(s)	leave graph
$(0_2; 2_2, 1_1, 2_1 - 1_2)$	$(0_2; 0_1, 1_1, 2_1 - 1_2)$	
$(1_2; 0_1, 0_2, 1_1 - 2_2)$	$(1_2; 0_1, 3_1, 1_1 - 2_2)$	$P_3 + K_2$
$(0_2; 2_2, 1_1, 2_1 - 1_2)$	$(0_2; 0_1, 1_1, 2_1 - 1_2)$	
$(1_2; 0_1, 0_2, 1_1 - 2_2)$	$(1_2; 0_1, 3_1, 1_1 - 2_2)$	
$(2_2; 0_1, 1_2, 3_1 - 0_2)$	$(2_2; 0_1, 2_1, 3_1 - 0_2)$	$3K_2$

Table 1: Substitutions in the packing $R \cup S$ to obtain different leave graphs for $n = 11$

A-decomposition, S, of the graph $K_3 \vee 3K_1$.

$$(0_2; 0_1, 1_1, 1_2 - 2_1)$$
$$(1_2; 0_1, 1_1, 2_2 - 2_1)$$
$$(2_2; 0_1, 1_1, 0_2 - 2_1)$$

The complete bipartite graph with one partite set $\{3_1, 4_1, 5_1, 6_1\}$ and another partite set $\{0_2, 1_2, 2_2\}$ has an A-decomposition, T, by Lemma 3. Therefore, $R \cup S \cup T$ forms a maximum A-packing of K_{11} with the leave graph S_3.

Case 2. $n \equiv 3 \pmod 8$, $T = P_5$

By Corollary 4, it suffices to achieve all possible leave graphs for $n = 11$. The leave graph has three edges by Theorem 2. Hence, the possible leave graphs are K_3, S_3, P_4, $P_3 + K_2$, and $3K_2$. The leave graph P_4 was achieved by Roditty [13]. In order to obtain the leave graph K_3, write $K_{11} = K_8 \vee K_3$. Label the vertices of K_8 with the elements of $\mathbb{Z}_8$ having subscript 1 and the vertices of K_3 with the elements of $\mathbb{Z}_3$ having subscript 2. Let R be a P_5-decomposition of K_8. The graph $K_{8,3}$ has a P_5-decomposition, S, by Lemma 3. Therefore, $R \cup S$ forms a maximum P_5-packing with the leave graph K_3.

In order to obtain the leave graph S_3, partition and label the vertices as above and let R be a P_5-decomposition of K_8. Consider the complete bipartite graph with one partite set $\{3_1, 4_1, 5_1, 6_1\}$ and the other partite set $\{0_2, 1_2, 2_2\}$. This graph has a P_5-decomposition, S', by Lemma 3. The set of vertices $\{0_1, 1_1, 2_1\}$, the set of vertices $\{0_2, 1_2, 2_2\}$, the edges between these two sets, and the edges within the second set form a graph $K_3 \vee 3K_1$. The following paths form a P_5-decomposition, T, for $K_3 \vee 3K_1$.

$$(1_2, 0_2, 0_1, 2_2, 1_1)$$
$$(2_2, 1_2, 1_1, 0_2, 2_1)$$
$$(0_2, 2_2, 2_1, 1_2, 0_1)$$

Therefore, $R \cup S' \cup T$ forms a maximum P_5-packing of K_{11} with the leave graph S_3.

The other leave graphs can be achieved by substituting some paths with other ones in the packing $R \cup S' \cup T$. (See Table 2.)

New path(s)	Previous path(s)	leave graph
$(0_2, 2_2, 7_1, 1_2, 0_1)$	$(0_2, 2_2, 2_1, 1_2, 0_1)$	$P_3 + K_2$
$(7_1, 1_2, 2_1, 2_2, 0_2)$	$(0_1, 1_2, 2_1, 2_2, 0_2)$	
$(7_1, 2_2, 0_1, 0_2, 1_2)$	$(1_1, 2_2, 0_1, 0_2, 1_2)$	
$(7_1, 0_2, 1_1, 1_2, 2_2)$	$(2_1, 0_2, 1_1, 1_2, 2_2)$	$3K_2$

Table 2: Substitutions in the packing $R \cup S$ to obtain different leave graphs for $n = 11$

Case 3. $n \equiv 4 \pmod 8$, $T = A$

By Theorem 2, the leave graph has 2 edges in this case. So, the only possible leave graphs are P_3 and $2K_2$. By Corollary 4, it suffices to achieve all possible leave graphs for K_{12}. Roditty showed how to obtain the leave graph P_3 [13].

In order to achieve the leave graph $2K_2$, write $K_{12} = K_8 \vee K_4$. Label the vertices of K_8 with the elements of $\mathbb{Z}_8$ having subscript 1 and the vertices of K_4 with the elements of $\mathbb{Z}_4$ having subscript 2. By Theorem 1, K_8 has an A-decomposition, R. Consider the set of vertices $\{0_1, 1_1, 2_1\}$, the set of vertices $\{0_2, 1_2, 2_2, 3_2\}$, the edges between these two sets, and the edges within the latter. These vertices and edges form a graph $K_4 \vee 3K_1$. The following trees construct a maximum A-packing, S, of the graph $K_4 \vee 3K_1$ with the leave graph $2K_2$. (See Figure 6.)

$$(0_2; 1_2, 0_1, 1_1 - 2_2)$$
$$(1_2; 2_2, 0_1, 1_1 - 3_2)$$
$$(2_2; 3_2, 0_1, 2_1 - 0_2)$$
$$(3_2; 0_2, 0_1, 2_1 - 1_2)$$

Now, consider the complete bipartite graph with partite sets $\{3_1, 4_1, 5_1, 6_1, 7_1\}$ and $\{0_2, 1_2, 2_2, 3_2\}$. By Lemma 3, this bipartite graph has an A-decomposition, T. Therefore, $R \cup S \cup T$ forms a maximum A-packing of K_{12} with the leave graph $2K_2$.

Case 4. $n \equiv 4 \pmod 8$, $T = P_5$

By Corollary 4, it suffices to achieve all leave graphs for $n = 12$. The leave graph has two edges in this case. Hence, the possible leave graphs are P_3 and $2K_2$. The leave graph $2K_2$ was obtained by Roditty [13]. In order

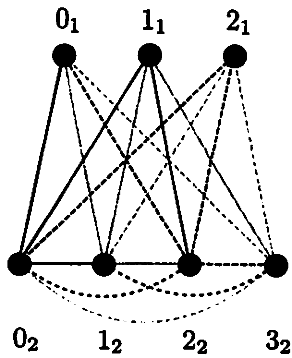

Figure 6: A-packing of $K_4 \vee 3K_1$ with the leave graph $2K_2$

to achieve the leave graph P_3, write $K_{12} = K_8 \vee K_4$. Label the vertices of K_8 with the elements of $\mathbb{Z}_8$ having subscript 1 and the vertices of K_4 with the elements of $\mathbb{Z}_4$ having subscript 2. Let R be a P_5-decomposition of K_8. The set of vertices $\{0_1, 1_1, 2_1\}$, the set of vertices $\{0_2, 1_2, 2_2, 3_2\}$, the edges between these two sets, and the edges within the second set form a graph $K_4 \vee 3K_1$. The following paths form a maximum P_5-packing, S, of $K_4 \vee 3K_1$ with the leave graph $2K_2$.

$$(1_2, 0_2, 0_1, 2_2, 1_1)$$
$$(2_2, 1_2, 1_1, 3_2, 2_1)$$
$$(3_2, 2_2, 2_1, 0_2, 1_1)$$
$$(0_2, 3_2, 0_1, 1_2, 2_1)$$

The complete bipartite graph with the partite sets $\{3_1, 4_1, 5_1, 6_1\}$ and $\{0_2, 1_2, 2_2, 3_2\}$ has a P_5-decomposition, T, by Lemma 3. Let U be the set consisting of the single path $(2_2, 0_2, 7_1, 3_2, 1_2)$. Therefore, $R \cup S \cup T \cup U$ forms a maximum P_5-packing of K_{12} with the leave graph P_3.

Case 5. $n \equiv 5 \pmod 8$, $T = A$

By Theorem 2, the leave graph has 2 edges in this case. The possible leave graphs are P_3 and $2K_2$. By Corollary 4 it suffices to obtain all leave graphs for $n = 13$. Roditty showed how to achieve the leave graph P_3. In order to gain the leave graph $2K_2$, write $K_{13} = K_{11} \vee K_2$. Label the vertices of K_{11} with the elements of $\mathbb{Z}_{11}$ having subscript 1 and the vertices of K_2 with the elements of $\mathbb{Z}_2$ having subscript 2. In case 1, we showed that there is a maximum A-packing of K_{11} with the leave graph $3K_2$. Let R be that packing and the edges of the leave graph be $\{5_1, 6_1\}$, $\{7_1, 8_1\}$, and $\{9_1, 10_1\}$. Consider the complete bipartite graph with one partite set $\{0_1, 1_1, 2_1, \ldots, 7_1\}$ and another partite set $\{0_2, 1_2\}$. This graph has an A-decomposition, S, by Lemma 3. Let T be formed by the trees $(0_2; 8_1, 9_1, 10_1 - 1_2)$ and $(1_2; 0_2, 8_1, 9_1 - 10_1)$. Therefore, $R \cup S \cup T$ forms a maximum A-packing of K_{13} with the leave graph $2K_2$.

Case 6. $n \equiv 5 \pmod 8$, $T = P_5$

By Corollary 4, it suffices to achieve all leave graphs for $n = 13$. The leave graph has two edges in this case. Hence, the possible leave graphs are P_3 and $2K_2$. Roditty obtained the leave graph P_3 [13]. In order to achieve the leave graph $2K_2$, write $K_{13} = K_9 \vee K_4$. Label the vertices of K_9 with the elements of $\mathbb{Z}_9$ having subscript 1 and the vertices of K_4 with the elements of $\mathbb{Z}_4$ having subscript 2. Let R be a P_5-decomposition of K_9. The set of vertices $\{0_1, 1_1, 2_1\}$, the set of vertices $\{0_2, 1_2, 2_2, 3_2\}$, the edges between these two sets, and the edges within the second set form a graph $K_4 \vee 3K_1$. The following paths form a maximum P_5-packing, S, of $K_4 \vee 3K_1$ with the leave graph $2K_2$.

$$(1_2, 0_2, 0_1, 2_2, 1_1)$$
$$(2_2, 1_2, 1_1, 3_2, 2_1)$$
$$(3_2, 2_2, 2_1, 0_2, 1_1)$$
$$(0_2, 3_2, 0_1, 1_2, 2_1)$$

The complete bipartite graph with one partite set $\{3_1, 4_1, 5_1, 6_1, 7_1, 8_1\}$ and the other partite set $\{0_2, 1_2, 2_2, 3_2\}$, has a P_5-decomposition, T, by Lemma 3. Therefore, $R \cup S \cup T$ forms a maximum P_5-packing of K_{13} with the leave graph $2K_2$.

Case 7. $n \equiv 6 \pmod 8$, $T = A$

By Theorem 2, the leave graph has three edges in this case. The possible leave graphs are those mentioned in case 2. By Corollary 4, it suffices to obtain all possible leave graphs for $n = 14$. Write $K_{14} = K_{11} \vee K_3$. Label the vertices of K_{11} with the elements of $\mathbb{Z}_{11}$ having subscript 1 and the vertices of K_3 with the elements of $\mathbb{Z}_3$ having subscript 2. Let H be any simple graph with three edges. By case 3, there is a maximum A-packing of K_{11}, R, with the leave graph H. The set of vertices $\{0_1, 1_1, 2_1\}$, the set of vertices $\{0_2, 1_2, 2_2\}$, the edges between these two sets, and the edges within the second set, forms a graph $K_3 \vee 3K_1$. The following trees form an A-decomposition, S, of the graph $K_3 \vee 3K_1$.

$$(0_2; 0_1, 1_1, 1_2 - 2_1)$$
$$(1_2; 0_1, 1_1, 2_2 - 2_1)$$
$$(2_2; 0_1, 1_1, 0_2 - 2_1)$$

The complete bipartite graph with one partite set $\{3_1, 4_1, 5_1, 6_1, 7_1, 8_1, 9_1,$

$10_1\}$ and the other partite set $\{0_2, 1_2, 2_2\}$, has an A-decomposition, T, by Lemma 3. Therefore, $R \cup S \cup T$ forms a maximum A-packing of K_{14} with the leave graph H. This completes the proof in this case.

Case 8. $n \equiv 6 \pmod 8$, $T = P_5$

The leave graph has three edges in this case by Theorem 2. Hence, the possible leave graphs are those mentioned in case 1. By Corollary 4, it suffices to achieve all possible leave graphs for $n = 14$. Let H be any possible leave graph with three edges. Write $K_{14} = K_{11} \vee K_3$. Label the vertices of K_{11} with the elements of $\mathbb{Z}_{11}$ having subscript 1 and the vertices of K_3 with the elements of $\mathbb{Z}_3$ having subscript 2. By case 1, there exists a maximum P_5-packing, R, of K_{11} with the leave graph H. The set of vertices $\{0_1, 1_1, 2_1\}$, the set of vertices $\{0_2, 1_2, 2_2\}$, the edges between these two sets, and the edges within the second set, form a graph $K_3 \vee 3K_1$. The following paths form a P_5-decomposition, S, of the graph $K_3 \vee 3K_1$.

$$(1_2, 0_2, 0_1, 2_2, 1_1)$$
$$(2_2, 1_2, 1_1, 0_2, 2_1)$$
$$(0_2, 2_2, 2_1, 1_2, 0_1)$$

The complete bipartite graph with partite sets $\{3_1, 4_1, 5_1, \ldots, 10_1\}$ and $\{0_2, 1_2,$
$2_2\}$, has a P_5-decomposition, T, by Lemma 3. Therefore, $R \cup S \cup T$ forms a maximum P_5-packing of K_{14} with the leave graph H. □

4 Spectrum of packing for trees with five edges

In this section, we will illustrate how to achieve all possible leave graphs in T-packings of K_n, where T is a tree with five edges, using the lemmas given in the preliminaries section.

Theorem 9. *Let $n \geq 9$ be an integer, T be any tree with five edges, and the leave graph in a T-packing of the complete graph K_n have i edges. For any graph H with i edges there exists a maximum T-packing of K_n such that the leave graph is isomorphic to H.*

Proof. Let $n \geq 9$ be an integer and T be any tree with five edges. As we stated, all non-isomorphic trees with five edges are S_5, B, C, D, E, and P_6, as shown in Figure 3, and for the tree S_5 the result is already established [3]. By Lemma 7, for each tree, it suffices to show the result for $n = 9, 10, 11, 12, 13$. For $n = 10$ and 11 and T any tree with five edges, there is a T-decomposition of K_n by Theorem 1. For $n = 9, 12$, the leave graph has a single edge by Theorem 5. Hence, the only possible leave graph will be K_2, which was given by Roditty [12]. Therefore, we only need to achieve all possible leave graphs for $n = 13$. By Theorem 5, the leave graph in a maximum T-packing of K_{13} has three edges. Hence, the possible leave graphs are K_3, S_3, P_4, P_3+K_2, and $3K_2$. Now, for each tree T, we construct maximum T-packings with each of these leave graphs.

Case 1. $T = B$

In order to obtain the leave graph K_3, write $K_{13} = K_{10} \vee K_3$. Label the vertices of K_{10} with the elements of $\mathbb{Z}_{10}$ having subscript 1 and the vertices of K_3 with the elements of $\mathbb{Z}_3$ having subscript 2. By Theorem 1, K_{10} has a B-decomposition, R. Moreover, the complete bipartite graph with one partite set $\{0_1, 1_1, 2_1, \ldots, 9_1\}$ and the other partite set $\{0_2, 1_2, 2_2\}$ has a B-decomposition, S, by Lemma 6. Let S consist of the following trees:

$$(0_2; 0_1, 1_1, 2_1, 3_1 - 1_2)$$
$$(1_2; 0_1, 1_1, 4_1, 2_1 - 2_2)$$
$$(2_2; 0_1, 1_1, 3_1, 4_1 - 0_2)$$
$$(0_2; 5_1, 6_1, 7_1, 8_1 - 1_2)$$
$$(1_2; 5_1, 6_1, 9_1, 7_1 - 2_2)$$
$$(2_2; 5_1, 6_1, 8_1, 9_1 - 0_2)$$

Therefore, $R \cup S$ forms a maximum B-packing of K_{13} with the leave graph K_3. We can obtain all the other possible leave graphs (except S_3) by making small changes to this construction. (See Table 3.)

New tree(s)	Previous tree(s)	leave graph
$(0_2; 2_2, 1_1, 2_1, 3_1 - 1_2)$	$(0_2; 0_1, 1_1, 2_1, 3_1 - 1_2)$	P_4
$(0_2; 2_2, 1_2, 2_1, 3_1 - 1_2)$	$(0_2; 0_1, 1_1, 2_1, 3_1 - 1_2)$	
$(2_2; 1_2, 0_1, 1_1, 4_1 - 0_2)$	$(2_2; 0_1, 1_1, 3_1, 4_1 - 0_2)$	$P_3 + K_2$
$(0_2; 2_2, 1_2, 2_1, 3_1 - 1_2)$	$(0_2; 0_1, 1_1, 2_1, 3_1 - 1_2)$	
$(2_2; 1_2, 0_1, 1_1, 4_1 - 0_2)$	$(2_2; 0_1, 1_1, 3_1, 4_1 - 0_2)$	
$(1_2; 0_2, 0_1, 4_1, 2_1 - 2_2)$	$(1_2; 0_1, 1_1, 4_1, 2_1 - 2_2)$	$3K_2$

Table 3: Substitutions in the packing $R \cup S$ to obtain different leave graphs for $n = 13$

In order to obtain the leave graph S_3, consider the same partition and labeling of the vertices of K_{13} and let R be the same B-decomposition of K_{10}. The complete bipartite graph with partite sets $\{4_1, 5_1, 6_1, 7_1, 8_1\}$ and $\{0_2, 1_2, 2_2\}$, has a B-decomposition, S', by Lemma 6. The set of vertices $\{0_1, 1_1, 2_1, 3_1\}$, the set of vertices $\{0_2, 1_2, 2_2\}$, the edges between these two sets, and the edges within the second set, form a graph $K_3 \vee 4K_1$. The following trees form a B-decomposition, T, of $K_3 \vee 4K_1$.

$$(i_2; 0_1, 1_1, 2_1, (i+1)_2 - 3_1),\ i \in \mathbb{Z}_3$$

Therefore, $R \cup S' \cup T$ forms a maximum B-packing of K_{13} with the leave graph S_3.

Case 2. $T = C$

In order to obtain the leave graph K_3, write $K_{13} = K_{10} \vee K_3$. Label the vertices of K_{10} with the elements of $\mathbb{Z}_{10}$ having subscript 1 and the vertices of K_3 with the elements of $\mathbb{Z}_3$ having subscript 2. Let R be the C-decomposition of K_{10}. The following trees form a C-decomposition, S, of the bipartite graph with one partite set $\{0_1, 1_1, 2_1, \ldots, 9_1\}$ and the other partite set $\{0_2, 1_2, 2_2\}$.

$$(0_2; 0_1, 1_1, 2_1 - 1_2 - 3_1)$$
$$(1_2; 0_1, 1_1, 4_1 - 2_2 - 2_1)$$
$$(2_2; 0_1, 1_1, 3_1 - 0_2 - 4_1)$$
$$(0_2; 5_1, 6_1, 7_1 - 1_2 - 8_1)$$
$$(1_2; 5_1, 6_1, 9_1 - 2_2 - 7_1)$$
$$(2_2; 5_1, 6_1, 8_1 - 0_2 - 9_1)$$

Therefore, $R \cup S$ forms a maximum C-packing of K_{13} with the leave graph K_3.

In order to obtain the leave graphs P_4, $P_3 + K_2$, and $3K_2$, we substitute some trees in the packing $R \cup S$ with new ones as shown in Table 4.

New tree(s)	Previous tree(s)	leave graph
$(0_2; 2_2, 1_1, 2_1 - 1_2 - 3_1)$	$(0_2; 0_1, 1_1, 2_1 - 1_2 - 3_1)$	P_4
$(0_2; 2_2, 1_1, 2_1 - 1_2 - 3_1)$ $(1_2; 0_1, 0_2, 4_1 - 2_2 - 2_1)$	$(0_2; 0_1, 1_1, 2_1 - 1_2 - 3_1)$ $(1_2; 0_1, 1_1, 4_1 - 2_2 - 2_1)$	$P_3 + K_2$
$(0_2; 2_2, 1_1, 2_1 - 1_2 - 3_1)$ $(1_2; 0_1, 0_2, 4_1 - 2_2 - 2_1)$ $(2_2; 1_2, 6_1, 8_1 - 0_2 - 9_1)$	$(0_2; 0_1, 1_1, 2_1 - 1_2 - 3_1)$ $(1_2; 0_1, 1_1, 4_1 - 2_2 - 2_1)$ $(2_2; 5_1, 6_1, 8_1 - 0_2 - 9_1)$	$3K_2$

Table 4: Substitutions in the packing $R \cup S$ to obtain different leave graphs for $n = 13$

In order to achieve the leave graph S_3, partition and label the vertices of K_{13} as above. Let R be the C-decomposition of K_{10}. The set of vertices $\{0_1, 1_1, 2_1, 3_1\}$, the set of vertices $\{0_2, 1_2, 2_2\}$, the edges between these two sets, and the edges within the second set, form a graph $K_3 \vee 4K_1$. The following trees form a C-decomposition, S', of $K_3 \vee 4K_1$.

$$(0_2; 1_2, 0_1, 1_1 - 2_2 - 2_1)$$
$$(1_2; 2_2, 0_1, 2_1 - 0_2 - 3_1)$$
$$(2_2; 0_2, 0_1, 3_1 - 1_2 - 1_1)$$

The complete bipartite graph with partite sets $\{4_1, 5_1, 6_1, 7_1, 8_1\}$ and $\{0_2, 1_2, 2_2\}$, have a C-decomposition, T, by Lemma 6. Therefore, $R \cup S' \cup T$ forms a maximum C-packing of K_{13} with the leave graph S_3.

Case 3. $T = D$

In order to achieve the leave graph K_3, write $K_{13} = K_{10} \vee K_3$. Label the vertices of K_{10} with the elements of $\mathbb{Z}_{10}$ having subscript 1 and the vertices of K_3 with the elements of $\mathbb{Z}_3$ having subscript 2. Let R be a D-decomposition of K_{10}. The graph $K_{10,3}$ has a D-decomposition, S, by Lemma 6. Therefore, $R \cup S$ forms a maximum D-packing of K_{13} with the leave graph K_3.

In order to obtain the leave graph S_3, partition and label the vertices of K_{13} as above. The set of vertices $\{0_1, 1_1, 2_1, 3_1\}$, the set of vertices $\{0_2, 1_2, 2_2\}$, the edges between these two sets, and the edges within the second set, form a graph $K_3 \vee 4K_1$. The following graphs form a D-decomposition, S', of the graph $K_3 \vee 4K_1$.

$$(0_2; 1_1, 0_1, 1_2 - 2_2, 2_1), (1_2; 0_1, 2_2, 3_1 - 2_1, 0_2), (2_2; 3_1, 0_2, 1_1 - 2_1, 1_2)$$

The complete bipartite graph with partite sets $\{4_1, 5_1, 6_1, 7_1, 8_1\}$ and $\{0_2, 1_2, 2_2\}$ has a D-decomposition, T, by Lemma 6. Therefore, $R \cup S' \cup T$ forms a maximum D-packing of K_{13} with the leave graph S_3.

Substitution of some graphs D with some others in the packing $R \cup S' \cup T$ leads to the leave graphs P_4, $P_3 + K_2$, and $3K_2$ (See Table 5).

New graph(s)	Previous graph(s)	leave graph
$(9_1; 0_2, 1_2, 2_2 - 0_1, 2_1)$	$(1_2; 0_1, 2_2, 3_1 - 2_1, 0_2)$	P_4
$(1_2; 9_1, 2_2, 3_1 - 2_1, 0_2)$	$(1_2; 0_1, 2_2, 3_1 - 2_1, 0_2)$	$P_3 + K_2$
$(1_2; 9_1, 2_2, 3_1 - 2_1, 0_2)$	$(1_2; 0_1, 2_2, 3_1 - 2_1, 0_2)$	
$(0_2; 9_1, 1_2, 0_1 - 2_1, 2_2)$	$(0_2; 1_1, 1_2, 0_1 - 2_1, 2_2)$	$3K_2$

Table 5: Substitutions in the packing $R \cup S$ to obtain different leave graphs for $n = 13$

Case 4. $T = E$

In order to obtain the leave graph K_3, write $K_{13} = K_{10} \vee K_3$. Label nine of the vertices of K_{10} with the elements of $\mathbb{Z}_9$ having subscript 1, the remaining vertex of K_{10} with ∞, and the vertices of K_3 with the elements of $\mathbb{Z}_3$ having subscript 2. The graph K_{10} has an E-decomposition, R. Let R contain the following trees. Note that the addition is taken modulo 9.

$$(i_1 - \infty, (i+1)_1; (i+2)_1 - (i+5)_1, (i+6)_1), i \in \mathbb{Z}_9$$

By Lemma 7, the graph $K_{10,3}$ has an E-decomposition, S. Therefore, $R \cup S$ forms a maximum E-packing of K_{13} with the leave graph K_3.

In order to obtain the leave graph S_3, divide and label the vertices of K_{13} as above. The set of vertices $\{0_1, 1_1, 2_1, 3_1\}$, the set of vertices $\{0_2, 1_2, 2_2\}$, the edges between these two sets, and the edges within the second set, form a graph $K_3 \vee 4K_1$. The following trees form an E-decomposition, T, of the graph $K_3 \vee 4K_1$.

$$(0_2 - 0_1, 1_1; 1_2 - 2_1, 3_1), (1_2 - 0_1, 1_1; 2_2 - 2_1, 3_1),$$
$$(2_2 - 0_1, 1_1; 0_2 - 2_1, 3_1)$$

The vertices $4_1, 5_1, 6_1, 7_1, 8_1$, the vertices $0_2, 1_2, 2_2$, and the edges between them, form a graph $K_{3,5}$, which has an E-decomposition, U, by Lemma 6. Therefore, $R \cup T \cup U$ forms a maximum E-packing of K_{13} with the leave graph S_3.

In order to obtain the remaining leave graphs, we substitute some trees for others in the packing $R \cup T \cup U$. Table 6 illustrates these substitutions.

New tree(s)	Previous tree(s)	leave graph
$(0_2 - 1_1, \infty; 1_2 - 2_1, 3_1)$	$(0_2 - 0_1, 1_1; 1_2 - 2_1, 3_1)$	$P_3 + K_2$
$(0_2 - 1_1, \infty; 1_2 - 2_1, 3_1)$	$(0_2 - 0_1, 1_1; 1_2 - 2_1, 3_1)$	
$(1_2 - 0_1, \infty; 2_2 - 2_1, 3_1)$	$(1_2 - 0_1, 1_1; 2_2 - 2_1, 3_1)$	$3K_2$
$(0_2 - 0_1, 1_1; \infty - 1_2, 2_2)$	$(0_2 - 0_1, 1_1; 1_2 - 2_1, 3_1)$	
$(2_1 - \infty, 1_2; 4_1 - 7_1, 8_1)$	$(2_1 - \infty, 3_1; 4_1 - 7_1, 8_1)$	P_4

Table 6: Substitutions in the packing $R \cup T \cup U$ to obtain different leave graphs for $n = 13$

Case 5. $T = P_6$

In order to obtain the leave graph K_3, write $K_{13} = K_9 \vee K_4$. Label the vertices of K_9 with the elements of $\mathbb{Z}_9$ having subscript 1 and the vertices of K_4 with the elements of $\mathbb{Z}_4$ having subscript 2. The graph K_9 has a maximum P_6-packing, R, with a single edge as the leave graph. Let the leave graph be the edge $\{7_1, 8_1\}$. By Parker's theorem [10], the complete bipartite graph with partite sets $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{0_2, 1_2, 2_2, 3_2\}$ has a P_6-decomposition, S. The set of vertices $\{5_1, 6_1, 7_1, 8_1\}$, the set of vertices

$\{0_2, 1_2, 2_2, 3_2\}$, the edges between them, and the edges within the second set, form a graph $K_4 \vee 4K_1$. The following paths form a maximum P_6-packing, T, of the graph $K_4 \vee 4K_1 \cup \{7_1, 8_1\}$ with the leave graph K_3.

$$(5_1, 0_2, 6_1, 1_2, 7_1, 8_1), (5_1, 1_2, 0_2, 2_2, 7_1, 3_2)$$
$$(7_1, 0_2, 3_2, 6_1, 2_2, 1_2), (0_2, 8_1, 1_2, 3_2, 5_1, 2_2)$$

Therefore, $R \cup S \cup T$ forms a maximum P_6-packing of K_{13} with the leave graph K_3. Table 7 demonstrates the substitutions needed in the packing $R \cup S \cup T$ in order to obtain the other leave graphs.

New path(s)	Previous path(s)	leave graph
$(5_1, 0_2, 6_1, 1_2, 7_1, 2_2)$	$(5_1, 0_2, 6_1, 1_2, 7_1, 8_1)$	
$(5_1, 1_2, 0_2, 2_2, 3_2, 7_1)$	$(5_1, 1_2, 0_2, 2_2, 7_1, 3_2)$	S_3
$(5_1, 1_2, 0_2, 2_2, 3_2, 8_1)$	$(5_1, 1_2, 0_2, 2_2, 7_1, 3_2)$	P_4
$(1_2, 0_2, 2_2, 3_2, 8_1, 7_1)$	$(5_1, 1_2, 0_2, 2_2, 7_1, 3_2)$	
$(5_1, 0_2, 6_1, 1_2, 7_1, 2_2)$	$(5_1, 0_2, 6_1, 1_2, 7_1, 8_1)$	$3K_2$
$(0_2, 6_1, 1_2, 7_1, 8_1, 3_2)$	$(5_1, 0_2, 6_1, 1_2, 7_1, 8_1)$	$P_3 + K_2$

Table 7: Substitutions in the packing $R \cup S \cup T$ to obtain different leave graphs for $n = 13$

□

5 Conclusion and Further Research

In this paper, we constructed the spectrum of packing for the complete graph with trees that have five edges or less. In the future, we are going to construct the spectrum of covering for the complete graph with trees that have up to five edges. We are also interested in the spectrum problem for packings of the complete graph with trees that have a higher number of edges.

Another direction that could be pursued is to consider decomposition (or packing) of the complete graph with different types of trees. For instance, consider the complete graph K_{16} and write $K_{16} = K_{10} \vee K_6$. Since the graphs K_{10}, K_6, and $K_{10,6}$ have a D-decomposition, S_3-decomposition, and E-decomposition respectively, The graph K_{16} can be decomposed with the trees S_3, D, and E. This idea might lead to a proof of the conjecture made in 1978 by Gyarfas and Lehel [4]. They conjectured that the complete graph K_n can be decomposed into any collection of trees $T_1, T_2, \ldots, T_{n-1}$, where each T_i is a tree with i vertices.

A *maximal G-packing* of H is a G-packing of H in which the leave graph contains no subgraph G. The difference between the maximal and maximum packing is that in a maximum packing the leave graph has the smallest possible number of edges, while in a maximal packing the leave graph can have any number of edges as long as it does not contain any subgraphs G. For example, Figure 7 demonstrates a maximal K_3-packing of K_6 which is not maximum. Another subject to consider is the maximal packing of the complete graph with small trees.

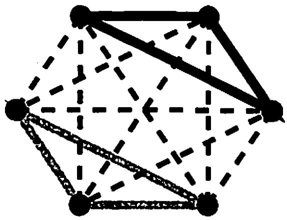

Figure 7: A maximal K_3-packing of K_6

References

[1] Buschmann T. and Bystrykh L.: Levenshtein error-correcting barcodes for multiplexed DNA sequencing. BMC Bioinformatics. 14, 272 (2013).

[2] Dyer D., Haghshenas S., Shalaby N.: Constructing the spectrum of packings and coverings for the complete graph with 4-stars. To appear in J. Combin. Math. and Combin. Computing.

[3] Dyer D., Haghshenas S., Shalaby N.: Constructing the spectrum of packings and coverings for the complete graph with stars with up to five edges. Graphs and Combinatorics 31, 1-19 (2015).

[4] Gyarfas A. and Lehel J.: Packing trees of different order into K_n. Combinatorics, A. Hajnal and V. T. Sos, eds., North-Holland, Amsterdam, 463-469 (1978).

[5] Hell P., Rosa A.: Graph decompositions, handcuffed prisoners and balanced P-designs. Discrete Math. 2 (no. 3), 229-252 (1972).

[6] Huang C., Rosa A.: Decomposition of complete graphs into trees. Ars Combin. 5, 23-63 (1978).

[7] Kirkman T.: Query VI. The Lady's and Gentleman's Diary, 48 (1850).

[8] Kovar P., Kubesa M., and Meszka M.: Factorizations of complete graphs into brooms. Discrete Math., 1084-1093 (2012).

[9] Levenshtein V.:On perfect codes in deletion and insertion metric. Discrete Math. Appl. 2 (no 3), 241-258 (1992).

[10] Parker C., Complete Bipartite Graph Path Decompositions, PhD Thesis, Auburn University (1998).

[11] Roditty Y.: Packing and covering of the complete graph with a graph G of four vertices or less. J. Combin. Theory Ser. B 34 (no. 2), 231-243 (1983).

[12] Roditty Y.: Packing and covering of the complete graph. II. The trees of order six. Ars Combin. 19, 81-93 (1985).

[13] Roditty Y.: The packing and covering of the complete graph. I. The forests of order five. Int. J. Math. Math. Sci. 9 (no. 2), 277-282 (1986).

[14] Roditty Y.: Packing and covering of the complete graph. IV. The trees of order seven. Ars Combin. 35, 33-64 (1993).

[15] Shalaby N., Wang J., Yin J.:Existence of perfect 4-deletion-correcting codes with length six. Designs, Codes, and Cryptography. 27, 145-156 (2002).

[16] Tarsi M.: Decomposition of a complete multigraph into simple paths: Nonbalanced handcuffed designs. J. Combin. Theory A. 34, 60-70 (1983).

[17] West D.: Introduction to Graph Theory (second edition). Prentice Hall (2001).

[18] Yamamoto S., Ikedo H., Shige-eda S., Ushio K., Hamada N.: On claw-decompositions of complete graphs and complete bigraphs. Hiroshima Math. J. 5, 33-42 (1975).