Cyclically Indecomposable Cyclic λ-fold Triple Systems that are Decomposable for $\lambda = 2, 3, 4$

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CANADA A1C 5S7

Abstract

A triple system is decomposable if the blocks can be partitioned into two sets, each of which is itself a triple system, it is cyclically decomposable if the resulting triple systems are themselves cyclic. In this paper, we prove that a cyclic two-fold triple system is cyclically indecomposable if and only if it is indecomposable. Moreover, we construct cyclic three-fold triple systems of order v which are cyclically indecomposable but decomposable for all $v \equiv 3 \mod 6$. The only known example of a cyclic three-fold triple system of order $v \equiv 1 \mod 6$ that is cyclically indecomposable but decomposable was a triple system on 19 points. We present a construction which yields infinitely many such triple systems of order $v \equiv 1 \mod 6$. We also give several examples of cyclically indecomposable but decomposable cyclic four-fold triple systems and few constructions that yields infinitely many such triple systems.

1 Introduction

A λ-fold triple system of order v, denoted $TS_\lambda(v)$, is a pair $(V, \mathcal{B})$ where V is a set with v elements (called points) and $\mathcal{B}$ is a collection of 3-subsets (called triples or blocks) from V, such that any given pair of elements in V lies in exactly λ triples. A $\mathrm{TS}_\lambda(v)$ is *simple* if it contains no repeated blocks.

A one-fold triple system is called a Steiner triple system and denoted by $STS(v)$. A $TS_\lambda(v)$ is *indecomposable* if its block set $\mathcal{B}$ cannot be partitioned into sets $\mathcal{B}_1, \mathcal{B}_2$ of blocks to form $TS_{\lambda_1}(v)$ and $TS_{\lambda_2}(v)$, where $\lambda_1 + \lambda_2 = \lambda$ with $\lambda_1, \lambda_2 \geq 1$.

A $TS_\lambda(v)$ is *cyclic*, denoted $CTS_\lambda(v)$, if its automorphism group contains a cycle of length v. A $CTS_\lambda(v)$ is called *cyclically indecomposable* if its block set $\mathcal{B}$ cannot be partitioned into sets $\mathcal{B}_1, \mathcal{B}_2$ of blocks to form $CTS_{\lambda_1}(v)$ and $CTS_{\lambda_2}(v)$, where $\lambda_1 + \lambda_2 = \lambda$, $\lambda_1, \lambda_2 \geq 1$.

The necessary conditions for a $CTS_\lambda(v)$ to exist are

1. $\lambda \equiv 1, 5, 7, 11 \mod 12$ and $v \equiv 1, 3 \mod 6$; or
2. $\lambda \equiv 2, 10 \mod 12$ and $v \equiv 0, 1, 3, 4, 7, 9 \mod 12$; or
3. $\lambda \equiv 3, 9 \mod 12$ and $v \equiv 1 \mod 2$; or
4. $\lambda \equiv 4, 8 \mod 12$ and $v \equiv 0, 1 \mod 3$; or
5. $\lambda \equiv 6 \mod 12$ and $v \equiv 0, 1, 3 \mod 4$; or
6. $\lambda \equiv 0 \mod 12$ and $v \geq 3$.

These conditions are shown to be sufficient with two definite exceptions $(v, \lambda) = (9, 1)$ or $(9, 2)$ by Colbourn and Colbourn [2].

Indecomposable triple systems have been investigated by many researchers; for example, by Archdeacon and Dinitz [1], Zhang [8] and Grüttmüller [3]. In [6], Rees and Shalaby constructed cyclic, simple and indecomposable two-fold triple systems for all admissible orders. They also introduced the notion of cyclically indecomposable triple systems as defined above. Two-fold cyclically indecomposable triple systems have been constructed by Grüttmüller, Rees and Shalaby [4] for all admissible orders. In addition, in that paper all cyclically indecomposable triple systems $CTS_2(v)$ for $v \leq 33$, $CTS_3(v)$ for $v \leq 21$ and $CTS_2(v)$ up to $v \leq 45$ that are constructed by Skolem-type and Rosa-type sequences are investigated by an exhaustive search method and the decomposable ones are identified. As a result of that search they found that all cyclically indecomposable cyclic two-fold triple systems of small order are also indecomposable. Moreover, they found new (but finitely many) cyclically indecomposable but decomposable $CTS_3(v)$. Grüttmüller, Rees and Shalaby [4] asked to find a $CTS_2(v)$ which is cyclically indecomposable but decomposable or to prove that this is impossible; and to determine the spectrum of those integers v for which there exists a cyclically indecomposable but decomposable $CTS_3(v)$.

In this paper, we will address these two questions. We shall prove in Section 2 that a cyclic two-fold triple system is cyclically indecomposable if and only if it is indecomposable. On the other hand, in Section 3 we construct cyclic three-fold triple systems of order v which are cyclically indecomposable but decomposable for all $v \equiv 3 \mod 6$. Currently, the only known example of a cyclic three-fold triple system of order $v \equiv 1 \mod 6$ that is cyclically indecomposable but decomposable was a triple system on 19 points. We present in Section 4 a construction which yields infinitely many such triple systems of order $v \equiv 1 \mod 6$. In Section 5, we give several examples of four-fold simple and cyclic triple systems that are cyclically indecomposable but decomposable and we present some constructions which yield infinitely many four-fold cyclic triple systems that are cyclically indecomposable but decomposable for $v \equiv 0 \text{ or } 1 \mod 3$.

We close this section by giving some more terminology. Throughout the paper we assume that $V = \mathbb{Z}_v$. Let $B = \{b_1, b_2, b_3\}$ be a block and $i \in \mathbb{Z}_v$, then the block $B + i = \{b_1 + i, b_2 + i, b_3 + i\} \mod v$ is called a *translate* of B. In a CTS the set of distinct translates of B forms a block orbit and an arbitrarily fixed block in a block orbit is called a *base block* R for this orbit. A base block R is *canonical* if it is lexicographically smallest in its block orbit and is said to be *short* if $R + i = R$ for some nonzero $i \in \mathbb{Z}_v$. It is easy to check that any canonical short block is of the form $\{0, v/3, 2v/3\}$, with $v \equiv 0 \mod 3$, and that a $CTS(v)$ with $v \equiv 3 \mod 6$ or a $CTS_{\lambda=2}(v)$ with $v \equiv 0, 3, 9 \mod 12$ must contain exactly 1 or 2 short base blocks, respectively.

To represent a CTS it suffices to list all its canonical base blocks. All blocks B in one orbit provide the same (multi) set of differences $\Delta B = \{\pm(b_2 - b_1), \pm(b_3 - b_1), \pm(b_3 - b_2)\}$. The multiset ∂B is defined by $\partial B = \{\pm(v/3)\}$ if B is short, otherwise $\partial B = \Delta B$. Given a block B and an integer w which is co-prime to v, we define $w \cdot B = \{wb_1, wb_2, wb_3\} \mod v$. Two CTS with lists of canonical base blocks $\mathcal{R}_1, \mathcal{R}_2$ are *equivalent* if there exist a bijection $\varphi : \mathcal{R}_1 \to \mathcal{R}_2$ and $w \in \mathbb{Z}_v$ such that $w \cdot R$ is a translate of $\varphi(R)$ for all $R \in \mathcal{R}_1$.

Let $\mathcal{R}$ be the list of canonical base blocks of a $CTS_\lambda(v)$. $r_d(R)$ counts how often difference d is repeated in the multiset ∂R, where $R \in \mathcal{R}$. Note that $r_d(R) = 0, 1, 2$. The *base block-difference graph* of the CTS is a (multi)graph with vertex set $\mathcal{R}$ which has an edge between base blocks R_i and R_j labeled with d if either $i \neq j$ and $d \in \partial R_i \cap \partial R_j$, or $i = j$ and $r_d(R_i) = 2$. Edges of the first kind are repeated $r_d(R_i) \cdot r_d(R_j)$ times, while loops are repeated $\binom{r_d(R_i)}{2} = 1$ times. Here, the edges with label d form for each difference $d \in \mathbb{Z}_v \setminus \{0\}$ a (possibly degenerated) λ-clique. Degenerated means that some vertices of the clique may collapse into one vertex generating multiple edges and loops. Note that, by definition, $\mathcal{R}$ is a multiset of blocks. However, it is also considered as the set of vertices of the base block difference graph. A block $R \in \mathcal{R}$ with multiplicity μ yields

μ vertices in the related graph.

A *Skolem sequence* of order n is a sequence $S_n = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers that satisfies the conditions:

1. for every $k \in \{1, 2, \ldots, n\}$ there are exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$, and

2. if $s_i = s_j = k$, $i < j$, then $j - i = k$.

Skolem sequences can also be written as collections of ordered pairs $\{(a_i, b_i) : 1 \leq i \leq n,\ b_i - a_i = i\}$ with $\cup_{i=1}^{n}\{a_i, b_i\} = \{1, 2, \ldots, 2n\}$. As an example, $S_5 = (1, 1, 3, 4, 5, 3, 2, 4, 2, 5)$ is a Skolem sequence of order 5 or, equivalently, the collection $\{(1, 2), (7, 9), (3, 6), (4, 8), (5, 10)\}$.

A *k-extended Skolem sequence of order n* is a sequence $ES_n = (s_1, s_2, \ldots, s_{2n+1})$ in which $s_k = 0$ and for each $j \in \{1, \ldots, n\}$, there exists a unique $i \in \{1, \ldots, 2n\}$ such that $s_i = s_{i+j} = j$. As an example, $(3, 1, 1, 3, 4, 2, 0, 2, 4)$ is a 7-extended Skolem sequence of order 4.

A *hooked Skolem sequence* of order n is an extended Skolem sequence of order n with $s_{2n} = 0$. As an example, $hS_6 = (1, 1, 2, 5, 2, 4, 6, 3, 5, 4, 3, 0, 6)$ is a hooked Skolem sequence of order 6.

A (gv, g, k, λ) *difference family* ((gv, g, k, λ)-DF in $\mathbb{Z}_{vg}$ for short) is a family $\mathcal{F}$ of k subsets (base blocks) of $\mathbb{Z}_{gv}$ with the property that its list of differences $\Delta\mathcal{F} = \cup_{B\in\mathcal{F}} \Delta B$ is λ times $\mathbb{Z}_{gv} \setminus \{0, v, 2v, \ldots, (g-1)v\}$, where $\Delta B = \{b_i - b_j | 1 \leq i, j \leq k, i \neq j\}$ if $B = \{b_1, \ldots, b_k\}$.

Let v, g, k, λ be positive integers and $\alpha \in [0, \lambda]$. A $(gv, \{g, k_\alpha\}, k, \lambda)$-difference family in $\mathbb{Z}_{gv}$ ($(gv, \{g, k_\alpha\}, k, \lambda)$-DF in $\mathbb{Z}_{gv}$ for short) is a family of k subsets (base blocks) of $\mathbb{Z}_{gv}$ with the property that its list of differences $\Delta\mathcal{F} = \cup_{B\in\mathcal{F}} \Delta B$ is $\lambda\mathbb{Z}_{gv} \setminus (\lambda\{0, v, \ldots, (g-1)v\} \cup \alpha\{0, gv/k, \ldots, (k-1)gv/k\})$, where $\Delta B = \{b_i - b_j : 1 \leq i, j \leq k, i \neq j\}$ if $B = \{b_1, b_2, \ldots, b_k\}$, such that $\{0, v, \ldots, (g-1)v\} \cap \{0, gv/k, \ldots, (k-1)gv/k\} = \{0\}$.

2 Cyclic Two-fold Triple Systems

We will make use of the following two lemmas.

Lemma 2.1 (Grüttmüller, Rees, Shalaby [4]) *A cyclic two-fold triple system $CTS_2(v)$ having a non-short base block R whose set of differences contains a repeated difference is indecomposable.*

Lemma 2.2 (Grüttmüller, Rees, Shalaby [4]) *A cyclic two-fold triple system $CTS_2(v)$ is cyclically decomposable if and only if its base block-difference graph is bipartite.*

We are now in the position to state the first main theorem.

Theorem 2.3 *A cyclic two-fold triple system $CTS_2(v)$ is cyclically indecomposable if and only if it is indecomposable.*

Proof There does not exist a $STS(v)$ with $v \equiv 0,4 \mod 12$, hence all $CTS_2(v)$ with $v \equiv 0,4 \mod 12$ are both indecomposable and cyclically indecomposable, and we only need to take care of the cases $v \equiv 1,3 \mod 6$. Moreover, it is evident from the definition that indecomposable implies cyclically indecomposable.

In the following we show that cyclically indecomposable implies indecomposable. Suppose first that $v \equiv 1 \mod 6$ and assume to the contrary that there exists a cyclic two-fold triple system (represented by its list of canonical base blocks $\mathcal{R}$) which is cyclically indecomposable but decomposable into two non-cyclic $STS(v)$ with block sets $\mathcal{B}_1$ and $\mathcal{B}_2$. For any base block $R \in \mathcal{R}$ define $u(R)$ to be the number of blocks from $\mathcal{B}_1$ which occur in the orbit of R. Lemma 2.1 implies that no base block contains a repeated difference. Thus, for each difference $d \in \mathbb{Z}_v \setminus \{0\}$ there are precisely two disjoint base blocks $R_1, R_2 \in \mathcal{R}$ providing this difference. This in turn implies that if $u = u(R_1)$, then exactly $v - u = u(R_2)$ blocks from $\mathcal{B}_1$ occur in the orbit of R_2 and the same is true for all base blocks which are adjacent to R_1 in the base block-difference graph. By this reasoning we know that for any path $R_1, R_2, \ldots, R_{2m}$ in the base block-difference graph the corresponding sequence $u(R_1), u(R_2), \ldots, u(R_{2m})$ is $u, v-u, u, v-u, \ldots, u, v-u$. Clearly, $u \neq v - u$ and, therefore, each cycle in the base block-difference graph has even length. This is equivalent to the statement that the base block-difference graph is bipartite and Lemma 2.2 implies that the $CTS_2(v)$ is cyclically decomposable, a contradiction.

A similar analysis can be applied in the case $v \equiv 3 \mod 6$. Recall that there are The only thing one needs to treat differently are the two short base blocks which necessarily occur in a $CTS_2(v)$ with $v \equiv 3 \mod 6$.. But these two short blocks form a component in the base block-difference graph which is trivially bipartite. ∎

3 Cyclic Three-fold Triple Systems $CTS_3(v)$ with $v \equiv 3 \mod 6$

Grüttmüller, Rees, Shalaby [4] found that there are exactly 3 inequivalent cyclically indecomposable but decomposable CTS_3 of order 9, exactly 45 of order 15, and exactly 7247 of order 21. They pointed out that many of the sub-STS are generated $+3 \mod v$. That is, the automorphism group contains no cycle of length v but a permutation which consists of three disjoint cycles of length $v/3$ each. In the following second main theorem we will present a class of cyclic three-fold triple systems which allow such a decomposition.

Theorem 3.1 *Let* $v \equiv 3 \mod 6$. *Then there exists a cyclically indecomposable but decomposable cyclic three-fold triple system of order* v.

Proof Let n be an odd positive integer. The following base blocks represent a cyclic three-fold triple system of order $3n$ as a short analysis of the differences in the base blocks shows:

$$\begin{array}{l} \{0,1,2\}, \\ \left.\begin{array}{l}\{0,3i+1,-3i+1\}, \\ \{1,3i+2,-3i+2\}, \\ \{2,3i,-3i\}\end{array}\right\} \text{ for } i=1,2,\ldots,(n-1)/2. \end{array}$$

Note that there is no short block. But any cyclic triple systems with $\lambda = 1$ or 2 needs to contain a short block. Thus, the triple system generated from the base blocks is cyclically indecomposable. Moreover, it can be decomposed into a non-cyclic triple system with $\lambda_1 = 1$ whose blocks are obtained by taking for each base block B the translates $B + 3m$ where $m = 0, 1, \ldots, n-1$. The complement of this $STS(v)$ is of course a $TS_2(v)$. ■

The above construction involves repeated blocks. We present below an example of a simple cyclically indecomposable but decomposable $\text{CTS}_3(21)$. We will make use of the following construction given in [7].

Construction 3.2 *[7] Let* $hS_n = (s_1, s_2, \ldots, s_{2n-1}, s_{2n+1})$ *be a hooked Skolem sequence of order* n *and let* $\{(a_i, b_i) | 1 \le i \le n\}$ *be the pairs of positions in* hS_n *for which* $b_i - a_i = i$. *Then the set* $\mathcal{F} = \{\{0, i, b_i + 1\} | 1 \le i \le n\}$ mod $2n+1$ *form the base blocks of a cyclic* $CTS_3(2n+1)$.

Lemma 3.3 *There exists a simple cyclically indecomposable but decomposable* $CTS_3(21)$.

Proof From $hS_{10} = (2,4,2,8,3,4,7,3,6,9,10,8,5,7,6,1,1,5,9,0,10)$, we take the base blocks of the form $\{0, i, b_i + 1\}$ mod 21 as in Construction 3.2. This will give the following canonical base blocks of a $CTS_3(21)$: $\{0,1,18\}$, $\{0,2,4\}$, $\{0,3,9\}$, $\{0,4,7\}$, $\{0,5,19\}$, $\{0,6,16\}$, $\{0,7,15\}$, $\{0,8,13\}$, $\{0,9,20\}$, $\{0,10,1\}$. Suppose there exists a decomposition into a $CTS_1(21)$ and a $CTS_2(21)$, denoted C_1 and C_2 respectively. Since the second and 8th base block contain repeated differences 2 and 8 respectively, these base blocks are forced to be in C_2. Moreover, because of the third difference of 2 and 8 the 5th and 7th base block must belong to C_1. But then, difference 7 occurs twice among the differences of the base blocks of C_1 which is impossible. Hence, the $CTS_3(21)$ above is cyclically indecomposable.

Furthermore, the design is decomposable since the following blocks which are chosen from the orbits of the base blocks above form an $STS(21)$: $\{0,1,10\}$, $\{0,2,4\}$, $\{0,3,17\}$, $\{0,5,19\}$, $\{0,6,13\}$, $\{0,7,15\}$, $\{0,8,16\}$, $\{0,9,20\}$, $\{0,11,12\}$, $\{0,14,18\}$, $\{1,2,19\}$, $\{1,3,8\}$, $\{1,4,5\}$, $\{1,6,14\}$, $\{1,7,17\}$, $\{1,9,15\}$, $\{1,11,16\}$, $\{1,12,13\}$, $\{1,18,20\}$, $\{2,3,12\}$, $\{2,5,11\}$, $\{2,6,9\}$, $\{2,7,13\}$, $\{2,8,20\}$, $\{2,10,15\}$, $\{2,14,17\}$, $\{2,16,18\}$, $\{3,4,13\}$, $\{3,5,7\}$, $\{3,6,20\}$, $\{3,9,16\}$, $\{3,10,18\}$, $\{3,11,19\}$, $\{3,14,15\}$, $\{4,6,11\}$, $\{4,7,8\}$, $\{4,9,17\}$, $\{4,10,20\}$, $\{4,12,18\}$, $\{4,14,19\}$, $\{4,15,16\}$, $\{5,6,15\}$, $\{5,8,14\}$, $\{5,9,12\}$, $\{5,10,16\}$, $\{5,13,18\}$, $\{5,17,20\}$, $\{6,7,16\}$, $\{6,8,10\}$, $\{6,12,19\}$, $\{6,17,18\}$, $\{7,9,14\}$, $\{7,10,11\}$, $\{7,12,20\}$, $\{7,18,19\}$, $\{8,9,18\}$, $\{8,11,17\}$, $\{8,12,15\}$, $\{8,13,19\}$, $\{9,10,19\}$, $\{9,11,13\}$, $\{10,12,17\}$, $\{10,13,14\}$, $\{11,14,20\}$, $\{11,15,18\}$, $\{12,14,16\}$, $\{13,15,20\}$, $\{13,16,17\}$, $\{15,17,19\}$, $\{16,19,20\}$. ■

4 Cyclic Three-fold Triple Systems $CTS_3(v)$ with $v \equiv 1 \mod 6$

It seems that cyclically indecomposable but decomposable cyclic three-fold triple systems of order $v \equiv 1 \mod 6$ are more difficult to find. It is known [4] that there is no such system for $v = 7$ or 13. To date the only known example is the following.

Lemma 4.1 (Grüttmüller, Rees, Shalaby [4]) *There exists a unique cyclically indecomposable but decomposable* $CTS_3(19)$.

Proof This design was found by a computer search described in [4]. Only the existence was mentioned in that paper, so we present here the canonical base blocks of the unique $CTS_3(19)$: $\{0,1,2\}$, $\{0,1,8\}$, $\{0,2,4\}$, $\{0,3,6\}$, $\{0,3,11\}$, $\{0,4,10\}$, $\{0,4,13\}$, $\{0,5,10\}$, $\{0,5,12\}$. Suppose there exists a decomposition into a $CTS_1(19)$ and a $CTS_2(19)$, denoted C_1 and C_2 respectively. Since the first and third base block contain a repeated difference, these base blocks are forced to be in C_2. But then, difference 2 occurs three times among the differences of the base blocks of C_2 which is impossible. Hence, the unique $CTS_3(19)$ is indeed cyclically indecomposable.

Moreover, it is decomposable since the following blocks which are chosen from the orbits of the base blocks above form a $STS(19)$: $\{0,1,2\}$, $\{0,17,18\}$, $\{2,3,4\}$, $\{4,5,6\}$, $\{6,7,8\}$, $\{8,9,10\}$, $\{10,11,12\}$, $\{12,13,14\}$, $\{14,15,16\}$, $\{5,16,17\}$, $\{1,3,5\}$, $\{1,16,18\}$, $\{5,7,9\}$, $\{9,11,13\}$, $\{13,15,17\}$, $\{1,4,17\}$, $\{4,7,10\}$, $\{5,8,11\}$, $\{10,13,16\}$, $\{11,14,17\}$, $\{0,3,11\}$, $\{0,8,16\}$, $\{1,9,12\}$, $\{2,5,13\}$, $\{2,10,18\}$, $\{3,6,14\}$, $\{4,12,15\}$,

$\{6,9,17\}$, $\{7,15,18\}$, $\{1,10,14\}$, $\{2,6,12\}$, $\{2,8,17\}$, $\{2,11,15\}$, $\{3,7,13\}$, $\{3,9,18\}$, $\{3,12,16\}$, $\{4,8,14\}$, $\{5,14,18\}$, $\{8,12,18\}$, $\{0,4,13\}$, $\{0,6,10\}$, $\{0,9,15\}$, $\{1,7,11\}$, $\{1,6,15\}$, $\{2,7,16\}$, $\{5,10,15\}$, $\{6,11,16\}$, $\{7,12,17\}$, $\{0,5,12\}$, $\{0,7,14\}$, $\{1,8,13\}$, $\{2,9,14\}$, $\{3,8,15\}$, $\{3,10,17\}$, $\{4,9,16\}$, $\{4,11,18\}$, $\{6,13,18\}$. ∎

The construction which we present now enables us to produce new cyclic λ-fold triple systems with the property of being cyclically indecomposable but decomposable from just one ingredient design with this property. A general construction can be stated as follows:

Construction 4.2 *Let* $v \equiv 1 \text{ or } 3 \mod 6$ *and let* h *be a divisor of* v. *There exists a cyclically indecomposable but decomposable* $CTS_\lambda(v)$ *if the following conditions are both satisfied:*

- *there exists a* $CTS(v)$ *with a cyclic subsystem of order* h*;*
- *there exists a cyclically indecomposable but decomposable* $CTS_\lambda(h)$.

Proof By assumption there exists a cyclic Steiner triple system of order v containing a cyclic sub-system of order h on the point set $0, n, 2n, \ldots, (h-1)n$, where $h \cdot n = v$. For any block B of the cyclic sub-system remove all blocks which are in the orbit of B. Let $\mathcal{B}'$ contain all remaining blocks. Taking each of these blocks λ times yields an incomplete $CTS_\lambda(v)$ with n holes, each hole containing h points. Let $\mathcal{R}$ be the set of canonical base blocks of the given ingredient $CTS_\lambda(h)$. Now insert the base block $n \cdot R$ for each $R \in \mathcal{R}$ and develop these blocks modulo v to fill all holes. This gives a $CTS_\lambda(v)$, say C.

It remains to show that this design C is cyclically indecomposable and to present a decomposition into a non-cyclic $TS_{\lambda_1}(v)$ and a $TS_{\lambda_2}(v)$.

Suppose first that there exists a decomposition of C into a $CTS_{\lambda_1}(v)$ and a $CTS_{\lambda_2}(v)$, with canonical base block lists $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. Let $\mathcal{R}'_1$ be the set of base blocks from $\mathcal{R}_1$ which cover a difference of the form $\pm dn$ where $d = 1, 2, \ldots, h-1$. By construction each base block from $\mathcal{R}'_1$ is a translate of a base block $n \cdot R$ with $R \in \mathcal{R}$. So define $\mathcal{R}'$ to be the set of all base blocks $R \in \mathcal{R}$ with $n \cdot R + i \in \mathcal{R}'_1$ for some $i \in \mathbb{Z}_v$. The base blocks in $\mathcal{R}'_1$ cover all differences of the form $\pm dn$ exactly λ_1 times and, therefore, the base blocks in $\mathcal{R}'$ cover all differences of the form $\pm d$ ($d \in \mathbb{Z}_h \setminus \{0\}$) exactly λ_1 times. This implies that the base blocks in $\mathcal{R}'$ generate a cyclic $CTS_{\lambda_1}(h)$ and the base blocks in $\mathcal{R} \setminus \mathcal{R}'$ generate a $CTS_{\lambda_2}(h)$ which form together a cyclic decomposition of the ingredient $CTS_\lambda(h)$ which is cyclically indecomposable by hypothesis, a contradiction.

A decomposition into a non-cyclic $TS_{\lambda_1}(v)$ and a $TS_{\lambda_2}(v)$ can be obtained as follows. By assumption, the ingredient $CTS_\lambda(h)$ can be decomposed into a $TS_{\lambda_1}(h)$ and a $TS_{\lambda_2}(h)$. Take all blocks $n \cdot B + i$, where B is a block from the $TS_{\lambda_1}(h)$ and $i \in \mathbb{Z}_n$, and adjoin the blocks in $\mathcal{B}'$ (which remained after the first step of the construction) each taken λ_1 times. This yields a $TS_{\lambda_1}(v)$, say S, whose blocks are clearly contained in the block set of C. The blocks of the $TS_{\lambda_2}(v)$ are just the complement of the blocks of S with respect to the blocks in C. ■

Construction 4.3 *Let $m, n \equiv 1 \mod 6$ and suppose that there exists a cyclically indecomposable but decomposable $CTS_3(m)$. Then there exists a cyclically indecomposable but decomposable cyclic three-fold triple system of order mn.*

Proof By Theorem 2.1 in [5] there exists a cyclic Steiner triple system of order mn containing a cyclic sub-system of order m on the point set $0, n, 2n, \ldots, (m-1)n$. The proof is similar to Construction 4.2. ■

Now, using the cyclically indecomposable but decomposable $CTS_3(m = 19)$ from Lemma 4.1 as ingredient design in Construction 4.2 we obtain an infinite family of cyclically indecomposable but decomposable $CTS_3(v)$ of order $v \equiv 1 \mod 6$.

Theorem 4.4 *There exists a cyclically indecomposable but decomposable $CTS_3(19n)$ for any $n \equiv 1 \mod 6$.*

5 Cyclic Four-fold Triple Systems

In this section we give a few examples of cyclically indecomposable but decomposable four-fold triple systems and a few constructions which yields infinitely many such triple systems of order $v \equiv 0 \text{ or } 1 \mod 3, v \neq 9$.

We will make use of the following constructions given in [7].

Construction 5.1 *[7] Let n be even. Let $ES_n = (s_1, s_2, \ldots, s_{\frac{3n}{2}}, 0, s_{\frac{3n}{2}+2}, \ldots, s_{2n+1})$ be an $(\frac{3n}{2}+1)$- extended Skolem sequence of order n and let $\{(a_i, b_i) | 1 \leq i \leq n\}$ be the pairs of positions in ES_n for which $b_i - a_i = i$. Then the set $\mathcal{F}$=$\{\{0, i, b_i\} | 1 \leq i \leq n\} \mod \frac{3n}{2}+1$ form the base blocks of a cyclic $CTS_4(\frac{3n}{2}+1)$.*

Construction 5.2 *[7] Let n be odd. Let $ES_n = (s_1, s_2, \ldots, s_{2n-\lfloor\frac{n}{2}\rfloor}, 0, s_{2n-\lfloor\frac{n}{2}\rfloor+2}, \ldots, s_{2n+1})$ be an $(2n - \lfloor\frac{n}{2}\rfloor + 1)$-extended Skolem sequence of order n and let $\{(a_i, b_i) | 1 \leq i \leq n\}$ be the pairs of positions in ES_n for which $b_i - a_i = i$. Then the set $\mathcal{F}$=$\{\{0, i, b_i\} | 1 \leq i \leq n\} \mod 2n - \lfloor\frac{n}{2}\rfloor + 1$ together with the block $\{0, \lfloor\frac{n}{2}\rfloor + 1, n+1\} \mod 2n - \lfloor\frac{n}{2}\rfloor + 1$ having a short orbit of length $\frac{2n-\lfloor\frac{n}{2}\rfloor+1}{3}$ form the base blocks of a cyclic $CTS_4(2n - \lfloor\frac{n}{2}\rfloor + 1)$.*

Lemma 5.3 *There exists a cyclically indecomposable but decomposable* $CTS_4(9)$.

Proof From the 9-extended Skolem sequence of order 5, $(5,3,1,1,3,5,4,2,0,2,4)$, take the base blocks of the form $\{0,i,b_i\}$ (mod 9) as in Construction 5.2. So, the canonical base blocks of the $CTS_4(9)$ are: $\{0,1,4\}$, $\{0,2,1\}$, $\{0,3,5\}$, $\{0,4,2\}$, $\{0,5,6\}$, and the short orbit $\{0,6,3\}$.

This triple system is cyclically indecomposable since there exists no $CTS_1(9)$ nor $CTS_2(9)$. This design is decomposable since the following blocks which are chosen from the orbits of the base blocks above form a $STS(9)$: $\{0,1,2\}$, $\{0,3,8\}$, $\{0,4,7\}$, $\{0,5,6\}$, $\{1,3,7\}$, $\{1,4,6\}$, $\{1,5,8\}$, $\{2,3,6\}$, $\{2,4,8\}$, $\{2,5,7\}$, $\{3,4,5\}$, $\{6,7,8\}$. The base blocks of the $TS_3(9)$ are just the complement of the blocks above with respect to the blocks in $CTS_4(9)$. ■

Lemma 5.4 *There exists a cyclically indecomposable but decomposable* $CTS_4(10)$.

Proof From the 10-extended Skolem sequence of order 6, $(5,3,1,1,3,5,6,4,2,0,2,4,6)$, take the base block of the form $\{0,i,b_i\}$ mod 10) as in Construction 5.1. So, the canonical base blocks of the $CTS_4(10)$ are: $\{0,1,4\}$, $\{0,2,1\}$, $\{0,3,5\}$, $\{0,4,2\}$, $\{0,5,6\}$, $\{0,6,3\}$. This triple system is cyclically indecomposable since there exists no $CTS_1(10)$ nor $CTS_2(10)$. On the other hand, this design is decomposable since the following blocks which are chosen from the orbits of the base blocks above form a $TS_2(10)$: $\{0,1,2\}$, $\{0,1,4\}$, $\{0,2,4\}$, $\{0,3,7\}$, $\{0,3,9\}$, $\{0,5,6\}$, $\{0,5,8\}$, $\{0,6,7\}$, $\{0,8,9\}$, $\{1,2,6\}$, $\{1,3,5\}$, $\{1,3,8\}$, $\{1,4,7\}$, $\{1,5,8\}$, $\{1,6,9\}$, $\{1,7,9\}$, $\{2,3,4\}$, $\{2,3,6\}$, $\{2,5,7\}$, $\{2,5,9\}$, $\{2,7,8\}$, $\{2,8,9\}$, $\{3,4,8\}$, $\{3,5,7\}$, $\{3,6,9\}$, $\{4,5,6\}$, $\{4,5,9\}$, $\{4,6,8\}$, $\{4,7,9\}$, $\{6,7,8\}$. The base blocks of the second $TS_2(10)$ are just the complement of the blocks above with respect to the blocks in $CTS_4(10)$. ■

Lemma 5.5 *There exists a cyclically indecomposable but decomposable* $CTS_4(12)$.

Proof From the 12-extended Skolem sequence of order 7, $(1,1,4,2,6,2,4,7,5,3,6,0,3,5,7)$, take the base block of the form $\{0,i,b_i\}$ mod 12 together with the short base block $\{0,4,8\}$ as in Construction 5.1. So, the canonical base blocks of the $CTS_4(12)$ are: $\{0,1,2\}$, $\{0,2,6\}$, $\{0,3,1\}$, $\{0,4,7\}$, $\{0,5,2\}$, $\{0,6,11\}$, $\{0,7,3\}$, and the short base block $\{0,4,8\}$. There exists no $CTS_1(12)$ and therefore no cyclic decomposition into a CTS_1 and a CTS_3. A decomposition into two $CTS_2(12)$ would require 2×2 short base blocks but there is only one. Therefore the design is cyclically indecomposable.

This design is decomposable since the following blocks which are chosen from the orbits of the base blocks above form a $TS_2(12)$: $\{0,1,2\}$, $\{0,1,3\}$, $\{0,2,5\}$, $\{0,3,7\}$, $\{0,4,7\}$, $\{0,4,8\}$, $\{0,5,9\}$, $\{0,6,8\}$, $\{0,6,11\}$, $\{0,9,10\}$, $\{0,10,11\}$, $\{1,2,4\}$, $\{1,3,6\}$, $\{1,4,8\}$, $\{1,5,9\}$, $\{1,5,11\}$, $\{1,6,7\}$, $\{1,7,9\}$, $\{1,8,10\}$, $\{1,10,11\}$, $\{2,3,4\}$, $\{2,3,9\}$, $\{2,5,10\}$, $\{2,6,9\}$, $\{2,6,11\}$, $\{2,7,8\}$, $\{2,7,11\}$, $\{2,8,10\}$, $\{3,4,5\}$, $\{3,5,8\}$, $\{3,6,10\}$, $\{3,7,10\}$, $\{3,8,11\}$, $\{3,9,11\}$, $\{4,5,11\}$, $\{4,6,9\}$, $\{4,6,10\}$, $\{4,7,11\}$, $\{4,9,10\}$, $\{5,6,7\}$, $\{5,6,8\}$, $\{5,7,10\}$, $\{7,8,9\}$, $\{8,9,11\}$. The base blocks of the second $TS_2(12)$ are just the complement of the blocks above with respect to the blocks in $CTS_4(12)$. ■

Lemma 5.6 *There exists a cyclically indecomposable but decomposable* $CTS_4(13)$.

Proof From the 13-extended Skolem sequence of order 8, $(1,1,6,4,2,8,2,4,6,7,5,3,0,8,3,5,7)$, take the base block of the form $\{0,i,b_i\}$ mod 13 as in Construction 5.2. So, the canonical base blocks of the $CTS_4(13)$ are: $\{0,1,2\}$, $\{0,2,7\}$, $\{0,3,2\}$, $\{0,4,8\}$, $\{0,5,3\}$, $\{0,6,9\}$, $\{0,7,4\}$, $\{0,8,1\}$. Here we have to show that neither a decomposition into a CTS_1 and a CTS_3, nor into two CTS_2 is possible. For a sub-$CTS_1(13)$ we need two base blocks whose differences give each of $\{1,2,\ldots,6\}$ exactly once. But these do not exist. Now, suppose there exists a decomposition into two $CTS_2(13)$, denoted C_1 and C_2 respectively. Assume w.l.o.g. that the base block $\{0,4,8\}$ providing the repeated difference 4 belongs to C_1. Then the two base blocks $\{0,6,9\}$, $\{0,7,4\}$ which cover difference 4 as well need to occur in C_2. Both blocks cover differences 3 and 6 each. Therefore, base blocks $\{0,2,7\}$ and $\{0,8,1\}$ are forced to be contained in C_1. But then difference 5 occurs three-times among the differences provided by base blocks of C_1, a contradiction. Hence, the $CTS_4(13)$ above is cyclically indecomposable.

The design is decomposable since the following blocks which are chosen from the orbits of the base blocks above form a $TS_2(13)$: $\{0,1,2\}$, $\{0,1,8\}$, $\{0,2,3\}$, $\{0,3,7\}$, $\{0,4,7\}$, $\{0,4,8\}$, $\{0,5,9\}$, $\{0,5,11\}$, $\{0,6,9\}$, $\{0,6,10\}$, $\{0,10,12\}$, $\{0,11,12\}$, $\{1,2,9\}$, $\{1,3,4\}$, $\{1,3,11\}$, $\{1,4,6\}$, $\{1,5,8\}$, $\{1,5,10\}$, $\{1,6,12\}$, $\{1,7,10\}$, $\{1,7,11\}$, $\{1,9,12\}$, $\{2,3,10\}$, $\{2,4,5\}$, $\{2,4,12\}$, $\{2,5,9\}$, $\{2,6,11\}$, $\{2,6,12\}$, $\{2,7,8\}$, $\{2,7,11\}$, $\{2,8,10\}$, $\{3,4,5\}$, $\{3,5,6\}$, $\{3,6,10\}$, $\{3,7,12\}$, $\{3,8,9\}$, $\{3,8,12\}$, $\{3,9,11\}$, $\{4,6,11\}$, $\{4,7,9\}$, $\{4,8,11\}$, $\{4,9,10\}$, $\{4,10,12\}$, $\{5,6,7\}$, $\{5,7,12\}$, $\{5,8,12\}$, $\{5,10,11\}$, $\{6,7,8\}$, $\{6,8,9\}$, $\{7,9,10\}$, $\{8,10,11\}$, $\{9,11,12\}$. The base blocks of the second $TS_2(13)$ are just the complement of the blocks above with respect to the blocks in $CTS_4(13)$. ■

Lemma 5.7 *There exists a cyclically indecomposable but decomposable* $CTS_4(15)$.

Proof From the 15-extended Skolem sequence of order 9, $(1,1,6,4,2,8,2,4,6,9,7,5,3,8,0,3,5,7,9)$, take the base block of the form $\{0,i,b_i\}$ mod 15 as in Construction 5.2. So, the canonical base blocks of the $CTS_4(15)$ are: $\{0,1,2\}$, $\{0,2,7\}$, $\{0,3,1\}$, $\{0,4,8\}$, $\{0,5,2\}$, $\{0,6,9\}$, $\{0,7,3\}$, $\{0,8,14\}$, $\{0,9,4\}$, together with the short orbit $\{0,5,10\}$. This design is decomposable since the following blocks which are chosen from the orbits of the base blocks above form a $STS(15)$: $\{0,1,3\}$, $\{0,2,5\}$, $\{0,4,9\}$, $\{0,6,10\}$, $\{0,7,11\}$, $\{0,8,14\}$, $\{0,12,13\}$, $\{1,2,10\}$, $\{1,4,14\}$, $\{1,5,9\}$, $\{1,6,12\}$, $\{1,7,8\}$, $\{1,11,13\}$, $\{2,3,11\}$, $\{2,4,7\}$, $\{2,6,13\}$, $\{2,8,9\}$, $\{2,12,14\}$, $\{3,4,6\}$, $\{3,5,8\}$, $\{3,7,12\}$, $\{3,9,13\}$, $\{3,10,14\}$, $\{4,5,13\}$, $\{4,8,12\}$, $\{4,10,11\}$, $\{5,6,14\}$, $\{5,7,10\}$, $\{5,11,12\}$, $\{6,7,9\}$, $\{6,8,11\}$, $\{7,13,14\}$, $\{8,10,13\}$, $\{9,10,12\}$, $\{9,11,14\}$. The base blocks of the second $TS_3(15)$ are just the complement of the blocks above with respect to the blocks in $CTS_4(15)$.

The design can be also decompose into two $TS_2(15)$. The first $TS_2(15)$ is: $\{0,1,2\}$, $\{0,1,3\}$, $\{0,2,7\}$, $\{0,3,9\}$, $\{0,4,8\}$, $\{0,4,11\}$, $\{0,5,10\}$, $\{0,5,13\}$, $\{0,6,9\}$, $\{0,6,12\}$, $\{0,7,11\}$, $\{0,8,14\}$, $\{0,10,14\}$, $\{0,13,14\}$, $\{1,2,3\}$, $\{1,4,8\}$, $\{1,4,14\}$, $\{1,5,9\}$, $\{1,5,10\}$, $\{1,6,11\}$, $\{1,6,14\}$, $\{1,7,10\}$, $\{1,7,13\}$, $\{1,8,12\}$, $\{1,9,12\}$, $\{1,11,13\}$, $\{2,3,5\}$, $\{2,4,7\}$, $\{2,4,9\}$, $\{2,5,11\}$, $\{2,6,10\}$, $\{2,6,13\}$, $\{2,8,11\}$, $\{2,8,14\}$, $\{2,9,13\}$, $\{2,10,12\}$, $\{2,12,14\}$, $\{3,4,6\}$, $\{3,4,12\}$, $\{3,5,8\}$, $\{3,6,10\}$, $\{3,7,11\}$, $\{3,7,12\}$, $\{3,8,13\}$, $\{3,9,13\}$, $\{3,10,14\}$, $\{3,11,14\}$, $\{4,5,7\}$, $\{4,5,13\}$, $\{4,6,9\}$, $\{4,10,11\}$, $\{4,10,13\}$, $\{4,6,12\}$, $\{4,8,11\}$, $\{4,8,15\}$, $\{4,10,11\}$, $\{4,10,13\}$, $\{4,12,14\}$, $\{5,6,7\}$, $\{5,6,14\}$, $\{5,8,12\}$, $\{5,9,14\}$, $\{5,11,12\}$, $\{6,7,8\}$, $\{6,8,11\}$, $\{6,12,13\}$, $\{7,8,9\}$, $\{7,9,12\}$, $\{7,10,14\}$, $\{7,13,14\}$, $\{8,9,10\}$, $\{8,10,13\}$, $\{9,10,11\}$, $\{9,11,14\}$, $\{11,12,13\}$. The base blocks of the second $TS_2(15)$ are just the complement of the blocks above with respect to the blocks in $CTS_4(15)$. With regard to cyclic decomposability it is easily seen that a decomposition into two $CTS_2(15)$ would require four short base blocks, but there is only one. Now, suppose there exists a decomposition into a $CTS_1(15)$ and a $CTS_3(15)$, denoted C_1 and C_2 respectively. The short base block has to occur in C_1 and therefore covers all pairs of points with difference 5. Consequently, the remaining differences $\{1,2,3,4,6,7\}$ have to be covered exactly once by two base blocks. But there are no such two base blocks. Hence, the $CTS_4(15)$ above is cyclically indecomposable. ■

Lemma 5.8 *There exists a cyclically indecomposable but decomposable* $CTS_4(16)$.

Proof From the 16-extended Skolem sequence of order 10, $(1,1,8,6,4,2,10,2,4,6,8,9,7,5,3,0,10,3,5,7,9)$, take the base block of the form $\{0,i,b_i\}$ mod 16 as in Construction 5.1. So, the canonical base blocks of the

$CTS_4(16)$ are: $\{0,1,2\}$, $\{0,2,8\}$, $\{0,3,2\}$, $\{0,4,9\}$, $\{0,5,3\}$, $\{0,6,10\}$, $\{0,7,4\}$, $\{0,8,11\}$, $\{0,9,5\}$, $\{0,10,1\}$. There does not exist a $CTS_1(16)$. So we only have to show that a decomposition into two CTS_2 is not possible. Suppose to the contrary that there exists a decomposition into two $CTS_2(13)$, denoted C_1 and C_2 respectively. Assume w.l.o.g. that the base block $\{0,1,2\}$ providing the repeated difference 1 belongs to C_1. Then the two base blocks $\{0,3,2\}$, $\{0,10,1\}$ which cover difference 1 as well need to occur in C_2. The second of these blocks covers difference 6. Therefore, base block $\{0,6,10\}$ with the repeated difference 6 has to be in C_1 and base block $\{0,2,8\}$ has to be in C_2. Moreover, the latter base block provides difference $8 = v/2$ in C_2 and thus base block $\{0,8,11\}$ with the second difference 8 is forced to be contained in C_1. We observe that difference 2 now already occurs twice in base blocks of C_2 and, therefore, base block $\{0,5,3\}$ belongs to C_1, covers a second difference 3 there and forces the 7th base block $\{0,7,4\}$ to be in C_2. Which in turn provides a second difference 7 in C_2 such that base blocks $\{0,4,9\}$ and $\{0,9,5\}$ need to be in C_1. But then difference 5 occurs three-times among the differences provided by base blocks of C_1, a contradiction. Hence, the $CTS_4(16)$ above is cyclically indecomposable.

Moreover, this design is decomposable since the following blocks which are chosen from the orbits of the base blocks above form a $TS_2(16)$: $\{0,1,2\}$, $\{0,1,10\}$, $\{0,2,3\}$, $\{0,3,5\}$, $\{0,4,7\}$, $\{0,4,9\}$, $\{0,5,12\}$, $\{0,6,12\}$, $\{0,6,14\}$, $\{0,7,11\}$, $\{0,8,10\}$, $\{0,8,11\}$, $\{0,9,13\}$, $\{0,13,15\}$, $\{0,14,15\}$, $\{1,2,11\}$, $\{1,3,9\}$, $\{1,3,14\}$, $\{1,4,9\}$, $\{1,4,13\}$, $\{1,5,11\}$, $\{1,5,12\}$, $\{1,6,10\}$, $\{1,6,14\}$, $\{1,7,8\}$, $\{1,7,15\}$, $\{1,8,13\}$, $\{1,12,15\}$, $\{2,3,4\}$, $\{2,4,5\}$, $\{2,5,10\}$, $\{2,6,9\}$, $\{2,6,13\}$, $\{2,7,14\}$, $\{2,7,15\}$, $\{2,8,12\}$, $\{2,8,14\}$, $\{2,9,13\}$, $\{2,10,13\}$, $\{2,11,15\}$, $\{3,4,13\}$, $\{3,5,11\}$, $\{3,6,8\}$, $\{3,6,11\}$, $\{3,7,12\}$, $\{3,7,13\}$, $\{3,8,12\}$, $\{3,9,15\}$, $\{3,10,14\}$, $\{3,10,15\}$, $\{4,5,14\}$, $\{4,6,7\}$, $\{4,6,12\}$, $\{4,8,11\}$, $\{4,8,15\}$, $\{4,10,11\}$, $\{4,10,14\}$, $\{4,12,15\}$, $\{5,6,7\}$, $\{5,6,15\}$, $\{5,7,13\}$, $\{5,8,10\}$, $\{5,8,13\}$, $\{5,9,14\}$, $\{5,9,15\}$, $\{6,8,9\}$, $\{6,10,13\}$, $\{6,11,15\}$, $\{7,8,9\}$, $\{7,9,10\}$, $\{7,10,12\}$, $\{7,11,14\}$, $\{8,14,15\}$, $\{9,10,11\}$, $\{9,11,12\}$, $\{9,12,14\}$, $\{10,13,15\}$, $\{11,12,13\}$, $\{11,13,14\}$, $\{12,13,14\}$. The base blocks of the second $TS_2(16)$ are just the complement of the blocks above with respect to the blocks in $CTS_4(16)$. ■

Lemma 5.9 *There exists a cyclically indecomposable but decomposable* $CTS_4(18)$.

Proof From the 18-extended Skolem sequence of order 11, $(11,9,7,5,3,1,1,3,5,7,9,11,10,8,6,4,2,0,2,4,6,8,10)$, take the base block of the form $\{0,i,b_i\}$ mod 18 as in Construction 5.1. So, the canonical base blocks of the $CTS_4(18)$ are: $\{0,1,7\}$, $\{0,2,1\}$, $\{0,3,8\}$, $\{0,4,2\}$, $\{0,5,9\}$,

$\{0,6,3\}$, $\{0,7,10\}$, $\{0,8,4\}$, $\{0,9,11\}$, $\{0,10,5\}$, $\{0,11,12\}$, and the short orbit $\{0,6,12\}$. This triple system is cyclically indecomposable since there exists no $CTS_1(18)$ nor $CTS_2(18)$. This design is decomposable since the following blocks which are chosen from the orbits of the base blocks above form a $TS_2(18)$: $\{0,1,2\}$, $\{0,1,7\}$, $\{0,2,4\}$, $\{0,3,6\}$, $\{0,3,8\}$, $\{0,4,8\}$, $\{0,5,13\}$, $\{0,5,15\}$, $\{0,6,17\}$, $\{0,7,10\}$, $\{0,9,11\}$, $\{0,9,14\}$, $\{0,10,13\}$, $\{0,11,12\}$, $\{0,12,15\}$, $\{0,14,16\}$, $\{0,16,17\}$, $\{1,2,8\}$, $\{1,3,10\}$, $\{1,3,17\}$, $\{1,4,7\}$, $\{1,4,12\}$, $\{1,5,9\}$, $\{1,5,14\}$, $\{1,6,11\}$, $\{1,6,16\}$, $\{1,8,11\}$, $\{1,9,14\}$, $\{1,10,15\}$, $\{1,12,13\}$, $\{1,13,16\}$, $\{1,15,17\}$, $\{2,3,4\}$, $\{2,3,9\}$, $\{2,5,8\}$, $\{2,5,10\}$, $\{2,6,10\}$, $\{2,6,16\}$, $\{2,7,15\}$, $\{2,7,17\}$, $\{2,9,12\}$, $\{2,11,13\}$, $\{2,11,16\}$, $\{2,12,15\}$, $\{2,13,14\}$, $\{2,14,17\}$, $\{3,4,10\}$, $\{3,5,7\}$, $\{3,5,12\}$, $\{3,6,14\}$, $\{3,7,11\}$, $\{3,8,13\}$, $\{3,9,15\}$, $\{3,11,16\}$, $\{3,12,17\}$, $\{3,13,16\}$, $\{3,14,15\}$, $\{4,5,6\}$, $\{4,5,11\}$, $\{4,6,13\}$, $\{4,7,15\}$, $\{4,8,12\}$, $\{4,9,13\}$, $\{4,9,17\}$, $\{4,10,16\}$, $\{4,11,14\}$, $\{4,14,17\}$, $\{4,15,16\}$, $\{5,6,12\}$, $\{5,7,14\}$, $\{5,8,16\}$, $\{5,9,13\}$, $\{5,10,15\}$, $\{5,11,17\}$, $\{5,16,17\}$, $\{6,7,8\}$, $\{6,7,13\}$, $\{6,8,15\}$, $\{6,9,12\}$, $\{6,9,17\}$, $\{6,10,14\}$, $\{6,11,15\}$, $\{7,8,14\}$,$\{7,9,11\}$, $\{7,9,16\}$, $\{7,10,13\}$, $\{7,12,16\}$, $\{7,12,17\}$, $\{8,9,10\}$, $\{8,9,15\}$, $\{8,10,17\}$, $\{8,10,14\}$, $\{8,12,16\}$, $\{8,13,17\}$, $\{9,10,16\}$, $\{10,11,12\}$, $\{10,11,17\}$, $\{10,12,14\}$, $\{11,13,15\}$, $\{12,13,14\}$, $\{13,15,17\}$, $\{14,15,16\}$. The base blocks of the second $TS_2(18)$ are just the complement of the blocks above with respect to the blocks in $CTS_4(18)$. ■

Lemma 5.10 *There exists a cyclically indecomposable but decomposable* $CTS_4(19)$.

Proof From the 19-extended Skolem sequence of order 12, $(11,9,7,5,3,1,1,3,5,7,9,11,12,10,8,6,4,2,0,2,4,6,8,10,12)$, take the base block of the form $\{0,i,b_i\}$ mod 19 as in Construction 5.2. So, the canonical base blocks of the $CTS_4(19)$ are: $\{0,1,7\}$, $\{0,2,1\}$, $\{0,3,8\}$, $\{0,4,2\}$, $\{0,5,9\}$, $\{0,6,3\}$, $\{0,7,10\}$, $\{0,8,4\}$, $\{0,9,11\}$, $\{0,10,5\}$, $\{0,11,12\}$, $\{0,12,6\}$. Again, we have to show that neither a decomposition into a CTS_1 and a CTS_3, nor into two CTS_2 is possible. First, suppose there exists a decomposition into a $CTS_1(19)$ and a $CTS_3(19)$, denoted C_1 and C_2 respectively. Then all base blocks with repeated differences must belong to C_2. These cover differences 4 and 2 three-times such that the remaining base blocks with differences 4 and 2, that is $\{0,5,9\}$ and $\{0,9,11\}$ need to occur in C_1. But then the difference 9 occurs twice in C_1, a contradiction. Now, suppose there exists a decomposition into two $CTS_2(19)$, denoted C_1 and C_2 respectively. Assume w.l.o.g. that the base block $\{0,2,1\}$ providing the repeated difference 1 belongs to C_1. Then the two base blocks $\{0,1,7\}$, $\{0,11,12\}$ which cover difference 1 as well need to occur in C_2. Both blocks cover difference 7 each. Therefore, 7th and 12th base block $\{0,7,10\}$ and

$\{0,12,6\}$ are forced to be contained in C_1. The later base block provides the repeated difference 6, hence the 6th base block $\{0,6,3\}$ must be in C_2 providing a repeated difference 3. This in turn forces the 3rd base block $\{0,3,8\}$ to occur in C_1. Continuing in this way difference 5 now forces the 10th base block $\{0,10,5\}$ to be in C_2 and the 5th base block $\{0,5,9\}$ to be in C_1. Finally difference 4 forces the 8th base block $\{0,8,4\}$ to be part of C_2 and the 4th base block $\{0,4,2\}$ to belong to C_1. The latter gives a repeated difference 2 such that together with the very first base block this difference occurs three-times among the differences provided by base blocks of C_1, a contradiction. Hence, the $CTS_4(19)$ above is cyclically indecomposable.

The design above is decomposable since the following blocks which are chosen from the orbits of the base blocks above form a $TS_2(19)$: $\{0,1,2\}$, $\{0,1,7\}$, $\{0,2,10\}$, $\{0,3,6\}$, $\{0,3,8\}$, $\{0,4,8\}$, $\{0,4,14\}$, $\{0,5,9\}$, $\{0,5,16\}$, $\{0,6,18\}$, $\{0,7,13\}$, $\{0,9,16\}$, $\{0,10,15\}$, $\{0,11,12\}$, $\{0,11,14\}$, $\{0,12,13\}$, $\{0,15,17\}$, $\{0,17,18\}$, $\{1,2,3\}$, $\{1,3,18\}$, $\{1,4,9\}$, $\{1,4,17\}$, $\{1,5,15\}$, $\{1,5,16\}$, $\{1,6,10\}$, $\{1,6,17\}$, $\{1,7,13\}$, $\{1,8,11\}$, $\{1,8,14\}$, $\{1,9,18\}$, $\{1,10,12\}$, $\{1,11,16\}$, $\{1,12,15\}$, $\{1,13,14\}$, $\{2,3,9\}$, $\{2,4,6\}$, $\{2,4,12\}$, $\{2,5,10\}$, $\{2,5,14\}$, $\{2,6,16\}$, $\{2,7,11\}$, $\{2,7,18\}$, $\{2,8,14\}$, $\{2,8,15\}$, $\{2,9,15\}$, $\{2,11,18\}$, $\{2,12,17\}$, $\{2,13,16\}$, $\{2,13,17\}$, $\{3,4,5\}$, $\{3,4,11\}$, $\{3,5,7\}$,$\{3,6,11\}$, $\{3,7,17\}$, $\{3,8,12\}$, $\{3,9,15\}$, $\{3,10,13\}$, $\{3,10,16\}$, $\{3,12,14\}$, $\{3,13,18\}$, $\{3,14,17\}$, $\{3,15,16\}$, $\{4,5,6\}$,$\{4,7,10\}$, $\{4,7,12\}$, $\{4,8,18\}$, $\{4,9,13\}$, $\{4,10,16\}$, $\{4,11,14\}$, $\{4,13,15\}$, $\{4,15,18\}$, $\{4,16,17\}$, $\{5,6,13\}$, $\{5,7,9\}$, $\{5,8,13\}$, $\{5,8,17\}$, $\{5,10,14\}$, $\{5,11,17\}$, $\{5,11,18\}$, $\{5,12,15\}$, $\{5,12,18\}$, $\{6,7,8\}$,$\{6,7,14\}$, $\{6,8,10\}$, $\{6,9,12\}$, $\{6,9,14\}$, $\{6,11,15\}$, $\{6,12,18\}$, $\{6,13,16\}$, $\{6,15,17\}$, $\{7,8,9\}$,$\{7,10,15\}$, $\{7,11,15\}$, $\{7,12,16\}$, $\{7,14,17\}$, $\{7,16,18\}$, $\{8,9,16\}$, $\{8,10,12\}$, $\{8,11,16\}$, $\{8,13,17\}$, $\{8,15,18\}$, $\{9,10,11\}$, $\{9,10,17\}$, $\{9,11,13\}$, $\{9,12,17\}$, $\{9,14,18\}$, $\{10,11,17\}$, $\{10,13,18\}$, $\{10,14,18\}$, $\{11,12,13\}$, $\{12,14,16\}$, $\{13,14,15\}$, $\{14,15,16\}$, $\{16,17,18\}$. The base blocks of the second $TS_2(19)$ are just the complement of the blocks above with respect to the blocks in $CTS_4(19)$. ■

Lemma 5.11 *There exists a cyclically indecomposable but decomposable* $CTS_4(21)$.

Proof From the 21-extended Skolem sequence of order 13, $(4,2,8,2,4,6,3,12,10,3,8,6,13,11,9,7,1,1,10,12,0,5,7,9,11,13,5)$, take the base block of the form $\{0,i,b_i\}$ mod 21 together with the short orbit $\{0,7,14\}$ as in Construction 5.2. So, the canonical base blocks of the $CTS_4(21)$ are: $\{0,1,18\}$, $\{0,2,4\}$, $\{0,3,10\}$, $\{0,4,5\}$, $\{0,5,6\}$, $\{0,6,12\}$, $\{0,7,2\}$, $\{0,8,11\}$, $\{0,9,3\}$, $\{0,10,19\}$, $\{0,11,4\}$, $\{0,12,20\}$, $\{0,13,5\}$. The existence of four short base blocks is a necessary condition for the existence of a decomposition into two two-fold triple systems which is clearly not satisfied. Therefore, if the design is cyclically decomposable, then it must be into a $CTS_1(21)$ and a $CTS_3(21)$, denoted C_1 and C_2. The short block with difference 7 belongs to C_1. Therefore, all other base blocks with difference 7 as well as all base blocks with repeated differences belong to C_2. These are the 2nd, 3rd, 6th, 7th, 11th and 13th base block. Among the differences provided by these blocks occurs 2 three-times. Hence, the 10th base block $\{0,10,19\}$ with difference 2 must occur in C_1 providing differences 9 and 10. This forces the 8th, 9th and 12th base block to be in C_2 with the consequence that difference 8 occurs now four-times in C_2, a contradiction.

The design is clearly decomposable since the following blocks which are chosen from the orbits of the base blocks above form a $TS_2(21)$: $\{0,1,13\}$, $\{0,1,16\}$, $\{0,2,4\}$, $\{0,2,7\}$, $\{0,3,9\}$, $\{0,3,10\}$, $\{0,4,11\}$, $\{0,5,6\}$, $\{0,5,19\}$, $\{0,6,12\}$, $\{0,7,18\}$, $\{0,8,9\}$, $\{0,8,13\}$, $\{0,10,18\}$, $\{0,11,14\}$, $\{0,12,15\}$, $\{0,14,16\}$, $\{0,15,20\}$, $\{0,17,19\}$, $\{0,17,20\}$, $\{1,2,14\}$, $\{1,2,18\}$, $\{1,3,8\}$, $\{1,3,13\}$, $\{1,4,5\}$, $\{1,4,10\}$, $\{1,5,12\}$, $\{1,6,7\}$, $\{1,6,14\}$, $\{1,7,19\}$, $\{1,8,15\}$, $\{1,9,12\}$, $\{1,9,17\}$, $\{1,10,16\}$, $\{1,11,19\}$, $\{1,11,20\}$, $\{1,15,17\}$, $\{1,18,20\}$, $\{2,3,19\}$, $\{2,3,20\}$, $\{2,4,9\}$, $\{2,5,6\}$, $\{2,5,15\}$, $\{2,6,13\}$, $\{2,7,8\}$, $\{2,8,17\}$, $\{2,9,19\}$, $\{2,10,11\}$, $\{2,10,15\}$, $\{2,11,13\}$, $\{2,12,16\}$, $\{2,12,20\}$, $\{2,14,17\}$, $\{2,16,18\}$, $\{3,4,16\}$, $\{3,4,20\}$, $\{3,5,7\}$, $\{3,5,10\}$, $\{3,6,7\}$, $\{3,6,13\}$, $\{3,8,16\}$, $\{3,9,15\}$, $\{3,11,12\}$, $\{3,11,14\}$, $\{3,12,18\}$, $\{3,14,17\}$, $\{3,15,18\}$, $\{3,17,19\}$, $\{4,5,17\}$, $\{4,6,8\}$, $\{4,6,16\}$, $\{4,7,14\}$, $\{4,7,17\}$, $\{4,8,9\}$, $\{4,10,19\}$, $\{4,11,18\}$, $\{4,12,15\}$, $\{4,12,20\}$, $\{4,13,15\}$, $\{4,13,19\}$, $\{4,14,18\}$, $\{5,7,12\}$, $\{5,8,14\}$, $\{5,8,18\}$, $\{5,9,10\}$, $\{5,9,16\}$, $\{5,11,17\}$, $\{5,11,20\}$, $\{5,13,16\}$, $\{5,13,18\}$, $\{5,14,20\}$, $\{5,15,19\}$, $\{6,8,18\}$, $\{6,9,10\}$, $\{6,9,19\}$, $\{6,10,11\}$, $\{6,11,19\}$, $\{6,12,18\}$, $\{6,14,15\}$, $\{6,15,17\}$, $\{6,16,20\}$, $\{6,17,20\}$, $\{7,8,20\}$, $\{7,9,11\}$, $\{7,9,14\}$, $\{7,10,16\}$, $\{7,10,17\}$, $\{7,11,18\}$, $\{7,12,13\}$, $\{7,13,19\}$, $\{7,15,16\}$, $\{7,15,20\}$, $\{8,10,15\}$, $\{8,10,20\}$, $\{8,11,12\}$, $\{8,11,17\}$, $\{8,12,19\}$, $\{8,13,14\}$, $\{8,16,19\}$, $\{9,11,16\}$, $\{9,12,13\}$, $\{9,13,20\}$, $\{9,14,15\}$, $\{9,17,18\}$, $\{9,18,20\}$, $\{10,12,14\}$, $\{10,12,17\}$, $\{10,13,14\}$, $\{10,13,20\}$, $\{10,18,19\}$, $\{11,13,15\}$, $\{11,15,16\}$, $\{12,14,19\}$, $\{12,16,17\}$, $\{13,16,17\}$, $\{13,17,18\}$, $\{14,16,18\}$, $\{14,19,20\}$, $\{15,18,19\}$, $\{16,19,20\}$. The base blocks of the second $TS_2(21)$ are just the complement of the blocks above with respect to the blocks in $CTS_4(21)$. ■

Next, we are going to construct infinitely many cyclically indecomposable but decomposable $CTS_4(v)$ using one ingredient design having this property.

Construction 5.12 *Let* $m \equiv 1 \text{ or } 3 \mod 6, m \neq 9$ *and* $n \equiv 1 \mod 6$ *and suppose that there exists a cyclically indecomposable but decomposable* $CTS_4(m)$. *Then there exists a cyclically indecomposable but decomposable cyclic four-fold triple system of order mn.*

Proof By Theorem 2.1 in [5] there exists a cyclic Steiner triple system of order mn containing a cyclic sub-system of order m on the point set $0, n, 2n, \ldots, (m-1)n$. The proof is similar to Construction 4.2. ■

Construction 5.13 *Let $m \equiv 3 \mod 6, m \neq 9$ and $n \equiv 3 \text{ or } 5 \mod 6, n \neq 3$ and suppose that there exists a cyclically indecomposable but decomposable $CTS_4(m)$. Then there exists a cyclically indecomposable but decomposable cyclic four-fold triple system of order mn.*

Proof By Theorem 2.1 in [5] there exists a cyclic Steiner triple system of order mn containing a cyclic sub-system of order m on the point set $0, n, 2n, \ldots, (m-1)n$. The proof is similar to Construction 4.2. ■

Construction 5.14 *Let $m \equiv 1 \mod 6, m \geq 7$ and suppose that there exists a cyclically indecomposable but decomposable $CTS_4(m)$. Then there exists a cyclically indecomposable but decomposable cyclic four-fold triple system of order $9m$.*

Proof By Theorem 2.1 in [5] there exists a cyclic Steiner triple system of order $9m$ containing a cyclic sub-system of order m on the point set $0, 9, 18, \ldots, 9(m-1)$. The proof is similar to Construction 4.2. ■

As corollaries of the above constructions we state the following proposition.

Proposition 5.15 *There exists cyclically indecomposable but decomposable $CTS_4(v)$ for $v = 13n, 15n, 19n, 21n$ and for any $n \equiv 1 \mod 6$. There exists cyclically indecomposable but decomposable $CTS_4(v)$ for $v = 15n, 21n$ and for any $n \equiv 3 \text{ or } 5 \mod 6, n \neq 3$. There exists cyclically indecomposable but decomposable $CTS_4(v)$ for $v = 13 \times 9 = 117$ and $v = 19 \times 9 = 171$.*

Proof There exists a cyclically indecomposable that is decomposable $\text{CTS}_4(v)$ for $v = 13$ (Lemma 5.6), $v = 15$ (Lemma 5.7), $v = 19$ (Lemma 5.10) and $v = 21$ (Lemma 5.11). Apply construction 5.12 for $m = 13, 15, 19$ and 21.

There exists a cyclically indecomposable that is decomposable $\text{CTS}_4(v)$ for $v = 15$ (Lemma 5.7) and for $v = 21$ (Lemma 5.11). Apply construction 5.13 for $m = 15$ and for $m = 21$.

There exists a cyclically indecomposable that is decomposable $\text{CTS}_4(v)$ for $v = 13$ (Lemma 5.6) and $v = 19$ (Lemma 5.10). Apply construction 5.14 for $m = 13$ and 19. ■

Proposition 5.16 *There exists a cyclically indecomposable but decomposable $CTS_4(16v)$ for every $v \equiv 1 \mod 3$.*

Proof By Lemma 3.8 in [9] there exists a $(16v, 16, 3, 1)$-DF for every $v \equiv 1 \mod 3$. By Lemma 5.8 there exists a cyclically indecomposable but decomposable $CTS_4(16)$. The rest of the proof is similar to the proof of Construction 4.2. ■

6 Conclusion

We proved that a cyclic two-fold triple system is cyclically indecomposable if and only if it is indecomposable.

A necessary condition for a $CTS_3(v)$ to be decomposable is that $v \equiv 1 \text{ or } 3 \mod 6$. We determined half of the spectrum of those parameters v which admit a cyclically indecomposable but decomposable $CTS_3(v)$. In order to determine the other half of the spectrum it would be sufficient to find a cyclically indecomposable but decomposable $CTS_3(p)$

a.) for all primes $p \equiv 1 \mod 6$ and $p = 7^2, 13^2, 7 \cdot 13$; and

b.) for all composite integers $p = n_1 \cdot n_2$ where $n_1, n_2 \equiv 5 \mod 6$.

So far no example of the latter is known.

We remark that our present constructions for $k = 3$ involve always repeated blocks. So it would be of interest to find the spectrum of simple, cyclically indecomposable but decomposable $CTS_3(v)$.

For $\lambda = 4$ we have several cyclic designs that are cyclically indecomposable but decomposable and few constructions which yield infinitely many such triple systems. We showed that there exists cyclically indecomposable but decomposable $CTS_4(v)$ for $v \equiv 1 \mod 3$ if $v = 10, 13, 16, 19, 78k + 13, 114k + 19, 48k + 16$ where k is a positive integer. There exists cyclically indecomposable but decomposable $CTS_4(v)$ for $v \equiv 0 \mod 3$ if $v = 9, 12, 15, 18, 21, 117, 171, 90k+75, 90k'+45, 126k'+63, 126k+105, 90k+15, 126k+21$ where k and k' are integers, $k \geq 0, k' \geq 1$.

Finally, it came to our attention that A. Wassermann and M. Buratti gave a talk on decomposable TSs in Italy (Conference Combinatorics 2014). They computed the number of inequivalent cyclically indecomposable simple cyclic triple systems for small parameters. It seems that some of the results in Section 5 have been independently obtained by the authors of the talk.

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