# The 2-steps Hamiltonian Subdivision Graphs of Cycles with a Chord

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**Abstract**

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A $(p,q)$-graph $G=(V,E)$ is said to be AL$(k)$-traversal if there exist a sequence of vertices $\{v_1,v_2,\ldots,v_p\}$ such that for each $i=1,2,\ldots,p-1$, the distance for $v_i$ and $v_{i+1}$ is equal to $k$. We call a graph $G$ a $k$-steps Hamiltonian graph if it has a AL$(k)$-traversal in $G$ and the distance between $v_p$ and $v_1$ is $k$. In this paper, we completely classify whether a subdivision graph of a cycle with a chord is 2-steps Hamiltonian.

## 1 Introduction

The Hamiltonicity of a graph is the problem of determining for a given graph whether it contains a path or a cycle that visits every vertex exactly once. Hamiltonian graphs are related to the traveling salesman problem. Thus, it has been a well-studied topic in graph theory. However, we know very little about Hamiltonian graphs. A good reference for recent development and open problems is [3].

Inspired by W.D. Wallis's Magic Graph [11], A.N.T. Lee in [8] initiated the study of AL$(k)$-traversal graphs and 2-steps Hamiltonian graphs defined as follows:

**Definition 1.** For $k>2$, a $(p,q)$-graph $G=(V,E)$ is said to have $k$-steps traversal if there exist a sequence of vertices, $v_1,v_2,\ldots,v_p$, such that, for each $i=1,2,\ldots,p-1$, the distance between $v_i$ and $v_{i+1}$ is equal to $k$. A graph admits a $k$-steps traversal is called the AL$(k)$-traversal graph.

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**Example 1.** The graph showed in the figure 1 is AL(2)-traversal, but not AL($k$)-traversal for all $k \geq 3$.

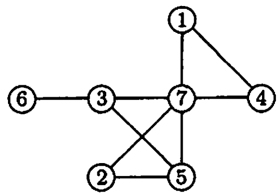


Figure 1: An AL(2)-traversal Graph

**Definition 2.** We call a graph $G$ a $k$-steps Hamiltonian graph if it has a AL($k$)-traversal in $G$ and the distance between vertices $v_p$ and $v_1$ is $k$.

Note here that in Figure 1 the distance between the vertices labeled 1 and 7 is not 2. Moreover, there is no labeling to make this graph 2-steps Hamiltonian.

**Example 2.** Figure 2 demonstrates a 2-steps Hamiltonian cubic graph.

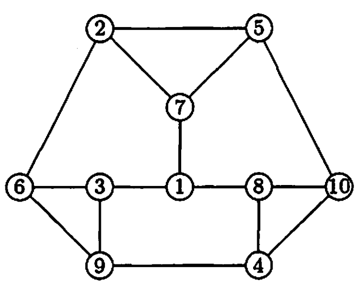


Figure 2: A 2-steps Hamiltonian cubic graph

**Definition 3.** For integer $k \geq 2$, and a graph $G$, we construct a new graph $D_k(G)$ as follows: $V(D_k(G)) = V(G)$ and $(u, v) \in E(D_k(G))$ if and only if $d(u, v) = k$ in $G$. We call $D_k(G)$ as the distance $k$ graph of $G$.

The following observation which would be useful in this paper was recorded in [7].

**Proposition 1.1.** *The cycle $C_n$ is $k$-steps Hamiltonian if and only if* $\gcd(n, k) = 1$.

**Proposition 1.2.** *The graph $G$ is $k$-steps Hamiltonian if and only if its distance $k$-graph is Hamiltonian.*

**Proposition 1.3.** *A bipartite graph is not* AL(2)*-traversal, thus, not 2-steps Hamiltonian.*

A Hamiltonian graph need not be $k$-steps Hamiltonian. One example is a cycle $C_n$ with $n = 0 \pmod{k}$ is Hamiltonian but not AL($k$)-traversal, hence cannot be $k$-steps Hamiltonian.

In this paper, we need the following lemma to determine whether a distance 2-graph is Hamiltonian.

**Lemma 1.4.** *If a distance 2-graph contains a subgraph $H$ consisted with all order 2 vertices and two order 3 vertices where the distance the two order 3 vertices is greater than 1, then it is not Hamiltonian. Moreover, if $H$ consists with 3 or more order 3 vertices and those order 3 vertices are adjacent to each other in two paths, then it is not Hamiltonian as well.*

**Proof.** For a labeling cycle, it must enter the subgraph $H$ through one of the two order 3 vertices. But, since the distance between two order 3 vertices is greater than 1, it is obvious that this cycle cannot be Hamiltonian.

Similarly, a path of adjacent order 3 vertices can be considered as one order 3 vertex in the purpose of our proof. □

**Definition 4.** For a graph $G$, let $S$ be a subset of $E(G)$ and $f : S \to \mathbb{N}$. The subdivision graph Sub$(G, S, f)$ is the graph obtained by for any $e$ in $S$, if $f(e) = m$, then we insert $m$ new $m$ vertices along in $e$.

For a cycle of order $n$, we denote its vertices by $v_1, v_2, \ldots, v_n$. If it has a chord between vertices $v_1$ and $v_k$, then we denote this graph by $C_n(k)$.

For a $\mathrm{Sub}(C_n(k), \{v_1, v_k\}, f(v_1, v_k) = h)$ graph, we denote the added vertices on the chord by $w_1, w_2, \ldots, w_h$. Note that the vertex $w_1$ is adjacent to the vertex $v_1$ and the vertex $w_h$ is adjacent to the vertex $v_k$.

Due to symmetry, $\mathrm{Sub}(C_n(k), \{v_1, v_k\}, f(v_1, v_k) = h)$ is isomorphic to $\mathrm{Sub}(C_n(n-k+1), \{v_1, v_{n-k+1}\}, f(v_1, v_{n-k+1}) = h)$. Thus, we can assume that $3 \le k \le \lfloor \frac{n}{2} \rfloor + 1$.

Note that in a $\mathrm{Sub}(C_n(k), \{v_1, v_k\}, f(v_1, v_k) = h)$ graph, the vertices $v_1$ and $v_k$ are the two end vertices of three paths of length $k$, $h+2$ and $n-k+2$. Thus, it is easy to see that

**Lemma 1.5.** *The graph* $\mathrm{Sub}(C_n(k), \{v_1, v_k\}, f(v_1, v_k) = h)$ *is isomorphic to the graph* $\mathrm{Sub}(C_{k+h}(k), \{v_1, v_k\}, f(v_1, v_k) = n-k)$ *as well as the graph* $\mathrm{Sub}(C_{n-k+2+h}(h), \{v_1, v_h\}, f(v_1, v_h) = k-2)$.

Since, in general, a subdivision graph of a cycle with a chord is not Hamiltonian, it would be interesting to see whether it is 2-steps Hamiltonian. Thus, in this paper, we investigate under what conditions the subdivision graph $\mathrm{Sub}(C_n(k), \{v_1, v_k\}, f(\{v_1, v_k\}) = h)$ is 2-steps Hamiltonian.

## 2 Cycles $C_n$ when $n \leq 8$

Let us start with small cycles. Note that for a cycle $C_n$, $n$ must be at least 4 to insert a chord.

**Theorem 2.1.** $\mathrm{Sub}\,(C_4(3), \{v_1, v_3\}, f(v_1, v_3) = h)$ *is 2-steps Hamiltonian if and only if $h$ is even.*

**Proof.** When $h$ is even, for a $\mathrm{Sub}\,(C_4(3), \{v_1, v_3\}, f(v_1, v_3) = h)$ graph, let us label the vertices from 1 to $4 + h$ by the order

$$v_1, v_3, w_{h-1}, w_{h-3}, \ldots, w_1, v_2, v_4, w_h, w_{h-2}, \ldots, w_2.$$

Since the distance between any two vertices in the order and between the last vertex $w_2$ and the first vertex $v_1$ are all 2, by definition, it is a 2-steps Hamiltonian graph.

When $h$ is odd, we can group the vertices $v_1, w_2, w_4, \ldots, w_{h-1}, v_3$ in a set and the other vertices in another set. It is easy to see that it becomes a bipartite graph. By Proposition 1.3, it is not a 2-steps Hamiltonian graph.

This completes the proof. □

**Theorem 2.2.** $\mathrm{Sub}\,(C_5(3), \{v_1, v_3\}, f(v_1, v_3) = h)$ *is 2-steps Hamiltonian for all $h \geq 1$.*

**Proof.** When $h$ is odd, for a $\mathrm{Sub}\,(C_5(3), \{v_1, v_3\}, f(v_1, v_3) = h)$ graph, let us label the vertices from 1 to $5 + h$ by the order

$$v_1, w_2, w_4, \ldots, w_{h-1}, v_3, v_5, w_1, w_3, \ldots, w_h, v_2, v_4.$$

Since the distance between any two vertices in the order and between the last vertex $v_4$ and the first vertex $v_1$ are all 2, by definition, it is a 2-steps Hamiltonian graph.

When $h$ is even, for a $\mathrm{Sub}\,(C_5(3), \{v_1, v_3\}, f(v_1, v_3) = h)$ graph, let us label the vertices from 1 to $5 + h$ by the order

$$v_1, v_3, w_{h-1}, w_{h-3}, \ldots, w_1, v_5, v_2, v_4, w_h, w_{h-2}, \ldots, w_2.$$

Since the distance between any two vertices in the order and between the last vertex $w_2$ and the first vertex $v_1$ are all 2, by definition, it is a 2-steps Hamiltonian graph.

This completes the proof. □

**Theorem 2.3.** $\mathrm{Sub}\,(C_6(3), \{v_1, v_3\}, f(v_1, v_3) = h)$ *is 2-steps Hamiltonian if and only if $h$ is even.*

**Proof.** When $h$ is even, for a Sub $(C_6(3), \{v_1, v_3\}, f(v_1, v_3) = h)$ graph, let us label the vertices from 1 to $6 + h$ by the order

$$v_1, w_2, w_4, \ldots, w_h, v_2, v_4, v_6, w_1, w_3, \ldots, w_{h-1}, v_3, v_5.$$

Since the distance between any two vertices in the order and between the last vertex $v_5$ and the first vertex $v_1$ are all 2, by definition, it is a 2-steps Hamiltonian graph.

When $h$ is odd, we can group the vertices $v_1, v_3, v_5, w_2, w_4, \ldots, w_{h-1}$ in a set and the other vertices in another set. It is easy to see that it becomes a bipartite graph. By Proposition 1.3, it is not a 2-steps Hamiltonian graph.

This completes the proof. □

**Theorem 2.4.** Sub $(C_6(4), \{v_1, v_4\}, f(v_1, v_4) = h)$ *is 2-steps Hamiltonian if and only if h is odd.*

**Proof.** When $h$ is odd, for a Sub $(C_6(4), \{v_1, v_4\}, f(v_1, v_4) = h)$ graph, let us label the vertices from 1 to $6 + h$ by the order

$$v_1, v_3, v_5, w_h, w_{h-2}, \ldots, w_1, v_2, v_6, v_4, w_{h-1}, w_{h-3}, \ldots, w_2.$$

Since the distance between any two vertices in the order and between the last vertex $w_2$ and the first vertex $v_1$ are all 2, by definition, it is a 2-steps Hamiltonian graph.

Note that when $h = 1$, the last vertex to label is $v_4$. Since the distance between $v_4$ and the first vertex $v_1$ is 2, it is still 2-steps Hamiltonian.

When $h$ is even, we can group the vertices $v_1, v_3, v_5, w_2, w_4, \ldots, w_h$ in a set and the other vertices in another set. It is easy to see that it becomes a bipartite graph. By Proposition 1.3, it is not a 2-steps Hamiltonian graph.

This completes the proof. □

**Theorem 2.5.** Sub $(C_7(3), \{v_1, v_3\}, f(v_1, v_3) = h)$ *is 2-steps Hamiltonian if and only if h is odd or* 2.

**Proof.** When $h$ is odd, for a Sub $(C_7(3), \{v_1, v_3\}, f(v_1, v_3) = h)$ graph, let us label the vertices from 1 to $7 + h$ by the order

$$v_1, w_2, w_4, \ldots, w_{h-1}, v_3, v_5, v_7, w_1, w_3, \ldots, w_h, v_2, v_4, v_6.$$

Since the distance between any two vertices in the order and between the last vertex $v_6$ and the first vertex $v_1$ are all 2, by definition, it is a 2-steps Hamiltonian graph.

When $h = 2$, for a Sub $(C_7(3), \{v_1, v_3\}, f(v_1, v_3) = 2)$ graph, let us label the vertices from 1 to 9 by the order

$$v_1, v_3, v_5, v_7, w_1, v_2, w_2, v_4, v_6.$$

Since the distance between any two vertices in the order and between the last vertex $v_6$ and the first vertex $v_1$ are all 2, by definition, it is a 2-steps Hamiltonian graph.

When $h = 4$, for a $\text{Sub}(C_7(3), \{v_1, v_3\}, f(v_1, v_3) = 4)$ graph, the subgraph with vertices, A, C, E, G, I, in the figure 3 demonstrates its distance 2-graph is not Hamiltonian by the Lemma 1.4. By Proposition 1.2, it is not 2-steps Hamiltonian.

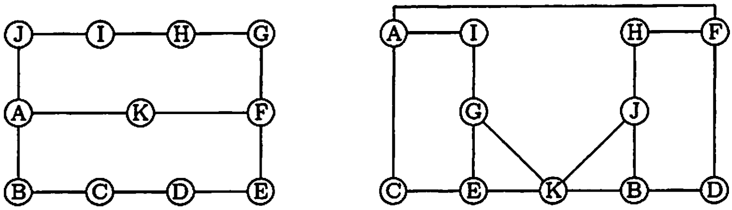


Figure 3: A $\text{Sub}(C_7(3), \{v_1, v_3\}, f(v_1, v_3) = 4)$ graph and its distance 2-graph

Similarly, when $h$ is even and greater than or equal to 6, the $D_2$ graph only gets a larger cycle and the same reason applies to show that it is not a 2-steps Hamiltonian.

This completes the proof. □

**Theorem 2.6.** $\text{Sub}(C_7(4), \{v_1, v_4\}, f(v_1, v_4) = h)$ *is 2-steps Hamiltonian if and only if $h$ is even or* 1.

**Proof.** When $h$ is even, for a $\text{Sub}(C_7(4), \{v_1, v_4\}, f(v_1, v_4) = h)$ graph, let us label the vertices from 1 to $7 + h$ by the order

$$v_1, w_2, w_4, \ldots, w_h, v_3, v_5, v_7, v_2, w_1, w_3, \ldots, w_{h-1}, v_4, v_6.$$

Since the distance between any two vertices in the order and between the last vertex $v_6$ and the first vertex $v_1$ are all 2, by definition, it is a 2-steps Hamiltonian graph.

When $h = 1$, for a $\text{Sub}(C_7(4), \{v_1, v_4\}, f(v_1, v_4) = 1)$ graph, let us label the vertices from 1 to 8 by the order

$$v_1, v_6, v_4, v_2, w_1, v_7, v_5, v_3.$$

Since the distance between any two vertices in the order and between the last vertex $v_3$ and the first vertex $v_1$ are all 2, by definition, it is a 2-steps Hamiltonian graph.

When $h = 3$, for a $\text{Sub}(C_7(4), \{v_1, v_4\}, f(v_1, v_4) = 3)$ graph, the subgraph with vertices, A, C, E, G, in the figure 4 demonstrates its distance

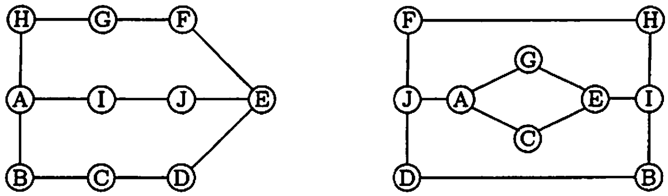


Figure 4: A $\text{Sub}(C_7(4), \{v_1, v_4\}, f(v_1, v_4) = 3)$ graph and its distance 2-graph

2-graph is not Hamiltonian by Lemma 1.4. By Proposition 1.2, it is not 2-steps Hamiltonian.

Similarly, when $h$ is odd and greater than or equal to 5, the $D_2$ graph only gets a larger cycle and the same reason applies to show that it is not a 2-steps Hamiltonian.

This completes the proof. □

**Theorem 2.7.** $\text{Sub}(C_8(3), \{v_1, v_3\}, f(v_1, v_3) = h)$ *is 2-steps Hamiltonian if and only if $h$ is even.*

**Proof.** When $h$ is even, for a $\text{Sub}(C_8(3), \{v_1, v_3\}, f(v_1, v_3) = h)$ graph, let us label the vertices from 1 to $8 + h$ by the order

$$v_1, w_2, w_4, \ldots, w_h, v_4, v_6, v_8, v_2, w_1, w_3, \ldots, w_{h-1}, v_3, v_5, v_7.$$

Since the distance between any two vertices in the order and between the last vertex $v_7$ and the first vertex $v_1$ are all 2, by definition, it is a 2-steps Hamiltonian graph.

When $h$ is odd, we can group the vertices $v_1, v_3, v_5, v_7, w_2, w_4, \ldots, w_{h-1}$ in a set and the other vertices in another set. It is easy to see that it becomes a bipartite graph. By Proposition 1.3, it is not a 2-steps Hamiltonian graph.

This completes the proof. □

**Theorem 2.8.** $\text{Sub}(C_8(4), \{v_1, v_4\}, f(v_1, v_4) = h)$ *is 2-steps Hamiltonian if and only if $h$ is odd.*

**Proof.** When $h$ is odd, for a $\text{Sub}(C_8(4), \{v_1, v_4\}, f(v_1, v_4) = h)$ graph, let us label the vertices from 1 to $8 + h$ by the order

$$v_1, w_2, w_4, \ldots, w_{h-1}, v_4, v_6, v_8, v_2, w_1, w_3, \ldots, w_h, v_3, v_5, v_7.$$

Since the distance between any two vertices in the order and between the last vertex $v_7$ and the first vertex $v_1$ are all 2, by definition, it is a 2-steps Hamiltonian graph.

When $h$ is even, we can group the vertices $v_1, v_3, v_5, v_7, w_2, w_4, \ldots, w_h$ in a set and the other vertices in another set. It is easy to see that it becomes a bipartite graph. By Proposition 1.3, it is not a 2-steps Hamiltonian graph.

This completes the proof. □

**Theorem 2.9.** Sub $(C_8(5), \{v_1, v_5\}, f(v_1, v_5) = h)$ *is not 2-steps Hamiltonian for all h.*

**Proof.** When $h$ is even, for a Sub $(C_8(5), \{v_1, v_5\}, f(v_1, v_5) = h)$ graph, the subgraph with vertices, A, C, E, G, in the figure 5 demonstrates its distance 2-graph is not Hamiltonian by Lemma 1.4. Note that the dotted line in the graph represents a path of even number order 2 vertices and the dotted line in its distance 2 graph represents a path of order 2 vertices. By Proposition 1.2, it is not 2-steps Hamiltonian.

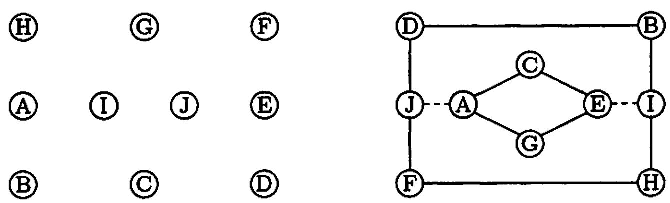


Figure 5: A Sub $(C_8(5), \{v_1, v_5\}, f(v_1, v_5) = h)$ graph when $h$ is even and its distance 2-graph

When $h$ is odd, we can group the vertices $v_1, v_3, v_5, v_7, w_2, w_4, \ldots, w_{h-1}$ in a set and the other vertices in another set. It is easy to see that it becomes a bipartite graph. By Proposition 1.3, it is not a 2-steps Hamiltonian graph.

This completes the proof. □

# 3 General Results

In this paper, when we say a subgraph of a graph $G$ is a path, we mean every vertex in the path except two end vertices is order 2 in $G$.

**Lemma 3.1.** *A graph $G$ with a subgraph $P$ which is a path of length 8 or more is 2-steps Hamiltonian if and only if the induced graph $H$ from $G$ by removing two non-end vertices from the path $P$ is 2-steps Hamiltonian.*

**Proof.** For convenience, we name the vertices of the path $P$ by $p_1, p_2, \ldots, p_t$ where $t \geq 8$. Now, consider a 2-steps Hamiltonian is to travel from vertex to vertex following its labeling from smallest number 1 to the largest number.

For a 2-steps Hamiltonian labeling, when it "enter" the path as deep as the third vertex, it has to "travel through" the path and "leave" the path through the other end. If not, then to label the middle vertices, it has to "enter" the path from the other side. After it labels all the middle vertices, it still has to "leave" the path. It is impossible because the first few vertices have been labeled. Since $t \geq 8$, $\lfloor \frac{t-1}{2} \rfloor \geq 3$. Thus, it has to label at least one more vertex before reaching the end of the path. Therefore, by removing two vertices from the middle of the path, it remains 2-steps Hamiltonian.

Since $t \geq 8$, even if you remove two middle vertices from the path, after labeling by the first "travel through", there are more than 3 vertices left to label. Thus, similarly, when it "enters" the path the second time, it has to "travel through" and label all vertices.

This completes the proof. □

**Corollary 3.2.** *If $h \leq 5$, $n \geq 14$ and $5 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$, then the graph* $\mathrm{Sub}\,(C_n(k), \{v_1, v_k\}, f(v_1, v_k) = h)$ *is 2-steps Hamiltonian if the smaller graph* $\mathrm{Sub}\,(C_{n-2}(k), \{v_1, v_k\}, f(v_1, v_k) = h)$ *is 2-steps Hamiltonian. Similarly, the graph* $\mathrm{Sub}\,(C_n(k), \{v_1, v_k\}, f(v_1, v_k) = h)$ *is 2-steps Hamiltonian if* $\mathrm{Sub}\,(C_{n-2}(k-2), \{v_1, v_{k-2}\}, f(v_1, v_{k-2}) = h)$ *under the same conditions.*

For $\mathrm{Sub}\,(C_n(k), \{v_1, v_k\}, f(v_1, v_k) = h)$, there are $k-2$ vertices in one side of the cycle part and $n-k$ vertices in another side. By Lemma 3.1, we only need to consider whether $\mathrm{Sub}\,(C_n(k), \{v_1, v_k\}, f(v_1, v_k) = h)$ is 2-steps Hamiltonian when $k-2$, $n-k$ and $h$ are all less than 6. Since section 2 has clarified all cases when $n \leq 8$, we only need to check when $8 < n < 14$ and $3 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$. At the same time, $n - k < 6$. Thus, we have

1. when $n = 9$, $n - k \geq 6$ if $k = 3$;
2. when $n = 10$, $n - k \geq 6$ if $3 \leq k \leq 4$;
3. when $n = 11$, $n - k \geq 6$ if $3 \leq k \leq 5$;
4. when $n = 12$, $n - k \geq 6$ if $3 \leq k \leq 6$;
5. when $n = 13$, $n - k \geq 6$ if $3 \leq k \leq 7$.

Thus, we only need to check the following graphs:

1. $\mathrm{Sub}\,(C_9(4), \{v_1, v_4\}, f(v_1, v_4) = h)$ where $h \leq 5$;
2. $\mathrm{Sub}\,(C_9(5), \{v_1, v_5\}, f(v_1, v_5) = h)$ where $h \leq 5$;
3. $\mathrm{Sub}\,(C_{10}(5), \{v_1, v_5\}, f(v_1, v_5) = h)$ where $h \leq 5$;
4. $\mathrm{Sub}\,(C_{10}(6), \{v_1, v_6\}, f(v_1, v_6) = h)$ where $h \leq 5$;
5. $\mathrm{Sub}\,(C_{11}(6), \{v_1, v_6\}, f(v_1, v_6) = h)$ where $h \leq 5$;

6. Sub $(C_{12}(7), \{v_1, v_7\}, f(v_1, v_7) = h)$ where $h \leq 5$.

**Theorem 3.3.** Sub $(C_9(4), \{v_1, v_4\}, f(v_1, v_4) = h)$ *is 2-steps Hamiltonian if and only if $h$ is even or* 1.

**Proof.** By Lemma 1.5, the graph Sub $(C_9(4), \{v_1, v_4\}, f(v_1, v_4) = h)$ is isomorphic to the graph Sub $(C_{h+4}(4), \{v_1, v_4\}, f(v_1, v_4) = 5)$. By Lemma 3.1, we only need to consider $h \leq 5$. Thus, we need to check

1. when $h = 1$, since Sub $(C_5(4), \{v_1, v_4\}, f(v_1, v_4) = 5)$ is isomorphic to Sub $(C_5(3), \{v_1, v_3\}, f(v_1, v_3) = 5)$, it is 2-steps Hamiltonian by Theorem 2.2;
2. when $h = 2$, since Sub $(C_6(4), \{v_1, v_4\}, f(v_1, v_4) = 5)$ is 2-steps Hamiltonian by Theorem 2.4;
3. when $h = 3$, since Sub $(C_7(4), \{v_1, v_4\}, f(v_1, v_4) = 5)$ is not 2-steps Hamiltonian by Theorem 2.6;
4. when $h = 4$, since Sub $(C_8(4), \{v_1, v_4\}, f(v_1, v_4) = 5)$ is 2-steps Hamiltonian by Theorem 2.8;
5. when $h = 5$, the subgraph with vertices, A, H, F, D, M, K, in the figure 6 demonstrates its distance 2-graph is not Hamiltonian by Lemma 1.4. By Proposition 1.2, it is not 2-steps Hamiltonian.

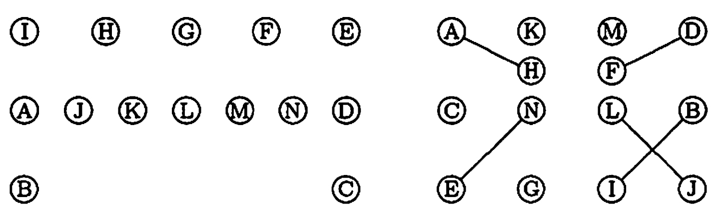


Figure 6: A Sub $(C_9(4), \{v_1, v_4\}, f(v_1, v_4) = 5)$ graph and its distance 2-graph

This completes the proof. □

**Theorem 3.4.** Sub $(C_9(5), \{v_1, v_5\}, f(v_1, v_5) = h)$ *is 2-steps Hamiltonian if and only if $h = 1$ or* 2.

**Proof.** By Lemma 1.5, the graph Sub $(C_9(5), \{v_1, v_5\}, f(v_1, v_5) = h)$ is isomorphic to the graph Sub $(C_{h+5}(5), \{v_1, v_5\}, f(v_1, v_5) = 4)$. By Lemma 3.1, we only need to consider $h \leq 5$. Thus, we need to check

1. when $h = 1$, since $\mathrm{Sub}\,(C_6(5), \{v_1, v_5\}, f(v_1, v_5) = 4)$ is isomorphic to $\mathrm{Sub}\,(C_6(3), \{v_1, v_3\}, f(v_1, v_3) = 4)$. Thus, it is 2-steps Hamiltonian by Theorem 2.3;
2. when $h = 2$, since $\mathrm{Sub}\,(C_7(5), \{v_1, v_5\}, f(v_1, v_5) = 4)$ is isomorphic to $\mathrm{Sub}\,(C_7(4), \{v_1, v_4\}, f(v_1, v_4) = 4)$. Thus, it is 2-steps Hamiltonian by Theorem 2.6;
3. when $h = 3$, since $\mathrm{Sub}\,(C_8(5), \{v_1, v_5\}, f(v_1, v_5) = 4)$ is not 2-steps Hamiltonian by Theorem 2.9;
4. when $h = 4$, the subgraph with vertices, E, G, I, J, L, in the figure 7 demonstrates its distance 2-graph is not Hamiltonian by Lemma 1.4. By Proposition 1.2, it is not 2-steps Hamiltonian.

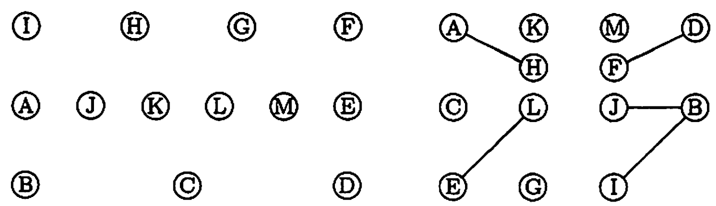


Figure 7: A $\mathrm{Sub}\,(C_9(5), \{v_1, v_5\}, f(v_1, v_5) = 4)$ graph and its distance 2-graph

5. when $h = 5$, the subgraph with vertices, A, C, E, M, K, in the figure 8 demonstrates its distance 2-graph is not Hamiltonian by Lemma 1.4. By Proposition 1.2, it is not 2-steps Hamiltonian.

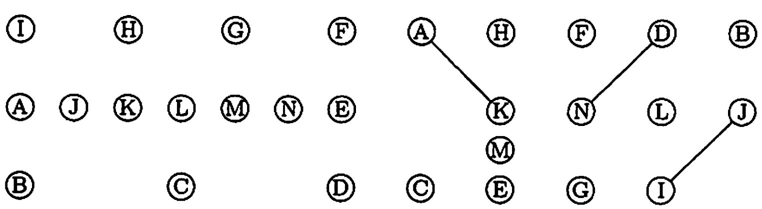


Figure 8: A $\mathrm{Sub}\,(C_9(5), \{v_1, v_5\}, f(v_1, v_5) = 5)$ graph and its distance 2-graph

This completes the proof. □

**Theorem 3.5.** $\mathrm{Sub}\,(C_{10}(5), \{v_1, v_5\}, f(v_1, v_5) = h)$ *is not* 2*-steps Hamiltonian for all* $h$.

**Proof.** By Lemma 1.5, the graph Sub $(C_{10}(5), \{v_1, v_5\}, f(v_1, v_5) = h)$ is isomorphic to the graph Sub $(C_{h+5}(5), \{v_1, v_5\}, f(v_1, v_5) = 5)$. By Lemma 3.1, we only need to consider $h \leq 5$. Thus, we need to check

1. when $h = 1$, since Sub $(C_6(5), \{v_1, v_5\}, f(v_1, v_5) = 5)$ is isomorphic to Sub $(C_6(3), \{v_1, v_3\}, f(v_1, v_3) = 5)$. Thus, it is not 2-steps Hamiltonian by Theorem 2.3;

2. when $h = 2$, since Sub $(C_7(5), \{v_1, v_5\}, f(v_1, v_5) = 5)$ is isomorphic to Sub $(C_7(4), \{v_1, v_4\}, f(v_1, v_4) = 5)$. Thus, it is not 2-steps Hamiltonian by Theorem 2.6;

3. when $h = 3$, since Sub $(C_8(5), \{v_1, v_5\}, f(v_1, v_5) = 5)$ is not 2-steps Hamiltonian by Theorem 2.9;

4. when $h = 4$, since Sub $(C_9(5), \{v_1, v_5\}, f(v_1, v_5) = 5)$ is not 2-steps Hamiltonian by Theorem 3.4;

5. when $h = 5$, we can group the vertices $v_1, v_3, v_5, v_7, v_9, w_2, w_4$ in a set and the other vertices in another set. It is easy to see that it becomes a bipartite graph. By Proposition 1.3, it is not a 2-steps Hamiltonian graph.

This completes the proof. □

**Theorem 3.6.** Sub $(C_{10}(6), \{v_1, v_6\}, f(v_1, v_6) = h)$ *is not* 2*-steps Hamiltonian for all h.*

**Proof.** By Lemma 1.5, the graph Sub $(C_{10}(6), \{v_1, v_6\}, f(v_1, v_6) = 1)$ is isomorphic to the graph Sub $(C_7(3), \{v_1, v_3\}, f(v_1, v_3) = 4)$. Thus, Theorem 2.5 tells us that when $h = 1$, it is not 2-steps Hamiltonian.

When $h$ is odd, for a Sub $(C_{10}(6), \{v_1, v_6\}, f(v_1, v_6) = h)$ graph, the subgraph with vertices, A, C, E, M, G, I, in the figure 9 demonstrates its distance 2-graph is not Hamiltonian by Lemma 1.4. Note that the dotted

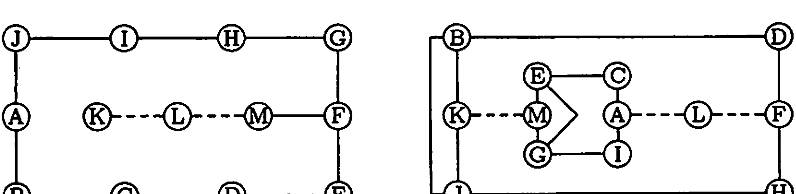


Figure 9: A Sub $(C_{10}(6), \{v_1, v_6\}, f(v_1, v_6) = h)$ graph when $h$ is odd and its distance 2-graph

line in the graph represents a path of even number order 2 vertices and the dotted line in its distance 2 graph represents a path of order 2 vertices. By Proposition 1.2, it is not 2-steps Hamiltonian.

When $h$ is even, we can group the vertices $v_1, v_3, v_5, v_7, v_9, w_2, w_4, \ldots, w_h$ in a set and the other vertices in another set. It is easy to see that it becomes a bipartite graph. By Proposition 1.3, it is not a 2-steps Hamiltonian graph.

This completes the proof. □

**Theorem 3.7.** *Sub* $(C_{11}(6), \{v_1, v_6\}, f(v_1, v_6) = h)$ *is 2-steps Hamiltonian if and only if* $h = 1$ *or* 2.

**Proof.** By Lemma 1.5, the graph Sub $(C_{11}(6), \{v_1, v_6\}, f(v_1, v_6) = h)$ is isomorphic to the graph Sub $(C_{h+6}(6), \{v_1, v_6\}, f(v_1, v_6) = 5)$. By Lemma 3.1, we only need to consider $h \leq 5$. Thus, we need to check

1. when $h = 1$, since Sub $(C_7(6), \{v_1, v_6\}, f(v_1, v_6) = 5)$ is isomorphic to Sub $(C_7(3), \{v_1, v_3\}, f(v_1, v_3) = 5)$. Thus, it is 2-steps Hamiltonian by Theorem 2.5;
2. when $h = 2$, since Sub $(C_8(6), \{v_1, v_6\}, f(v_1, v_6) = 5)$ is isomorphic to Sub $(C_8(4), \{v_1, v_4\}, f(v_1, v_4) = 5)$. Thus, it is 2-steps Hamiltonian by Theorem 2.8;
3. when $h = 3$, since Sub $(C_9(6), \{v_1, v_6\}, f(v_1, v_6) = 5)$ is isomorphic to Sub $(C_9(5), \{v_1, v_5\}, f(v_1, v_5) = 5)$. Thus, it is not 2-steps Hamiltonian by Theorem 3.4;
4. when $h = 4$, since Sub $(C_{10}(6), \{v_1, v_6\}, f(v_1, v_6) = 5)$ is not 2-steps Hamiltonian by Theorem 3.6;
5. when $h = 5$, the subgraph with vertices, A, M, O, F, H, J, in the figure 10 demonstrates its distance 2-graph is not Hamiltonian by Lemma 1.4. By Proposition 1.2, it is not 2-steps Hamiltonian.

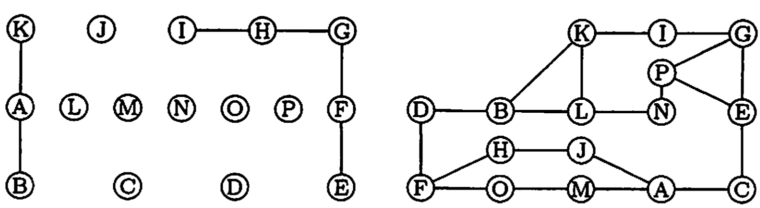


Figure 10: A Sub $(C_{11}(6), \{v_1, v_6\}, f(v_1, v_6) = 5)$ graph and its distance 2-graph

This completes the proof. □

**Theorem 3.8.** $\mathrm{Sub}(C_{12}(7), \{v_1, v_7\}, f(v_1, v_7) = h)$ *is not* 2*-steps Hamiltonian for all* $h$.

**Proof.** When $h$ is odd, we can group the vertices $v_1, v_3, v_5, v_7, v_9, v_{11}, w_2, w_4, \ldots, w_{h-1}$ in a set and the other vertices in another set. It is easy to see that it becomes a bipartite graph. By Proposition 1.3, it is not a 2-steps Hamiltonian graph.

When $h = 2$, since $\mathrm{Sub}(C_{12}(7), \{v_1, v_7\}, f(v_1, v_7) = 2)$ is isomorphic to $\mathrm{Sub}(C_9(4), \{v_1, v_4\}, f(v_1, v_4) = 5)$. Thus, it is not 2-steps Hamiltonian by Theorem 3.3.

When $h = 4$, since $\mathrm{Sub}(C_{12}(7), \{v_1, v_7\}, f(v_1, v_7) = 4)$ is isomorphic to $\mathrm{Sub}(C_{11}(6), \{v_1, v_6\}, f(v_1, v_6) = 5)$. Thus, it is not 2-steps Hamiltonian by Theorem 3.7.

This completes the proof. □

Finally, by Corollary 3.2, since we cover all the basic cases needed to classify whether a graph $\mathrm{Sub}(C_n(k), \{v_1, v_k\}, f(v_1, v_k) = h)$ is 2-steps Hamiltonian, we can summarize the results here:

**Theorem 3.9.** *The graph* $\mathrm{Sub}(C_n(k), \{v_1, v_k\}, f(v_1, v_k) = h)$ *is* 2*-steps Hamiltonian if and only if* $n$, $k$ *and* $h$ *satisfy one of the following conditions:*

1. $k = 3$, $n = 2t$ *where* $t \geq 2$ *and* $h$ *is even.*
2. $k = 3$, $n = 5$ *and* $h \geq 1$.
3. $k = 3$, $n = 2t + 1$ *where* $t \geq 3$ *and* $h$ *is odd or* 2.
4. $k = 4$, $n = 2t$ *where* $t \geq 3$ *and* $h$ *is odd.*
5. $k = 4$, $n = 2t + 1$ *where* $t \geq 3$ *and* $h$ *is even or* 1.
6. $k = t + 1$, $n = 2t + 1$ *where* $t \geq 4$ *and* $h = 1, 2$.

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