Decompositions of λK_n into LEO and ELO Graphs

Derek W. Hein and Dinesh G. Sarvate

ABSTRACT. The authors previously defined the Stanton–type graph $S(n, m)$ and showed how to decompose λK_n (for the appropriate minimal values of λ) into Stanton–type graphs $S(4, 3)$ of the LOE–, OLE–, LEO– and ELO–types. Sarvate and Zhang showed that for all possible values of λ, the necessary conditions are sufficient for LOE– and OLE–decompositions. In this paper, we show that for all possible values of λ, the necessary conditions are sufficient for LEO– and ELO–decompositions.

1. Introduction

A *simple graph* G is an ordered pair (V, E) where V is an n–set (of *points*), and E is a nonempty subset of the set of $\binom{n}{2}$ pairs of distinct elements of V (called *edges*). This can be generalized to a *multigraph* (without loops) by allowing E to be a multiset, where edges can occur with *frequencies* greater than or equal to 1. A complete multigraph λK_n (for $\lambda \geq 1$) is a graph on n points with λ edges between every pair of distinct points. A complete bipartite multigraph $\lambda K_{m,n}$ (for $\lambda \geq 1$) has λ copies of each edge in a complete bipartite graph $K_{m,n}$ (also denoted $K_{S,T}$ when $|S| = m$ and $|T| = n$.)

Decomposition of graphs into subgraphs is a well–known classical problem; for an excellent survey on graph decompositions, see [1]. Recently several people including Chan [4], El–Zanati, Lapchinda, Tangsupphathawat and Wannasit [5], Hein [6, 7], Sarvate, Winter [9, 10] and Zhang [11] have worked on decomposing λK_n into

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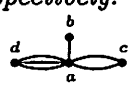
multigraphs. In fact, similar decompositions have been attempted earlier in various papers; see Priesler and Tarsi **[8]**. Ternary designs also provide such decompositions; see Billington **[2, 3]**.

The following definitions and examples are from **[7]**:

DEFINITION 1. *Let* $V = \{a, b, c, d\}$. *An* LEO graph $|a, b, c, d|$ *on* V *is a graph with 6 edges where the frequencies of edges* $\{a, b\}$, $\{b, c\}$ *and* $\{c, d\}$ *are 1, 3 and 2 respectively.*

DEFINITION 2. *Let* $V = \{a, b, c, d\}$. *An* ELO graph (a, b, c, d) *on* V *is a graph with 6 edges where the frequencies of edges* $\{a, b\}$, $\{a, c\}$ *and* $\{a, d\}$ *are 1, 2 and 3 respectively.*

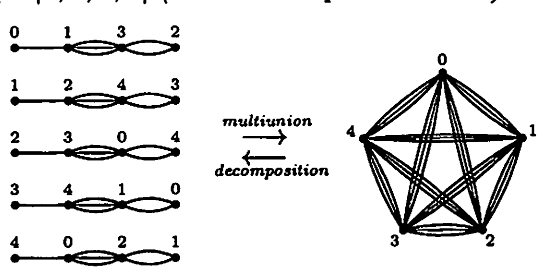


DEFINITION 3. *For any positive integers* $n \geq 4$ *and* $\lambda \geq 3$, *an* LEO–decomposition *of* λK_n *(denoted* $LEO(n, \lambda)$*) is a collection of LEO graphs such that the multiunion of their edge sets contains* λ *copies of all edges in a* K_n.

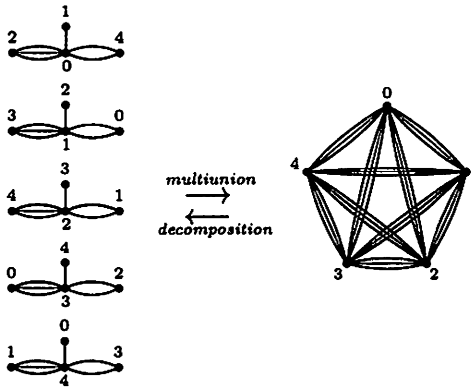
DEFINITION 4. *For any positive integers* $n \geq 4$ *and* $\lambda \geq 3$, *an* ELO–decomposition *of* λK_n *(denoted* $ELO(n, \lambda)$*) is a collection of ELO graphs such that the multiunion of their edge sets contains* λ *copies of all edges in a* K_n.

One of the powerful techniques to construct combinatorial designs is based on *difference sets* and *difference families*; see Stinson **[12]** for details. This technique is modified to achieve our decompositions of λK_n; in general, we exhibit the *base graphs*, which can be developed (modulo either n or $n - 1$) to obtain the decomposition.

EXAMPLE 1. *Considering the set of points to be* $V = \mathbb{Z}_5$, *the LEO base graph* $|0, 1, 3, 2|$ *(when developed modulo 5) constitutes an* $LEO(5, 3)$.

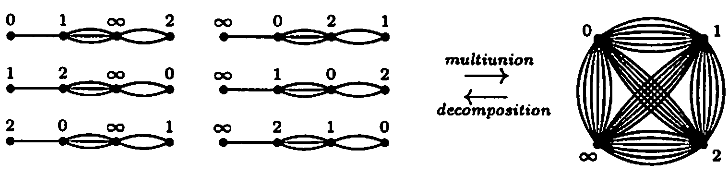


EXAMPLE 2. *Considering the set of points to be* $V = \mathbb{Z}_5$, *the ELO base graph* $(0,1,4,2)$ *(when developed modulo 5) constitutes an* $ELO(5,3)$.



We note that special attention is needed with the base graphs containing the "dummy element" ∞; the non–∞ elements are developed, while ∞ is simply rewritten each time.

EXAMPLE 3. *Considering the set of points to be* $V = \mathbb{Z}_3 \cup \{\infty\}$, *the LEO base graphs* $|0,1,\infty,2|$ *and* $|\infty,0,2,1|$ *(when developed modulo 3) constitute an* $LEO(4,6)$.



THEOREM 1.1. [7] *Let integers* $\lambda \geq 3$ *and* $n \geq 4$. *An* $LEO(n,\lambda)$ *and an* $ELO(n,\lambda)$ *exist for the minimum value of* λ, *which is*

a) $\lambda = 3$, *when* $n \equiv 0,1,4,5,8,9 \pmod{12}$,
b) $\lambda = 4$, *when* $n \equiv 3,6,7,10 \pmod{12}$ *and*
c) $\lambda = 6$, *when* $n \equiv 2,11 \pmod{12}$

with the exception of an $LEO(4,\lambda)$, *which has a minimum* λ *of* 6.

2. The Necessary Conditions

Since there are $\frac{\lambda n(n-1)}{2}$ edges in a λK_n, and 6 edges in an LEO or an ELO graph, we must have that $\lambda n(n-1) \equiv 0 \pmod{12}$ (where $\lambda \geq 3$ and $n \geq 4$). Specifically, we have

THEOREM 2.1. [11] *For either an* $LEO(n,\lambda)$ *or an* $ELO(n,\lambda)$, *the necessary conditions for* n *are:*

1) $n \equiv 0,1,4,9 \pmod{12}$ *when* $\lambda \equiv 1,5 \pmod 6$
2) $n \equiv 0,1 \pmod 3$ *when* $\lambda \equiv 2,4 \pmod 6$
3) $n \equiv 0,1 \pmod 4$ *when* $\lambda \equiv 3 \pmod 6$
4) *There is no condition for* n *when* $\lambda \equiv 0 \pmod 6$.

3. LEO–Decompositions

As a special case, we first consider LEO$(4,\lambda)$.

LEMMA 3.1. *In an* $LEO(4,\lambda)$ *we must have that* $\lambda \equiv 0 \pmod 3$.

PROOF. We let $V = \{v_1,\ldots,v_4\}$, $e = \{v_1,v_2\}$ and $f = \{v_3,v_4\}$. Suppose that an LEO$(4,\lambda)$ exists. We note each edge must occur λ times in this decomposition. If e occurs in the decomposition x times with multiplicity 2, then f occurs x times with multiplicity 1. If e occurs y times with multiplicity 1, then f occurs y times with multiplicity 2. Suppose that e occurs (without f) z_1 times with multiplicity 3, and that f occurs (without e) z_2 times with multiplicity 3. Therefore we must have a solution to $2x+y+3z_1 = \lambda$ and $x+2y+3z_2 = \lambda$ for some non–negative integers x, y, z_1 and z_2. Adding these equations, we have $3x+3y+3z_1+3z_2 = 2\lambda$, or $3(x+y+z_1+z_2) = 2\lambda$. This implies that 3 divides λ. Hence, we must have that $\lambda \equiv 0 \pmod 3$ when an LEO$(4,\lambda)$ exists. ■

NOTE 1. [7] *There does not exist an* $LEO(4,3)$.

PROOF. We let $V = \{v_1,\ldots,v_4\}$, $e = \{v_1,v_2\}$ and $f = \{v_3,v_4\}$. Suppose that an LEO$(4,3)$ exists. We note that we must have 3 LEO graphs in this decomposition. Let an edge (say e) occur in an LEO graph (say G_1) with frequency 1. Then, f must occur in G_1 with frequency 2. Thus, f has to occur in another LEO graph (say G_2) with frequency 1. Then, e will occur in G_2 with frequency 2. We see that graphs must come in pairs in this decomposition; that is, there must be an even number of graphs in this decomposition. However, 3 is not an even number. Hence, an LEO$(4,3)$ does not exist. ■

We recall that an LEO$(4,6)$ is given in Example 3.

EXAMPLE 4. *The LEO graphs* $|0,1,2,3|$, $|0,1,3,2|$, $|0,3,2,1|$, $|1,3,0,2|$, $|2,0,3,1|$, $|2,3,1,0|$, $|3,0,1,2|$, $|3,0,2,1|$ *and* $|3,2,0,1|$ *constitute an* $LEO(4,9)$ *with point set* $V = \{0,\ldots,3\}$. ▲

THEOREM 3.1. *An* $LEO(4,\lambda)$ *exists for all necessary* $\lambda \geq 6$.

PROOF. Let $\lambda \geq 6$. From Lemma 3.1 the necessary condition for an LEO$(4,\lambda)$ is that $\lambda \equiv 0 \pmod 3$. Let $\lambda = 3t$ for $t \geq 2$.

If $t \geq 2$ is even (that is, if $t = 2s$ for $s \geq 1$), then $\lambda = 6s$. By taking s copies of an LEO$(4,6)$, we have an LEO$(4,6s)$ in this case.

If $t \geq 3$ is odd (that is, if $t = 2s+1$ for $s \geq 1$), then $\lambda = 6s+3 = 6(s-1)+9$. By taking $s-1$ copies of an LEO$(4,6)$ if necessary, and adjoining an LEO$(4,9)$, we have an LEO$(4,6s+3)$ in this case. ■

We now consider LEO(n,λ) for $n \geq 5$. The following examples play important roles in the sequel:

EXAMPLE 5. *The LEO graphs* $|0,1,4,7|$, $|0,5,3,4|$, $|0,5,8,2|$, $|0,7,2,3|$, $|0,7,8,6|$, $|1,0,3,6|$, $|1,0,5,7|$, $|1,2,4,6|$, $|1,2,5,8|$, $|1,3,6,0|$, $|3,1,2,0|$, $|3,1,5,6|$, $|3,4,7,1|$, $|3,8,2,5|$, $|3,8,6,7|$, $|4,3,7,2|$, $|4,3,8,1|$, $|4,5,6,1|$, $|4,5,7,0|$, $|4,6,1,5|$, $|6,2,0,1|$, $|6,2,3,7|$, $|6,4,5,3|$, $|6,4,8,0|$, $|6,7,1,4|$, $|7,0,4,8|$, $|7,6,0,3|$, $|7,6,2,4|$, $|7,8,0,4|$ *and* $|7,8,1,3|$ *constitute an LEO*$(9,5)$ *with point set* $V = \{0,\ldots,8\}$. ▲

EXAMPLE 6. *The LEO graphs* $|0,10,4,13|$, $|0,10,6,2|$, $|1,11,5,14|$, $|1,11,7,3|$, $|2,12,6,0|$, $|2,12,8,4|$, $|3,13,7,1|$, $|3,13,9,5|$, $|4,14,8,2|$, $|4,14,10,6|$, $|10,0,9,3|$, $|10,5,1,12|$, $|10,5,14,8|$, $|11,1,10,4|$, $|11,6,0,9|$, $|11,6,2,13|$, $|12,2,11,5|$, $|12,7,1,10|$, $|12,7,3,14|$, $|13,3,12,6|$, $|13,8,2,11|$, $|13,8,4,0|$, $|14,4,13,7|$, $|14,9,3,12|$, $|14,9,5,1|$, $|\infty,0,8,1|$, $|\infty,0,11,7|$, $|\infty,0,12,9|$, $|\infty,0,13,11|$, $|\infty,0,14,13|$, $|\infty,1,0,14|$, $|\infty,1,9,2|$, $|\infty,1,12,8|$, $|\infty,1,13,10|$, $|\infty,1,14,12|$, $|\infty,2,0,13|$, $|\infty,2,1,0|$, $|\infty,2,10,3|$, $|\infty,2,13,9|$, $|\infty,2,14,11|$, $|\infty,3,0,12|$, $|\infty,3,1,14|$, $|\infty,3,2,1|$, $|\infty,3,11,4|$, $|\infty,3,14,10|$, $|\infty,4,0,11|$, $|\infty,4,1,13|$, $|\infty,4,2,0|$, $|\infty,4,3,2|$, $|\infty,4,12,5|$, $|\infty,5,0,10|$, $|\infty,5,2,14|$, $|\infty,5,3,1|$, $|\infty,5,4,3|$, $|\infty,5,13,6|$, $|\infty,6,1,11|$, $|\infty,6,3,0|$, $|\infty,6,4,2|$, $|\infty,6,5,4|$, $|\infty,6,14,7|$, $|\infty,7,0,8|$, $|\infty,7,2,12|$, $|\infty,7,4,1|$, $|\infty,7,5,3|$, $|\infty,7,6,5|$, $|\infty,8,1,9|$, $|\infty,8,3,13|$, $|\infty,8,5,2|$, $|\infty,8,6,4|$, $|\infty,8,7,6|$, $|\infty,9,2,10|$, $|\infty,9,4,14|$, $|\infty,9,6,3|$, $|\infty,9,7,5|$, $|\infty,9,8,7|$, $|\infty,10,3,11|$, $|\infty,10,5,0|$, $|\infty,10,7,4|$, $|\infty,10,8,6|$, $|\infty,10,9,8|$, $|\infty,11,4,12|$, $|\infty,11,6,1|$, $|\infty,11,8,5|$, $|\infty,11,9,7|$, $|\infty,11,10,9|$, $|\infty,12,5,13|$, $|\infty,12,7,2|$, $|\infty,12,9,6|$, $|\infty,12,10,8|$, $|\infty,12,11,10|$, $|\infty,13,6,14|$, $|\infty,13,8,3|$, $|\infty,13,10,7|$, $|\infty,13,11,9|$, $|\infty,13,12,11|$, $|\infty,14,7,0|$, $|\infty,14,9,4|$, $|\infty,14,11,8|$, $|\infty,14,12,10|$ *and* $|\infty,14,13,12|$ *constitute an LEO*$(16,5)$ *with point set* $V = \{0,\ldots,14,\infty\}$. ▲

We note that the multiplicity of the edges is fixed by position, as per the LEO graph. Also, $\lambda \geq 3$ and $n \geq 4$. We begin with small cases of λ. Recall from Theorem 1.1 that an LEO$(4,3)$, an LEO$(4,4)$ and an LEO$(4,5)$ do not exist.

LEMMA 3.2. *There exists an LEO$(n,3)$ for the necessary $n \geq 5$.*

PROOF. From case 3 of Theorem 2.1, the necessary condition is $n \equiv 0, 1 \pmod 4$. Such LEO$(n,3)$ are given in [7, Theorem 2]. ■

LEMMA 3.3. *There exists an LEO$(n,4)$ for the necessary $n \geq 5$.*

PROOF. From case 2 of Theorem 2.1, the necessary condition is $n \equiv 0, 1 \pmod 3$. We expand this to $n \equiv 0, 1, 3, 4, 6, 7, 9, 10 \pmod{12}$. Notice that the cases $n \equiv 3, 6, 7, 10 \pmod{12}$ have been done in [7]; the only cases that remain are $n \equiv 0, 1, 4, 9 \pmod{12}$.

Let $n = 12t$ (for $t \geq 1$). We consider the set V as $\mathbb{Z}_{12t-1} \cup \{\infty\}$. The number of graphs required for an LEO$(12t, 4)$ is $\frac{4(12t)(12t-1)}{12} = 4t(12t-1)$. Thus, we need $4t$ base graphs (modulo $12t-1$). The differences we must achieve (modulo $12t-1$) are $1, 2, \ldots, 6t-1$. We use the base graphs $|8t-1, 2t, 0, \infty|$, $|8t-1, 2t+1, 0, \infty|$, $|8t-1, 2t+2, 0, 1|$, $|8t-1, 2t+3, 0, 1|$, $|8t-1, 2t+4, 0, 2|$, $|8t-1, 2t+5, 0, 2|, \ldots, |8t-1, 6t-2, 0, 2t-1|$ and $|8t-1, 6t-1, 0, 2t-1|$. ▲

Let $n = 12t+1$ (for $t \geq 1$). We consider the set V as $\mathbb{Z}_{12t+1}$. The number of graphs required for an LEO$(12t+1, 4)$ is $\frac{4(12t+1)(12t)}{12} = 4t(12t+1)$. Thus, we need $4t$ base graphs (modulo $12t+1$). The differences we must achieve (modulo $12t+1$) are $1, 2, \ldots, 6t$. We use the base graphs $|8t+1, 2t+1, 0, 1|$, $|8t+1, 2t+2, 0, 1|$, $|8t+1, 2t+3, 0, 2|$, $|8t+1, 2t+4, 0, 2|, \ldots, |8t+1, 6t-1, 0, 2t|$ and $|8t+1, 6t, 0, 2t|$. ▲

Let $n = 12t+4$ (for $t \geq 1$). We consider the set V as $\mathbb{Z}_{12t+4}$. The number of graphs required for an LEO$(12t+4, 4)$ is $\frac{4(12t+4)(12t+3)}{12} = (4t+1)(12t+4)$. Thus, we need $4t+1$ base graphs (modulo $12t+4$). The differences we must achieve (modulo $12t+4$) are $1, 2, \ldots, 6t+2$. We use the base graphs $|8t+2, 2t+1, 0, 1|$, $|8t+2, 2t+2, 0, 1|$, $|8t+2, 2t+3, 0, 2|$, $|8t+2, 2t+4, 0, 2|, \ldots, |8t+2, 6t-1, 0, 2t|$, $|8t+2, 6t, 0, 2t|$ and $|8t+2, 6t+1, 0, 6t+2|$. ▲

Let $n = 12t+9$ (for $t \geq 0$). We consider the set V as $\mathbb{Z}_{12t+8} \cup \{\infty\}$. The number of graphs required for an LEO$(12t+9, 4)$ is $\frac{4(12t+9)(12t+8)}{12} = (4t+3)(12t+8)$. Thus, we need $4t+3$ base graphs (modulo $12t+8$). The differences we must achieve (modulo $12t+8$) are $1, 2, \ldots, 6t+4$. If $t = 0$ (so that $n = 9$), we use the base graphs $|2, 0, 3, \infty|$, $|3, 0, 2, \infty|$ and $|2, 1, 0, 4|$. If $t \geq 1$, we use the base graphs $|8t+1, 6t, 0, 1|$, $|8t+1, 6t-1, 0, 1|$, $|8t+1, 6t-2, 0, 2|$, $|8t+1, 6t-3, 0, 2|, \ldots, |8t+1, 2t+2, 0, 2t|$, $|8t+1, 2t+1, 0, 2t|$, $|6t+2, 0, 6t+3, \infty|$, $|6t+3, 0, 6t+2, \infty|$ and $|12t+2, 6t+1, 0, 6t+4|$. ■

OBSERVATION 1. *We see that the LEO graphs* $|a_2, b_1, a_1, b_2|$, $|b_2, a_2, b_1, a_1|$, $|b_2, a_1, b_3, a_2|$, $|b_1, a_2, b_2, a_1|$ *and* $|b_2, a_2, b_3, a_1|$ *constitute an LEO–decomposition of* $5K_{\{a_1,a_2\},\{b_1,b_2,b_3\}}$. ♦

Sarvate, Winter and Zhang **[9, 10]** have obtained several results on such multigraph decompositions of bipartite graphs.

LEMMA 3.4. *There exists an* $LEO(n, 5)$ *for the necessary* $n \geq 5$.

PROOF. From case 1 of Theorem 2.1, the necessary condition is $n \equiv 0, 1, 4, 9 \pmod{12}$.

Let $n = 12t$ (for $t \geq 1$). We consider the set V as $\mathbb{Z}_{12t-1} \cup \{\infty\}$. The number of graphs required for an LEO$(12t, 5)$ is $\frac{5(12t)(12t-1)}{12} = 5t(12t-1)$. Thus, we need $5t$ base graphs (modulo $12t-1$). The differences we must achieve (modulo $12t-1$) are $1, 2, \ldots, 6t-1$. When $t = 1$ (so that $n = 12$), we use the base graphs $|\infty, 0, 1, 6|$, $|\infty, 0, 2, 6|$, $|\infty, 0, 3, 6|$, $|\infty, 0, 4, 6|$ and $|\infty, 0, 5, 6|$. When $t = 2$ (so that $n = 24$), we use the base graphs $|\infty, 0, 1, 11|$, $|\infty, 0, 2, 11|$, $|\infty, 0, 3, 11|$, $|\infty, 0, 4, 11|$, $|\infty, 0, 5, 11|$, $|0, 11, 17, 22|$, $|0, 11, 18, 22|$, $|0, 11, 19, 22|$, $|0, 11, 20, 22|$ and $|0, 11, 21, 22|$. When $t \geq 3$, we use the base graphs $|\infty, 0, 1, 5t+1|$, $|\infty, 0, 2, 5t+1|$, $|\infty, 0, 3, 5t+1|$, $|\infty, 0, 4, 5t+1|$, $|\infty, 0, 5, 5t+1|$, $|6t-1, 0, 6, 5t+1|$, $|6t-1, 0, 7, 5t+1|$, $|6t-1, 0, 8, 5t+1|$, $|6t-1, 0, 9, 5t+1|$, $|6t-1, 0, 10, 5t+1|, \ldots, |5t+2, 0, 5t-9, 5t+1|$, $|5t+2, 0, 5t-8, 5t+1|$, $|5t+2, 0, 5t-7, 5t+1|$, $|5t+2, 0, 5t-6, 5t+1|$, $|5t+2, 0, 5t-5, 5t+1|$ as well as $|0, 5t+1, 10t-3, 10t+2|$, $|0, 5t+1, 10t-2, 10t+2|$, $|0, 5t+1, 10t-1, 10t+2|$, $|0, 5t+1, 10t, 10t+2|$ and $|0, 5t+1, 10t+1, 10t+2|$. ▲

Let $n = 12t+1$ (for $t \geq 1$). We consider the set V as $\mathbb{Z}_{12t+1}$. The number of graphs required for an LEO$(12t+1, 5)$ is $\frac{5(12t+1)(12t)}{12} = 5t(12t+1)$. Thus, we need $5t$ base graphs (modulo $12t+1$). The differences we must achieve (modulo $12t+1$) are $1, 2, \ldots, 6t$. When $t = 1$ (so that $n = 13$), we use the base graphs $|0, 6, 11, 12|$, $|0, 6, 10, 12|$, $|0, 6, 9, 12|$, $|0, 6, 8, 12|$ and $|0, 6, 7, 12|$. When $t \geq 2$, we use the base graphs $|0, 5t+1, 10t+1, 10t+2|$, $|0, 5t+1, 10t, 10t+2|$, $|0, 5t+1, 10t-1, 10t+2|$, $|0, 5t+1, 10t-2, 10t+2|$, $|0, 5t+1, 10t-3, 10t+2|$, $|5t+2, 0, 5t-5, 5t+1|$, $|5t+2, 0, 5t-6, 5t+1|$, $|5t+2, 0, 5t-7, 5t+1|$, $|5t+2, 0, 5t-8, 5t+1|$, $|5t+2, 0, 5t-9, 5t+1|, \ldots, |6t, 0, 5, 5t+1|$, $|6t, 0, 4, 5t+1|$, $|6t, 0, 3, 5t+1|$, $|6t, 0, 2, 5t+1|$ and $|6t, 0, 1, 5t+1|$. ▲

Let $n = 12t+4 = 12(t-1)+16$ (for $t \geq 1$). We consider the set V as $\{a_1, a_2, \ldots, a_{16}, b_1, b_2, \ldots, b_{12(t-1)}\}$. To obtain an LEO$(16+12(t-1), 5)$, we use an LEO$(16, 5)$ on $\{a_1, a_2, \ldots, a_{16}\}$ (given in Example 6), an LEO$(12(t-1), 5)$ on $\{b_1, b_2, \ldots, b_{12(t-1)}\}$ (given two cases above) if necessary, and an LEO–decomposition of $5K_{\{a_{2i-1},a_{2i}\},\{b_{3j+1},b_{3j+2},b_{3j+3}\}}$ for all $i = 1, 2, \ldots, 8$ and for all $j = 0, 1, \ldots, 4t-1$ (given in Observation 1) if necessary. ▲

Let $n = 12t + 9$ (for $t \geq 0$). We consider the set V as $\{a_1, a_2, \ldots, a_{12t}, b_1, b_2, \ldots, b_9\}$. To obtain an LEO$(12t+9, 5)$, we use an LEO$(12t, 5)$ on $\{a_1, a_2, \ldots, a_{12t}\}$ (given three cases above) if necessary, an LEO$(9, 5)$ on $\{b_1, b_2, \ldots, b_9\}$ (given in Example 5), and an LEO–decomposition of $5K_{\{a_{2i-1},a_{2i}\},\{b_{3j+1},b_{3j+2},b_{3j+3}\}}$ for all $i = 1, 2, \ldots, 6t$ and for all $j = 0, 1, 2$ (given in Observation 1) if necessary. ■

LEMMA 3.5. *There exists an LEO$(n, 6)$ for any $n \geq 5$.*

PROOF. We consider cases when $n \geq 5$ is odd or even.

Let $n = 2t + 1$ (for $t \geq 2$). We consider the set V as $\mathbb{Z}_{2t+1}$. The number of graphs required for an LEO$(2t + 1, 6)$ is $\frac{6(2t+1)(2t)}{12} = t(2t+1)$. Thus, we need t base graphs (modulo $2t+1$). The differences we must achieve (modulo $2t + 1$) are $1, 2, \ldots, t$. We use the base graphs $|0, 1, t+1, t|$, $|0, 2, t+1, 1|$, $|0, 3, t+1, 2|, \ldots, |0, t, t+1, t-1|$. ▲

Let $n = 2t$ (for $t \geq 3$). We consider the set V as $\mathbb{Z}_{2t-1} \cup \{\infty\}$. The number of graphs required for an LEO$(2t, 6)$ is $\frac{6(2t)(2t-1)}{12} = t(2t-1)$. Thus, we need t base graphs (modulo $2t - 1$). The differences we must achieve (modulo $2t - 1$) are $1, 2, \ldots, t-1$. If $t = 3$ (that is, if $n = 6$), we use the base graphs $|0, 2, \infty, 1|$, $|\infty, 2, 0, 3|$ and $|0, 4, 3, 2|$. If $t \geq 4$, we use the base graphs $|0, t-1, \infty, 1|$, $|\infty, 0, t-1, 2t-2|$, $|0, 1, t-1, t-2|$, $|0, 2, t-1, 1|$, $|0, 3, t-1, 2|, \ldots, |0, t-2, t-1, t-3|$. ■

THEOREM 3.2. *Let $\lambda \geq 3$ and $n \geq 4$. An LEO(n, λ) exists for all λ and necessary n.*

PROOF. We recall from Theorem 1.1 that there do not exist LEO$(4, 3)$, LEO$(4, 4)$ or LEO$(4, 5)$. We proceed by cases on λ (mod 6).

For $\lambda \equiv 0 \pmod 6$ (so that $\lambda = 6t$ for $t \geq 1$), by taking t copies of an LEO$(n, 6)$ (given in Lemma 3.5), we have an LEO$(n, 6t)$. ▲

For $\lambda \equiv 1 \pmod 6$ (so that $\lambda = 6t+1 = 6(t-1)+7 = 6(t-1)+(3+4)$ for $t \geq 1$), we first take an LEO$(n, 3)$ (given in Lemma 3.2) and an LEO$(n, 4)$ (given in Lemma 3.3). (This gives us $\lambda = 7$ thus far.) We then adjoin this to $t - 1$ copies of an LEO$(n, 6)$ (given in Lemma 3.5) if necessary. Hence, we have an LEO$(n, 6t + 1)$. ▲

For $\lambda \equiv 2 \pmod 6$ (so that $\lambda = 6t + 2 = 6(t - 1) + 8$ for $t \geq 1$), we first take two copies of an LEO$(n, 4)$ (given in Lemma 3.3). (This gives us $\lambda = 8$ thus far.) We then adjoin this to $t - 1$ copies of an LEO$(n, 6)$ (given in Lemma 3.5). Hence, we have an LEO$(n, 6t+2)$. ▲

For $\lambda \equiv 3 \pmod 6$ (so that $\lambda = 6t + 3 = 3(2t + 1)$ for $t \geq 0$), by taking $2t + 1$ copies of an LEO$(n, 3)$ (given in Lemma 3.2) we have an LEO$(n, 6t + 3)$. ▲

For $\lambda \equiv 4 \pmod 6$ (so that $\lambda = 6t+4$ for $t \geq 0$), we first take an LEO$(n, 4)$ (given in Lemma 3.3). (This gives us $\lambda = 4$ thus far.) We then adjoin this to t copies of an LEO$(n, 6)$ (given in Lemma 3.5) if necessary. Hence, we have an LEO$(n, 6t + 4)$. ▲

For $\lambda \equiv 5 \pmod 6$ (so that $\lambda = 6t+5$ for $t \geq 0$), we first take an LEO$(n, 5)$ (given in Lemma 3.4). (This gives us $\lambda = 5$ thus far.) We then adjoin this to t copies of an LEO$(n, 6)$ (given in Lemma 3.5) if necessary. Hence, we have an LEO$(n, 6t + 5)$. ■

4. ELO–Decompositions

The following examples play important roles in the sequel:

EXAMPLE 7. *The set of ELO graphs* (0, 8, 1, 4), (0, 8, 2, 3), (0, 8, 3, 2), (0, 8, 4, 1), (1, 6, 2, 5), (1, 6, 3, 4), (1, 6, 4, 3), (1, 6, 5, 2), (2, 4, 6, 3), (2, 5, 3, 6), (3, 8, 4, 7), (3, 8, 5, 6), (3, 8, 6, 5), (3, 8, 7, 4), (4, 2, 5, 8), (4, 2, 6, 7), (4, 2, 7, 6), (4, 2, 8, 5), (5, 2, 0, 6), (5, 2, 6, 0), (5, 2, 7, 8), (5, 2, 8, 7), (6, 1, 8, 0), (6, 7, 0, 8), (7, 6, 0, 1), (7, 6, 1, 0), (7, 6, 2, 8), (7, 6, 8, 2), (8, 0, 2, 1) *and* (8, 3, 1, 2) *constitute an ELO*$(9, 5)$ *on* $V = \{0, \ldots, 8\}$. ▲

EXAMPLE 8. *The set of ELO graphs* $(0, 7, 3, 4)$, $(0, \infty, 1, 6)$, $(0, \infty, 2, 5)$, $(0, \infty, 4, 3)$, $(0, \infty, 5, 2)$, $(0, \infty, 6, 1)$, $(1, 2, 3, 6)$, $(1, 2, 4, 5)$, $(1, 2, 5, 4)$, $(1, 2, 6, 3)$, $(1, 2, 8, \infty)$, $(1, 7, \infty, 8)$, $(2, 9, 5, 6)$, $(2, \infty, 3, 8)$, $(2, \infty, 4, 7)$, $(2, \infty, 6, 5)$, $(2, \infty, 7, 4)$, $(2, \infty, 8, 3)$, $(3, 4, 5, 8)$, $(3, 4, 6, 7)$, $(3, 4, 7, 6)$, $(3, 4, 8, 5)$, $(3, 4, \infty, 10)$, $(3, 9, 10, \infty)$, $(4, 11, 7, 8)$, $(4, \infty, 5, 10)$, $(4, \infty, 6, 9)$, $(4, \infty, 8, 7)$, $(4, \infty, 9, 6)$, $(4, \infty, 10, 5)$, $(5, 6, 7, 10)$, $(5, 6, 8, 9)$, $(5, 6, 9, 8)$, $(5, 6, 10, 7)$, $(5, 6, \infty, 12)$, $(5, 11, 12, \infty)$, $(6, 13, 9, 10)$, $(6, \infty, 7, 12)$, $(6, \infty, 8, 11)$, $(6, \infty, 10, 9)$, $(6, \infty, 11, 8)$, $(6, \infty, 12, 7)$, $(7, 0, 8, 13)$, $(7, 0, 13, 8)$, $(7, 0, 14, \infty)$, $(7, 0, \infty, 14)$, $(7, 1, 9, 12)$, $(7, 1, 10, 11)$, $(7, 1, 11, 10)$, $(7, 1, 12, 9)$, $(8, 10, 0, \infty)$, $(8, 10, 9, 14)$, $(8, 10, 12, 11)$, $(8, 10, 14, 9)$, $(8, 10, \infty, 0)$, $(8, 13, 11, 12)$, $(9, 2, 0, 10)$, $(9, 2, 1, \infty)$, $(9, 2, 10, 0)$, $(9, 2, \infty, 1)$, $(9, 3, 11, 14)$, $(9, 3, 12, 13)$, $(9, 3, 13, 12)$, $(9, 3, 14, 11)$, $(10, 13, 0, 12)$, $(10, 13, 1, 11)$, $(10, 13, 2, \infty)$, $(10, 13, 11, 1)$, $(10, 13, \infty, 2)$, $(10, 14, 12, 0)$, $(11, 4, 2, 12)$, $(11, 4, 3, \infty)$, $(11, 4, 12, 2)$, $(11, 4, \infty, 3)$, $(11, 5, 0, 14)$, $(11, 5, 1, 13)$, $(11, 5, 13, 1)$, $(11, 5, 14, 0)$, $(12, 2, 1, 0)$, $(12, 2, 3, 13)$, $(12, 2, 4, \infty)$, $(12, 2, 13, 3)$, $(12, 2, \infty, 4)$, $(12, 14, 0, 1)$, $(13, 6, 0, 3)$, $(13, 6, 1, 2)$, $(13, 6, 2, 1)$, $(13, 6, 3, 0)$, $(13, 8, 4, 14)$, $(13, 8, 5, \infty)$, $(13, 8, 14, 4)$, $(13, 8, \infty, 5)$, $(14, 10, 0, 5)$, $(14, 10, 5, 0)$, $(14, 10, 6,$

∞), (14, 10, ∞, 6), (14, 12, 1, 4), (14, 12, 2, 3), (14, 12, 3, 2) *and* (14, 12, 4, 1) *constitute an* $ELO(16,5)$ *on* $V = \{0, \ldots, 14, \infty\}$. ▲

We note that the multiplicity of the edges is fixed by position, as per the ELO graph. Also, $\lambda \geq 3$ and $n \geq 4$. We begin with small cases of λ.

LEMMA 4.1. *There exists an* $ELO(n,3)$ *for the necessary* $n \geq 4$.

PROOF. From case 3 of Theorem 2.1, the necessary condition is $n \equiv 0, 1 \pmod 4$. Such ELO$(n, 3)$ are given in [**7**, Theorem 3]. ■

LEMMA 4.2. *There exists an* $ELO(n,4)$ *for the necessary* $n \geq 4$.

PROOF. From case 2 of Theorem 2.1, the necessary condition is $n \equiv 0, 1 \pmod 3$.

Let $n = 3t$ (for $t \geq 2$). We consider the set V as $\mathbb{Z}_{3t-1} \cup \{\infty\}$. The number of graphs required for an ELO$(3t, 4)$ is $\frac{4(3t)(3t-1)}{12} = t(3t-1)$. Thus, we need t base graphs (modulo $3t-1$). We consider cases when t is even or odd. If $t = 2s$ (that is, if $n = 6s$ for $s \geq 1$), then we need $2s$ base graphs (modulo $6s-1$). The differences we must achieve (modulo $6s-1$) are $1, 2, \ldots, 3s-1$. We use the base graphs $(0, 3s-2, 3s-1, \infty)$, $(0, \infty, 3s-1, 3s-2)$ and the pairs $(0, 3x-2, 3x, 3x-1)$, $(0, 3x-1, 3x, 3x-2)$ for $x = 1, \ldots, s-1$ if necessary. If $t = 2s+1$ (that is, if $n = 6s+3$ for $s \geq 1$), then we need $2s+1$ base graphs (modulo $6s+2$). The differences we must achieve (modulo $6s+2$) are $1, 2, \ldots, 3s+1$. We use the base graphs $(0, 3s-2, 3s-1, \infty)$, $(0, \infty, 3s-1, 3s-2)$, $(0, 3s, 3s+1, 3s+2)$ and the pairs $(0, 3x-2, 3x, 3x-1)$ and $(0, 3x-1, 3x, 3x-2)$ for $x = 1, \ldots, s-1$. ▲

Let $n = 3t+1$ (for $t \geq 1$). We consider the set V as $\mathbb{Z}_{3t+1}$. The number of graphs required for an ELO$(3t+1, 4)$ is $\frac{4(3t+1)(3t)}{12} = t(3t+1)$. Thus, we need t base graphs (modulo $3t+1$). We consider cases when t is even or odd. If $t = 2s$ (that is, if $n = 6s+1$), then we need $2s$ base graphs (modulo $6s+1$). The differences we must achieve (modulo $6s+1$) are $1, 2, \ldots, 3s$. We use the base graphs $(0, 3x-2, 3x, 3x-1)$ and $(0, 3x-1, 3x, 3x-2)$ for $x = 1, \ldots, s$. If $t = 2s+1$ (that is, if $n = 6s+4$), then we need $2s+1$ base graphs (modulo $6s+4$). The differences we must achieve (modulo $6s+4$) are $1, 2, \ldots, 3s+2$. We use the base graphs $(0, 3x-2, 3x, 3x-1)$ and $(0, 3x-1, 3x, 3x-2)$ for $x = 1, \ldots, s$ and $(0, 3s+1, 3s+2, 3s+3)$. ■

OBSERVATION 2. *We see that the ELO graphs* (a, b_6, b_1, b_2), (a, b_6, b_2, b_3), (a, b_6, b_3, b_4), (a, b_6, b_4, b_5) *and* (a, b_6, b_5, b_1) *constitute an ELO–decomposition of* $5K_{\{a\},\{b_1,b_2,\dots,b_6\}}$. ♦

LEMMA 4.3. *There exists an ELO*$(n, 5)$ *for the necessary* $n \geq 4$.

PROOF. From case 1 of Theorem 2.1, the necessary condition is $n \equiv 0, 1, 4, 9 \pmod{12}$.

Let $n = 12t$ (for $t \geq 1$). We consider the set V as $\mathbb{Z}_{12t-1} \cup \{\infty\}$. The number of graphs required for an ELO$(12t, 5)$ is $\frac{5(12t)(12t-1)}{12} = 5t(12t-1)$. Thus, we need $5t$ base graphs (modulo $12t-1$). The differences we must achieve (modulo $12t-1$) are $1, 2, \dots, 6t-1$. When $t = 1$, we use the base graphs $(0, \infty, 1, 2)$, $(0, \infty, 2, 3)$, $(0, \infty, 3, 4)$, $(0, \infty, 4, 5)$ and $(0, \infty, 5, 1)$. When $t \geq 2$, we additionally use the base graphs $(0, 6x, 6x{+}1, 6x{+}2)$, $(0, 6x, 6x{+}2, 6x{+}3)$, $(0, 6x, 6x{+}3, 6x{+}4)$, $(0, 6x, 6x+4, 6x+5)$ and $(0, 6x, 6x+5, 6x+1)$ for $x = 1, \dots, t-1$. ▲

Let $n = 12t+1$ (for $t \geq 1$). We consider the set V as $\mathbb{Z}_{12t+1}$. The number of graphs required for an ELO$(12t+1, 5)$ is $\frac{5(12t+1)(12t)}{12} = 5t(12t+1)$. Thus, we need $5t$ base graphs (modulo $12t+1$). The differences we must achieve (modulo $12t+1$) are $1, 2, \dots, 6t$. We use the base graphs $(0, 6x+1, 6x+2, 6x+3)$, $(0, 6x+1, 6x+3, 6x+4)$, $(0, 6x+1, 6x+4, 6x+5)$, $(0, 6x+1, 6x+5, 6x+6)$ and $(0, 6x+1, 6x+6, 6x+2)$, for $x = 0, \dots, t-1$. ▲

Let $n = 12t+4$ (for $t \geq 0$). If $t = 0$ (so that $n = 4$), we use the ELO graphs $(0, 1, 2, 3)$, $(1, 0, 2, 3)$, $(1, 3, 2, 0)$, $(2, 1, 3, 0)$ and $(3, 1, 0, 2)$. If $t \geq 1$, we let $n = 12(t-1) + 16$. We consider the set V as $\{a_1, \dots, a_{16}, b_1, \dots b_{12(t-1)}\}$. To obtain an ELO$(16 + 12(t-1), 5)$, we use an ELO$(16, 5)$ on $\{a_1, a_2, \dots, a_{16}\}$ (given in Example 8), an ELO$(12(t-1), 5)$ on $\{b_1, b_2, \dots, b_{12(t-1)}\}$ (given two cases above) if necessary, and an ELO–decomposition of $5K_{\{a_i\},\{b_{6j+1},b_{6j+2},b_{6j+3},b_{6j+4},b_{6j+5},b_{6j+6}\}}$ for all $i = 1, 2, \dots, 16$ and for all $j = 0, 1, \dots, 2t-3$ (given in Observation 2) if necessary. ▲

Let $n = 12t+9$ (for $t \geq 0$). We consider the set V as $\{a_1, \dots, a_9, b_1, \dots, b_{12t}\}$. To obtain an ELO$(9 + 12t, 5)$, we use an ELO$(9, 5)$ on $\{a_1, a_2, \dots, a_9\}$ (given in Example 7), an ELO$(12t, 5)$ on $\{b_1, b_2, \dots, b_{12t}\}$ (given three cases above) if necessary, and an ELO–decomposition of $5K_{\{a_i\},\{b_{6j+1},b_{6j+2},b_{6j+3},b_{6j+4},b_{6j+5},b_{6j+6}\}}$ for all $i = 1, \dots, 9$ and for all $j = 0, 1, 2, \dots, 2t-1$ (given in Observation 2) if necessary. ■

LEMMA 4.4. *There exists an ELO*$(n, 6)$ *for any* $n \geq 4$.

PROOF. We consider cases when n is odd or even.

Let $n = 2t+1$ (for $t \geq 2$). We consider the set V as $\mathbb{Z}_{2t+1}$. The number of graphs required for an ELO$(2t+1,6)$ is $\frac{6(2t+1)(2t)}{12} = t(2t+1)$. Thus, we need t base graphs (modulo $2t+1$). The differences we must achieve (modulo $2t+1$) are $1, 2, \ldots, t$. If $t = 2$ (so that $n = 5$), we use the base graphs $(0,1,2,4)$ and $(0,3,1,2)$. If $t \geq 3$, we use the base graphs $(0, t-1, t, 1)$, $(0, t, 1, 2)$ and $(0, s, s+1, s+2)$ for $s = 1, \ldots, t-2$. ▲

Let $n = 2t$ (for $t \geq 2$). We consider the set V as $\mathbb{Z}_{2t-1} \cup \{\infty\}$. The number of graphs required for an ELO$(2t,6)$ is $\frac{6(2t)(2t-1)}{12} = t(2t-1)$. Thus, we need t base graphs (modulo $2t-1$). The differences we must achieve (modulo $2t-1$) are $1, 2, \ldots, t-1$. If $t = 2$ (that is, if $n = 4$), we use two copies of the base graph $(0,1,2,\infty)$. If $t = 3$ (that is, if $n = 6$), we use the base graphs $(0,2,1,\infty)$, $(0,\infty,2,1)$ and $(0,1,\infty,2)$. If $t = 4$ (that is, if $n = 8$), we use the base graphs $(0,3,2,\infty)$, $(0,\infty,3,1)$, $(0,1,\infty,2)$ and $(0,2,1,3)$. If $t = 5$ (that is, if $n = 10$), we use the base graphs $(0,1,4,\infty)$, $(0,\infty,3,4)$, $(0,2,\infty,3)$, $(0,3,1,2)$ and $(0,4,2,1)$. If $t \geq 6$, we use the base graphs $(0, t-1, t-2, \infty)$, $(0, \infty, t-1, t-2)$, $(0, t-2, \infty, t-1)$, $(0, t-4, t-3, 1)$, $(0, t-3, 1, 2)$ and $(0, s, s+1, s+2)$ for $s = 1, \ldots, t-5$. ■

THEOREM 4.1. *Let $\lambda \geq 3$ and $n \geq 4$. An ELO(n,λ) exists for all λ and necessary n.*

PROOF. We proceed by cases on λ (mod 6).

For $\lambda \equiv 0$ (mod 6) (so that $\lambda = 6t$ for $t \geq 1$), by taking t copies of an ELO$(n,6)$ (given in Lemma 4.4), we have an ELO$(n,6t)$. ▲

For $\lambda \equiv 1$ (mod 6) (so that $\lambda = 6t+1 = 6(t-1)+7 = 6(t-1)+(3+4)$ for $t \geq 1$), we first take an ELO$(n,3)$ (given in Lemma 4.1) and an ELO$(n,4)$ (given in Lemma 4.2). (This gives us $\lambda = 7$ thus far.) We then adjoin this to $t-1$ copies of an ELO$(n,6)$ (given in Lemma 4.4) if necessary. Hence, we have an ELO$(n,6t+1)$. ▲

For $\lambda \equiv 2$ (mod 6) (so that $\lambda = 6t+2 = 6(t-1)+8$ for $t \geq 1$), we first take two copies of an ELO$(n,4)$ (given in Lemma 4.2). (This gives us $\lambda = 8$ thus far.) We then adjoin this to $t-1$ copies of an ELO$(n,6)$ (given in Lemma 4.4) if necessary. Hence, we have an ELO$(n,6t+2)$. ▲

For $\lambda \equiv 3$ (mod 6) (so that $\lambda = 6t+3 = 3(2t+1)$ for $t \geq 0$), by taking $2t+1$ copies of an ELO$(n,3)$ (given in Lemma 4.1) we have an ELO$(n,6t+3)$. ▲

For $\lambda \equiv 4$ (mod 6) (so that $\lambda = 6t+4$ for $t \geq 0$), we first take an ELO$(n,4)$ (given in Lemma 4.2). (This gives us $\lambda = 4$ thus far.) We then adjoin this to t copies of an ELO$(n, 6)$ (given in Lemma 4.4) if necessary. Hence, we have an ELO$(n, 6t + 4)$. ▲

For $\lambda \equiv 5 \pmod 6$ (so that $\lambda = 6t+5$ for $t \geq 0$), we first take an ELO$(n, 5)$ (given in Lemma 4.3). (This gives us $\lambda = 5$ thus far.) We then adjoin this to t copies of an ELO$(n, 6)$ (given in Lemma 4.4) if necessary. Hence, we have an ELO$(n, 6t + 5)$. ■

5. Summary

In this paper, we addressed the necessary conditions for the existence of LEO– and ELO–decompositions in general (that is, for all possible values λ). In all cases, we proved that the necessary conditions established in Section 2 are sufficient for these decompositions.

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