Mortal and Eternal Vertex Covers

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Abstract

A *vertex cover* of a graph $G = (V, E)$ is a subset $S \subseteq V$ such that every edge is incident with at least one vertex in S, and $\alpha(G)$ is the cardinality of a smallest vertex cover. For a given vertex cover S, a defense by S to an attack on an edge $e = \{v, w\}$, where $v \in S$, is a one-to-one function $f : S \to V$, such that (1) $f(v) = w$, and (2) for each $s \in S - v$, $f(s) \in N[s]$. Informally, a set is an *eternal vertex cover* if it can defend an "attack" on any edge and the process can be repeated indefinitely. The cardinality of a smallest eternal vertex cover is denoted $\alpha_m^\infty(G)$. A set of vertices which is not an eternal vertex cover is *mortal*. A formal definition of eternal vertex cover is provided and demonstrated to be equivalent to a characterization using closed families of vertex covers. Eternal vertex covers are shown to be closed under taking supersets and a lower bound for $\alpha_m^\infty(G)$ is given which depends on the vertex connectivity number and the independent domination number. A corresponding upper bound is given for the size of a mortal set. The *death spiral number* of a mortal vertex cover is defined and used to partition the collection of all mortal sets. Mortal sets are shown to be closed under taking subsets implying the collection of mortal sets for a graph with at least one edge is an independence system. The death spiral number of a graph is the maximum of the death spiral numbers of all mortal sets. An optimal attack/defense strategy is determined for a set of size $\alpha_m^\infty(T) - 1$ in a tree T, along with a polynomial labeling algorithm which computes its death spiral number.

Keywords: vertex cover; eternal protection; mortal vertex cover; tree covers; independence system; death spiral number

1 Introduction

A relatively recent area of research involves relating an "eternal" concept to a standard graphical invariant. Goddard, Hedetniemi, and Hedetniemi [6] introduced this idea by applying it to domination. If $G = (V, E)$ is a

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graph, a subset $S \subseteq V$ is a *dominating set*, not necessarily minimum, when every vertex not in S is adjacent to a vertex in S. The set S can be thought of as a collection of "guards," where each guard protects its own vertex as well as the vertices adjacent to it. An "attack" is the selection of a vertex w in $V - S$ and a "defense" is the shift of a guard from a neighbor v of w to w. Informally, S is an *eternally secure set* if the new configuration of guards $(S - \{v\}) \cup \{w\}$ is a dominating set and this process can be repeated without ever producing a non-dominating set, regardless of the attack strategy. The smallest number of guards required in an eternally secure set is the *eternal security number.* Work on eternally secure sets can be found in [1, 3, 11, 12].

This concept has been extended in three ways: (1) Allowing more than a single guard to shift in each defense, (2) considering graphical invariants other than domination, and (3) protecting edges from attack as opposed to vertices. Such extensions have been studied in [5, 7, 8, 9, 10, 13, 14, 15, 16]. This paper deals with *eternal vertex covers* as described in [2, 5, 10, 13, 16].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We assume $E(G) \neq \emptyset$ and none of the edges of G are loops. Given $S \subseteq V(G)$, an edge $e \in E(G)$ is *covered by S* (or simply *covered* if S is understood) if e is incident with a vertex in S. The edge e is *uncovered* if e is not covered. The set S is a *vertex cover* of G if every edge of G is covered by S. The minimum size of a vertex cover is the *vertex cover number* of G, denoted $\alpha(G)$.

In the context of vertex covers, an *attack* occurs on a single edge and a *defense* by S to the attack on the edge $e = \{v, w\}$, where $v \in S$, is a one-to-one function $f : S \to V$, such that (1) $f(v) = w$, and (2) for each $s \in S - v$, $f(s) \in N[s]$. Informally, we say that "the guard on s shifts to $f(s)$". More generally, if $\{v = x_1, x_2, \ldots, x_{t+1} = w\} \subseteq S$ and $f(x_i) = x_{i+1}$ for $1 \leq i \leq t$ we say "f shifts guards along a path from v to w". An *attack strategy* is a function $A : \mathcal{P}(V(G)) \to E(G)$ where $\mathcal{P}(V(G))$ is the power set of $V(G)$. Informally, a set S is an *eternal vertex cover* if there is no attack strategy that can force an uncovered edge in some finite number of attacks.

In Section 2, we will provide a formal definition of eternal vertex cover, show its equivalence to a simple characterization, and use that characterization to prove some basic results.

The *eternal vertex cover number* of G, denoted $\alpha_m^{\infty}(G)$, is the smallest cardinality of an eternal vertex cover. This number is well-defined since for any graph $V(G)$ is an eternal vertex cover and, under our assumption that G is loopless, any set with $|V(G)| - 1$ vertices is an eternal vertex cover. For any tree T, let $I(T)$ be the set of non-leaf vertices of T. Klostermeyer and Mynhardt [13] show the following:

Theorem 1 *For any tree* T, $\alpha_m^\infty(T) = |I(T)| + 1$.

If a guard set is not an eternal vertex cover, we say it is *mortal*. A similar concept for domination is discussed by Burger, et. al. [4]. For mortal guard sets, there is an attack strategy that will force an uncovered edge after a finite number of attacks. For a mortal guard set S of size k the minimum number of attacks needed to uncover an edge of G against an optimal defense by S is the *death spiral number of* S denoted $ds(G, S)$. The maximum of the death spiral numbers taken over all mortal sets of size k is called the k^{th} *death spiral number of* G, denoted $ds^k(G)$, i.e. $ds^k(G) = \max\{ds(G, S) : S \text{ is mortal and } |S| = k\}$.

In Section 3, we make these notions precise by partitioning the mortal sets and present some general results. In Section 4 we provide a complete analysis of death spirals and death spiral numbers for trees T when $|S| = |I(T)|$. We will show that a best initial configuration for the defender is $S = I(T)$, and from then on an optimal attack strategy paired with an optimal defense strategy will cause every resultant vertex cover to contain exactly one leaf. In addition, we provide a polynomial labeling algorithm which computes $ds^k(T)$ for any tree.

2 General Graphs - Eternal Vertex Covers

The study of mortal and eternal vertex covers of arbitrary graphs appears to be difficult. In this section we introduce basic facts related to such a study. We assume G is an arbitrary connected graph with $n = |V(G)|$.

Let $VC(G)$ be the collection of all vertex covers of G. A *defense strategy* is a function $D : VC(G) \times E(G) \to \mathcal{P}(V(G))$ such that $D(S, e) = f(S)$ for some defense f by S to an attack on e. Given $S \in VC(G)$, a defense strategy D, and an attack strategy A, we recursively define a sequence of sets by $S^0_{D,A} = S$ and $S^{i+1}_{D,A} = D(S^i_{D,A}, A(S^i_{D,A}))$ for all $i \geq 0$ such that $S^i_{D,A} \in VC(G)$. We are now able to provide a precise definition for eternal vertex cover.

Definition 2 *A set* S *is an* **eternal vertex cover** *if (1)* S *is a vertex cover and (2) there exists a defense strategy* D *such that for any attack strategy* A, $S^i_{D,A} \in VC(G)$ *for all* $i \geq 0$.

This definition formalizes the common definitions found in the literature. The introduction of the S^i's provides a basis for sound inductive arguments. However, an alternative simpler characterization exists based on the following definition.

Definition 3 *A collection of vertex covers, $\mathcal{T}$, is said to be a* **closed family of vertex covers** *if there exists a defense strategy, D, such that $D(S,e) \in \mathcal{T}$ for all $S \in \mathcal{T}$ and for all $e \in E(G)$.*

The following theorem shows the equivalence of Definition 2 to a characterization of eternal vertex covers using closed families of vertex covers.

Theorem 4 *A set S is an eternal vertex cover if and only if S is in some closed family of vertex covers.*

Proof: Suppose S is an eternal vertex cover. Let D be the defense strategy specified in Definition 2. Let $\mathcal{T} = \{S^i_{D,A}$: A is an attack strategy and $i \geq 0\}$. Let $T \in \mathcal{T}$ and let $e \in E(G)$. By the definition of $\mathcal{T}$, $T = S^i_{D,A}$ for some attack strategy A and for some $i \geq 0$. Let i_m be the minimum such i. By Definition 2, T is a vertex cover. Define an attack strategy A' by $A'(T) = e$ and $A'(X) = A(X)$ for $X \in \mathcal{P}(V(G)) - T$. By the minimality of i_m, $S^k_{D,A} = S^k_{D,A'}$ for $0 \leq k \leq i_m$. Hence, $T = S^{i_m}_{D,A} = S^{i_m}_{D,A'}$ and $A'(S^{i_m}_{D,A'}) = A'(T) = e$. By substitution, $D(T,e) = D(S^{i_m}_{S,A'}, A'(S^{i_m}_{S,A'})) = S^{i_m+1}_{D,A'} \in \mathcal{T}$. By the arbitrary choice of T and e, for all $T \in \mathcal{T}$ and for all $e \in E(G)$, T is a vertex cover and $D(T,e) \in \mathcal{T}$. By definition 3, $\mathcal{T}$ is a closed family of vertex covers.

Now, suppose $S \in \mathcal{T}$ where $\mathcal{T}$ is a closed family of vertex covers. By the definition of a closed family of vertex covers, there exists a defense strategy D such that $D(T,e) \in \mathcal{T}$ for all $T \in \mathcal{T}$ and all $e \in E(G)$. Let A be an attack strategy. By definition, $S^0_{D,A} = S \in \mathcal{T}$. Suppose $S^i_{D,A} \in \mathcal{T}$. By the choice of D, $S^{i+1}_{D,A} = D[(S^i(D,A), A(S^i_{D,A})] \in \mathcal{T}$. Hence, by induction, $S^i_{D,A} \in \mathcal{T}$ for all $i \geq 0$. Since every set in a closed family of vertex covers is a vertex cover, $S^i_{D,A} \in VC(G)$ for all $i \geq 0$. By the arbitrary choice of A, S is an eternal vertex cover by definition 2. □

The proof in Theorem 4 relies on the functional nature of Definition 2 but it is not restricted to the vertex cover property. Characterizations similar to the one given in the theorem exist for any type of eternal security which lends itself to such a definition. In particular, given a definition of a secure set and a specification of allowable responses by a secure set to an attack, a set will be eternally secure if and only if it belongs to a family of secure sets which is closed under the allowable responses. We will use Theorem 4 and the following lemma to analyze subset relations between sets of guards.

Lemma 5 *Let X' and X be vertex covers. If $X' \subseteq X$ and f' is a defense by X' to an attack on an edge e, then there exists a defense f by X to an attack on e such that $f'(X') \subseteq f(X)$.*

Proof: If $X' = X$ let $f = f'$. Otherwise, $X' \subset X$ and it is sufficient to show the result holds when $X' = X - \{x\}$ for an arbitrary $x \in X$. Let f' be a defense by X' to an attack on an arbitrary edge e. Let $x_1, x_2, ..., x_p$ be vertices in X of a maximal path, P, subject to the conditions $x_1 = x$ and $f'(x_t) = x_{t-1}$ for $2 \leq t \leq p$. (Note that if $x \notin f'(X')$ then $p = 1$.)

If e is not an edge of P, define $f : X \to V(G)$ by $f(v) = v$ for $v \in \{x_1, x_2, ..., x_p\}$ and $f(v) = f'(v)$ for $v \notin \{x_1, x_2, ..., x_p\}$. If $e = \{x_i, x_{i+1}\}$ is an edge of P, define $f : X \to V(G)$ by $f(x_i) = x_{i+1}$, $f(x_{i+1}) = x_i$, $f(v) = v$ for $v \in \{x_1, x_2, ..., x_p\} - \{x_i, x_{i+1}\}$, and $f(v) = f'(v)$ for $v \notin \{x_1, x_2, ..., x_p\}$. In both cases, f is a defense by X to an attack on e and $f'(X') = f(X) - x_p \subset f(X)$. □

Theorem 4 and Lemma 5 can be used to prove that the set of eternal vertex covers is closed under the operation of taking supersets.

Theorem 6 *If X' is an eternal vertex cover and $X' \subseteq X$, then X is an eternal vertex cover.*

Proof: Suppose X' is an eternal vertex cover and $X' \subseteq X$. By Theorem 4, $X' \in \mathcal{T}'$ where $\mathcal{T}'$ is a closed family of vertex covers. Let $\mathcal{T}$ be the collection of proper supersets of sets in $\mathcal{T}'$. Every set in $\mathcal{T}$ is a vertex cover since it contains a vertex cover in $\mathcal{T}'$. Also, $X \in \mathcal{T}$ since $X' \subseteq X$. Let Y be an arbitrary element in $\mathcal{T}$. By definition of $\mathcal{T}$ there exists $Y' \in \mathcal{T}'$ such that $Y' \subseteq Y$. By Theorem 4 for any edge e there exists a defense f' by Y' such that $f'(Y') \in \mathcal{T}'$. By Lemma 5, there exists a defense f by Y to an attack on e such that $f'(Y') \subset f(Y)$. Since $f'(Y') \in \mathcal{T}'$ and $f'(Y') \subseteq f(Y)$, $f(Y) \in \mathcal{T}$. Hence, $\mathcal{T}$ is a closed family of vertex covers and X is an eternal vertex cover by Theorem 4. □

Theorem 4 also provides a proof of the following statement involving $\kappa(G)$, the *vertex connectivity number* of G, and $\gamma_i(G)$, the *independent domination number* of G.

Theorem 7 *If X is a vertex cover of G and $|X| > n - \min\{\kappa(G), \gamma_i(G)\}$, then X is an eternal vertex cover.*

Proof: Suppose X is a vertex cover and $|X| > n - \min\{\kappa(G), \gamma_i(G)\}$. Equivalently, $|V(G) - X| < \min\{\kappa(G), \gamma_i(G)\}$. The condition $|V(G) - X| < \kappa(G)$ implies that $[X]$, the subgraph of G induced by X, is connected. Also, by the definition of vertex cover, $V(G) - X$ is an independent set of vertices; so, $|V(G) - X| < \gamma_i(G)$ implies $V(G) - X$ does not dominate G. Hence, there is a vertex $x \in X$ such that $N(x) \cap (V(G) - X) = \emptyset$. Equivalently, $N(x) \subseteq X - \{x\}$.

Let $\mathcal{T}$ be the set of all vertex covers Y with $|Y| = |X|$. Suppose an attack is made on the edge $\{u, v\}$ where $v \in V(G) - X$. By definition of vertex cover $u \in X$ and since $[X]$ is connected there exists a path in $[X]$ from x to u. Hence, there is a defense f which shifts guards along a path from x to v and leaves all other guards fixed. For this defense, $X - \{x\} \subset f(X)$, so X being a vertex cover and $N(x)$ being a subset of $X - \{x\}$ implies $f(X)$ is also a vertex cover. Since $|f(X)| = |X|$, we conclude $f(X) \in \mathcal{T}$. By the arbitrary choice of e, $\mathcal{T}$ is a closed family of vertex covers; hence, X is an eternal vertex cover by Theorem 4. □

3 General Graphs - Mortal Vertex Covers

The contrapositive of Theorem 7 gives an upper bound on the size of a mortal vertex cover.

Corollary 8 *If X is a mortal vertex cover of G, then* $|X| \leq n - \min\{\kappa(G), \gamma_i(G)\}$.

For each value of $k \geq 1$, we partition the set of mortal vertex sets of size k by the following recursive definition. Let $\mathcal{T}_0^k$ be the set of all sets of size k which are not vertex covers. For $j > 0$, let $\mathcal{T}_j^k$ be the set of all vertex covers S of size k such that (1) for every attack there is at least one defense f with $f(S)$ not in $\mathcal{T}_i^k$ for any $i < j - 1$, and (2) there exists some attack so that for every defense, f, $f(S)$ is in $\mathcal{T}_i^k$, where $i \leq j - 1$. Intuitively, a best attack followed by a best response reduces the "life expectancy" by one. The set S is in $\mathcal{T}_j^k$ if and only if $|S| = k$ and $ds(G, S) = j$. Also, $ds^k(G)$ is the largest value of j for which $\mathcal{T}_j^k$ is not empty. For $S \in \mathcal{T}_j^k$, we say (A, D) is an *S-optimal attack/defense pair* if A is an attack strategy, D is a defense strategy, and $S_{D,A}^i \in \mathcal{T}_{j-i}^k$ for $0 \leq i \leq j$. Further, (A, D) is a *k-optimal attack/defense pair* if (A, D) is an *S-optimal attack/defense pair* for some set S of size k such that $ds(G, S) = ds^k(G)$,

The following observation states the properties of $\mathcal{T}_i^k$'s for easy reference.

Observation 9 *If X is a mortal vertex cover of G and $X \in \mathcal{T}_i^k$, then:*

1. *For an attack on any edge e, there is a* best *defense, i.e., there is a defense f such that $f(X) \in \mathcal{T}_j^k$ for some $j \geq i - 1$ or $f(X)$ is an eternal vertex cover;*
2. *There is a* best *attack, i.e., there is an edge e such that if f is any defense to an attack on e then $f(X) \in \mathcal{T}_j^k$ for some $j \leq i - 1$;*
3. *For a* best *attack, there is a* best *defense, i.e., a defense f such that $f(X) \in \mathcal{T}_{i-1}^k$.*

Note that Part 3 of Observation 9 follows from parts 1 and 2. If u and v are adjacent vertices and $S \subset V(G) - \{u, v\}$ then S is not a vertex cover. Hence, $k \le n-2$ implies $\mathcal{T}_0^k \ne \emptyset$. Also, since there are only a finite number of subsets of $V(G)$ of size k, Observation 9 part 3 implies the following corollary.

Corollary 10 *For each positive integer $k \le n-2$, there is a finite value I_k for which $\mathcal{T}_i^k \ne \emptyset$ for $0 \le i \le I_k$ and $\mathcal{T}_i^k = \emptyset$ for $I_k < i < \infty$.*

The next observation follows immediately from the definition of $\alpha(G)$.

Observation 11 *If $k < \alpha$ then $I_k = 0$.*

If $S \in \mathcal{T}_i^k$ then by Part 2 of Observation 9 there exists an edge e such that for every defense f to an attack on e, $f(S) \in \mathcal{T}_j^k$ where $j < i$. By Part 1 of Observation 9, $S \notin \mathcal{T}_{i'}^k$ for any $i' > i$. By definition, $\mathcal{T}_0^k$ is disjoint from $\mathcal{T}_i^k$ for all $i > 0$; hence, the collection of $\mathcal{T}_i^k$'s are disjoint by induction. We show that the union of the $\mathcal{T}_i^k$'s contains all of the mortal vertex sets of size k by using Theorem 4 to prove the contrapositive.

Theorem 12 *A set S is an eternal vertex cover if and only if $S \notin \mathcal{T}_j^{|S|}$ for any $j \ge 0$.*

Proof: Let $\mathcal{T} = \{S : S \notin \mathcal{T}_j^{|S|}$, for any $j \ge 0\}$ and let $S \in \mathcal{T}$. Suppose there exists an edge e such that for every defense f by S to an attack on e, $f(S) \in \mathcal{T}_j^{|S|}$ for some $j \ge 0$. Let A be the set of all such edges, and for each $e \in A$, let D_e be the set of defenses by S to an attack on e. Let $m = \min_{e \in A}(\max_{f \in D_e}(\{j \ge 0 : f(S) \in \mathcal{T}_j^{|S|}\}))$. By this choice of m for every attack there is at least one defense f such that $f(S) \notin \mathcal{T}_i^{|S|}$, for any $i < m$ and there exists some attack such that for every defense f, $f(S) \in \mathcal{T}_i^{|S|}$, for some $i \le m$. By definition, $S \in \mathcal{T}_{m+1}^{|S|}$ which contradicts the definition of $\mathcal{T}$. Hence, for every e there exists a defense f by S such that $f(S) \notin \mathcal{T}_j^{|S|}$ for all $j \ge 0$, i.e., $f(S) \in \mathcal{T}$. Therefore, $\mathcal{T}$ is a closed family of vertex covers and by Theorem 4, every set in $\mathcal{T}$ is an eternal vertex cover.

On the other hand, suppose there exists an eternal vertex cover $S \in \mathcal{T}_j^{|S|}$ for some j. Let m be the minimum integer such that $\mathcal{T}_m^{|S|}$ contains an eternal vertex cover and let S be such a set. Clearly, $m > 0$. By the definition of $\mathcal{T}_m^{|S|}$ there exists an attack such that for every defense f, $f(S) \in \mathcal{T}_i^{|S|}$ for some $i < m$. By the choice of m, $f(S)$ is not an eternal vertex cover. Hence, S is not in any closed family of vertex covers, which contradicts Theorem 4. $\square$

Corollary 13 *The collection of $\mathcal{T}_i^k$'s partition the mortal vertex sets of size k.*

By Theorem 12, when $I_k = 0$, there is at least one set of k vertices which is not a vertex cover and every set of k vertices which is a vertex cover must be eternal.

Theorem 6 and Lemma 5 can be used to verify the natural and expected relation between a mortal set and its subsets expressed in the next theorem.

Theorem 14 *If $X \in \mathcal{T}_i^{|X|}$ and $X' \subseteq X$ then $X' \in \mathcal{T}_{i'}^{|X'|}$, for some $i' \leq i$.*

Proof: The result is trivial if $X' = X$ so we assume $X' \subset X$. We let $X \in \mathcal{T}_i^{|X|}$ and induct on i, the length of the death spiral. If $i = 0$ then $X \in \mathcal{T}_0^{|X|}$ and X is not a vertex cover. Hence, $X' \subset X$ is not a vertex cover, so $X' \in \mathcal{T}_0^{|X'|}$ and the result holds.

Let $i > 0$ and assume that the result holds for all $l < i$ (i.e., if $X \in \mathcal{T}_l^{|X|}$ and $X' \subset X$, then $X' \in \mathcal{T}_{l'}^{|X'|}$ for some $l' \leq l$). From Observation 9 part 2, there exists an edge e such that attacking e is a best attack against X (i.e., for any defense f by X to an attack on e, $f(X) \in \mathcal{T}_k^{|f(X)|}$ for some $k \leq i - 1$). By the contrapositive to Theorem 6, X' is mortal and, by Theorem 12, $X' \in \mathcal{T}_{i'}^{|X'|}$ for some i'. Let f' be a best defense by X' to an attack on e. From Observation 9 part 1, $f'(X') \in \mathcal{T}_{j'}^{|f(X')|}$ for some $j' \geq i' - 1$. By Lemma 5 there exists a defense f by X to an attack on e such that $f'(X') \subset f(X)$. By the choice of e, $f(X) \in \mathcal{T}_j^{|f(X)|}$ for some $j \leq i - 1$. By induction, $j' \leq j$. We have $i' - 1 \leq j' \leq j \leq i - 1$; hence, $i' \leq i$. □

The following corollary is the contrapositive to Theorem 6 and also follows from Theorem 14.

Corollary 15 *If X is a mortal set and $X' \subset X$ then X' is a mortal set.*

If G has a least one edge then the empty set is mortal and Corollary 15 implies the collection of mortal subsets of G is an independence system (see [17] page 1484).

4 Trees with $\alpha_m^\infty - 1$ Guards

If the number of guards in a tree T is not greater than the number of internal vertices, then, by Theorem 1, the guard set is mortal. In this case, a best defense will place guards so as to maximize the death spiral number. Here, we investigate the case where there are exactly $|I(T)| = \alpha_m^\infty - 1$

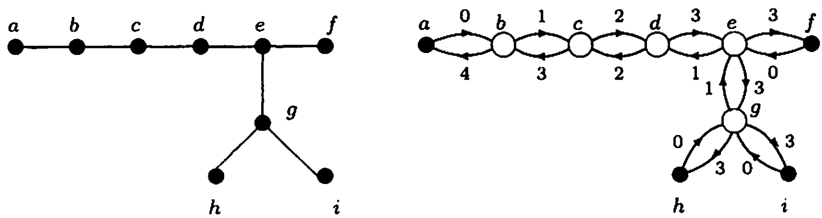
guards. Throughout the section we will assume that T is a tree with at least one edge.

Suppose e is an edge of a tree T with end vertices u and v. Associate with e the two ordered pairs, or directed edges, (u, v) and (v, u). Suppose C is the component of $T - e$ which contains u. If $d = (u, v)$ then the *tail-graph* of d, denoted T_d, is the subgraph of T which is induced by $\{v\} \cup V(C)$. We will also refer to T_d as a tail-graph of v. The *Fan* of d, denoted $Fan(d)$, is the set of directed edges of the form (y, u) with $y \neq v$.

We assign a label to each directed edge $d = (u, v)$ recursively as follows. Once all the edges in $Fan(d)$ have been assigned labels, we let $m = min_{d' \in Fan(d)}\{L(d')\}$ and $M = max_{d' \in Fan(d)}\{L(d')\}$ and assign the label $L(d)$ to d according to the equation:

$$L(d) = \begin{cases} 0 & \text{if } deg(u) = 1 \\ M & \text{if } m < M \\ 1 + M & \text{otherwise} \end{cases}$$

Figure 1 shows a simple tree with the associated labeled digraph. The vertices b, c, d, e, g are the interior vertices of the tree. In the digraph a possible guard set is indicated by the white vertices and $Fan((e, g)) = \{(d, e), (f, e)\}$. Notice that u is an interior vertex of T if and only if the edge (u, v) is labeled using the second or third condition of the recursive definition. Also, both of these conditions imply the existence of an edge d' in $Fan(d)$ such that $L(d') < L(d)$.



(a) Tree

(b) Associated digraph and labeling with initial assignment of guards

Figure 1: Example of a labeling and initial guard assignment

The directed edges (x, y) and (u, v) have the *same direction* if and only if there is a directed path in T which contains both (x, y) and (u, v). The next lemma establishes an important monotonicity condition on the labeling for edges with the same direction.

Lemma 16 *Let $d = (u, v)$ be a directed edge in T. If $d' \in T_d$ is in the same direction as d then $L(d') \leq L(d)$.*

Proof: Let P be a longest directed path from a leaf of $T_d - v$ to u. We induct on the length of P. If the length of P is 0 then u is a leaf and d is the only edge in T_d so the result holds vacuously. If the length of P is greater than 0, then the label on d is generated by either the second or third condition in the recursive definition and $L(e) \leq L(d)$ for all $e \in Fan(d)$. By induction, if $e \in Fan(d)$ and $d' \in T_e$ is in the same direction as e then $L(d') \leq L(e)$. Since every edge in $Fan(d)$ is in the same direction as d, the result follows by transitivity of inequality. □

The following observation provides useful properties of the labeling.

Observation 17 *Suppose $d = (u, v)$ is a directed edge of T. The following conditions are equivalent:*

1. *u is an interior vertex of T;*
2. *$L(d)$ is generated using the second or third condition of the recursive definition of the labeling;*
3. *There exists a directed edge $d' \in$ Fan(d) such that $L(d') < L(d)$;*
4. *$L(d) \geq 1$.*

Notice that Lemma 16 implies that all the labelings are non-negative. Thus, Part 3 of the observation implies Part 4. The rest of the implications are immediate from the recursive definition.

Let d be a directed edge and let P be a directed path $d_m, \ldots, d_1, d_0 = d$. We say P is an *$L(d)$-labeled path* if $L(d_i) = L(d)$ for all $m \geq i \geq 0$. We define *Fan* of P, denoted *$Fan(P)$*, to be *$Fan(d_m)$*. The path P is *non-extendable* if $L(d') < L(d)$ for all $d' \in Fan(P)$. When $L(d) \neq 0$ and P is non-extendable, d_m must have been labeled via the third condition in the recursive definition of the labeling. This leads to the following observation.

Observation 18 *If P is a non-extendable $L(d)$-labeled path, then $L(d') = L(d) - 1$ for all $d' \in Fan(P)$.*

Note that if $d = (u, v)$ and $L(d) = 0$, then u is a leaf, $P = d$, and $Fan(P) = \emptyset$; so Observation 18 is vacuously true .

Let S be a subset of $V(T)$ and let $d = (u, v)$ be a directed edge. We say d is *S-vulnerable* if $|S \cap V(T_d)| \leq |I(T_d)|$, that is, the guard set, when restricted to T_d, is mortal. Note that if $v \in I(T)$, then $I(T_d) \neq V(T_d) \cap I(T)$ since $v \notin I(T_d)$. We say d is *exposed by S* (or simply *exposed* if S is understood) if $v \notin S$.

The *Labeling Attack Strategy* is to attack an S-vulnerable exposed edge d such that $L(d)$ is a minimum among such edges. A key idea for the corresponding defense strategy is to respond to such an attack by shifting guards along a non-extendable L(d)-labeled path. The precise formulation includes some technicalities so we delay it for the moment. First we will show that the Labeling Attack Strategy provides an upper bound on the death spiral number of T. We begin with a counting argument.

Lemma 19 *Let $S \subseteq V(T)$. Let $d = (u, v)$ be a directed edge with $L(d) \geq 1$. If (i) for all $d' \in Fan(d)$ $|S \cap (V(T_{d'}) - \{u\})| \geq |I(T_{d'})|$, and (ii) there exists $d' \in Fan(d)$ such that $|S \cap (V(T_{d'}) - \{u\})| > |I(T_{d'})|$, then $|S \cap (V(T_d) - \{v\})| \geq |I(T_d)|$.*

Proof: By the definition of $Fan(d)$, $S \cap (V(T_d) - \{v\})$ equals the disjoint union of $S \cap \{u\}$ with $\cup_{d' \in Fan(d)}(S \cap (V(T_{d'}) - \{u\}))$. Hence, $|S \cap (V(T_d) - \{v\})| = |S \cap \{y\}| + |\cup_{d' \in Fan(d)} (S \cap (V(T_{d'}) - \{u\}))| \geq |\cup_{d' \in Fan(d)} (S \cap (V(T_{d'}) - \{u\}))|$. Since $|\cup_{d' \in Fan(d)} (S \cap (V(T_{d'}) - \{y\}))| = \Sigma_{d' \in Fan(d)} |S \cap (V(T_{d'}) - \{u\})|$, the assumption implies $\Sigma_{d' \in Fan(d)} |S \cap (V(T_{d'}) - \{u\})| \geq 1 + \Sigma_{d' \in Fan(d)} |I(T_{d'})|$. The result follows since $I(T_d) = \{u\} \cup (\cup_{d' \in Fan(d)} I(T_{d'}))$. □

The next lemma guarantees the existence of a suitable next edge to be attacked.

Lemma 20 *Let $S \subseteq V(T)$ and $d = (u, v)$ be a directed edge with $L(d) \geq 1$. If $|S \cap (V(T_d) - \{v\})| < |I(T_d)|$, then there exists an S-vulnerable directed edge $d' = (u', v')$ in T_d such that d' is in the same direction as d and $L(d') < L(d)$.*

Proof: We induct on $|I(T_d)|$. If $|I(T_d)| = 1$ then T_d is a star centered at u. Also, $|S \cap (V(T_d) - \{v\})| < |I(T_d)|$ implies $|S \cap (V(T_d) - \{v\})| = 0$. In this case, every directed edge in $T_d - \{v\}$ which is in the same direction as d has the form (x, u) where x is a leaf. By the definition of the labeling, all such directed edges are labeled 0 and they all satisfy the conclusion of the lemma.

Let $|I(T_d)| > 1$. Suppose u is in S. Among the components of T_d induced by $S \cap (V(T_d) - \{v\})$ let C_u be the one that contains u. Since $V(C_u) \subseteq S \cap (V(T_d) - \{v\})$, the condition $|S \cap (V(T_d) - \{v\})| < |I(T_d)|$ implies $I(T_d) - V(C_u) \neq \emptyset$. Let $\{d_1, \ldots, d_k\}$ be the set of all directed edges such that for each i, $d_i = (y_i, x_i)$ where $y_i \in I(T_d) - V(C_u)$ and $x_i \in V(C_u)$ (the $y_i's$ are distinct but the $x_i's$ need not be). See Figure 2 for an example of this construction. For all i we observe that $d_i \in T_d$ and is in the same direction as d, hence, $L(d_i) \leq L(d)$ by Lemma 16. This implies, again by Lemma 16, that if $d' \in Fan(d_i)$ then $L(d') \leq L(d)$.

If for some i there exists $d' \in Fan(d_i)$ such that $|S \cap V(T_{d'})| \leq |I(T_{d'})|$ and $L(d') < L(d_i)$ then d' is S-vulnerable and satisfies the conclusion of the lemma since d', d_i, and d are all in the same direction and $L(d_i) \leq L(d)$ implies $L(d') < L(d)$. Also, If for some i there exists $d' \in Fan(d_i)$ such that $|S \cap (V(T_{d'}) - \{y_i\})| < |I(T_{d'})|$, then by induction, since $|I(T_{d'})| < |I(T_d)|$, there exists an S-vulnerable directed edge d'' in $T_{d'}$ such that d'' is in the same direction as d' and $L(d'') < L(d')$. Since d'', d', and d are all in the same direction as d and $L(d'') < L(d') \leq L(d)$, d'' satisfies the conclusion of the Lemma.

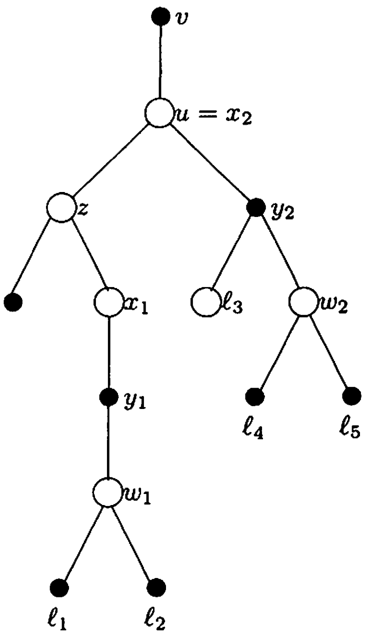
Otherwise, if $1 \leq i \leq k$ and $d' \in Fan(d_i)$ we have: (i) $|S \cap (V(T_{d'}) - \{y_i\})| \geq |I(T_{d'})|$ and (ii) $L(d') < L(d)$ implies $|S \cap V(T_{d'})| \geq |I(T_{d'})| + 1$. Suppose $1 \leq i \leq k$. By Observation 17, there exists $\widehat{d} \in Fan(d_i)$ such that $L(\widehat{d}) < L(d_i)$. By (ii), $|S \cap V(T_{\widehat{d}})| \geq |I(T_{\widehat{d}})| + 1$. This implies $|S \cap V(T_{\widehat{d}}) - \{y_i\}| \geq |I(T_{\widehat{(d)}})| + 1$ since $y_i \notin S$. By construction y_i is an interior vertex so $L(d_i) \geq 1$, by Observation 17, which verifies the hypotheses of Lemma 19 for d_i. Lemma 19 and the arbitrary choice of i imply $|S \cap (V(T_{d_i}) - \{x_i\}| \geq |I(T_{d_i})|$ for $1 \leq i \leq k$. These sets are disjoint so we may sum the inequalities to obtain $\Sigma_{i=1}^{k} |S \cap (V(T_{d_i}) - \{x_i\})| \geq \Sigma_{i=1}^{k} |I(T_{d_i}|$. This implies $|S \cap (V(T_d) - \{v\})| = |V(C_u)| + \Sigma_{i=1}^{k} |S \cap (V(T_{d_i}) - \{x_i\})| \geq |V(C_u)| + \Sigma_{i=1}^{k} |I(T_{d_i})|$. Since $V(C_u) \cup (\cup_{i=1}^{k} I(T_{(d_i)})) \supseteq I(T_d)$, we have $|S \cap (V(T_d) - \{v\})| \geq |I(T_d)|$, a contradiction to the hypothesis of the lemma, so this situation cannot occur.

If u is not in S then, as in the argument above, the absence of a directed edge which satisfies the conclusion of the lemma implies the edges in $Fan(d)$ satisfy the hypotheses of Lemma 19; hence $|S \cap (V(T_d) - \{v\})| \geq |I(T_d)|$ once again contradicting the hypothesis. □

Corollary 21 *Let $S \subseteq V(T)$ and $d = (u, v)$ be a directed edge with $L(d) \geq 1$. If $|S \cap (V(T_d) - \{v\})| < |I(T_d)|$ then there exists an exposed S-vulnerable directed edge $d' = (u', v')$ in T_d such that d' is in the same direction as d and $L(d') < L(d)$.*

Proof: By Lemma 20 there exists an S-vulnerable directed edge $d' = (u', v')$ in T_d such that d' is in the same direction as d and $L(d') < L(d)$. If d' is exposed, we're done. Hence, we assume d' is not exposed which implies $|S \cap (V(T_{d'}) - \{v'\})| < |I(T_d)|$. If $L(d') \geq 1$ we're done by induction on $L(d)$. If $L(d') = 0$ then being S-vulnerable implies d' is exposed (indeed in this case S is not even a vertex cover). □

Corollary 22 *If $S \subseteq V(T)$ and $|S| \leq |I(T)|$ then there exists an exposed S-vulnerable directed edge.*



$$V(C_u) = \{u, z, x_1\}$$

$$V(T_{d_1}) = \{x_1, y_1, w_1, \ell_1, \ell_2\}$$

$$V(T_{d_2}) = \{x_2, y_2, w_2, \ell_3, \ell_4, \ell_5\}$$

Figure 2: Construction from Lemma 20

Proof: Let v be a leaf of T and $d = (u, v)$ be a directed edge. Since $I(T_d) = I(T)$, d is S-vulnerable. If d is exposed, we're done. Consequently, we assume $v \in S$ which implies $|S \cap (V(T_d) - \{v\})| < |S| \leq |I(T_d)|$. If $L(d) = 0$ then by Observation 17 u is not an interior vertex. This implies T is a K_2, $S = \emptyset$ and d is exposed. Otherwise, $L(d) \geq 1$ and the result follows from Corollary 21. □

For $S \subseteq V(T)$ we let $Exp(S)$ be the set of exposed S-vulnerable directed edges and define the life expectancy of S, denoted $LE(S)$, by $LE(S) = min(\{L(d) : d \in Exp(S)\})$. The next lemma relates the death spiral number of a mortal vertex cover S to the life expectancy of S.

Lemma 23 *If $S \subset V(T)$ with $|S| \leq \alpha_m^\infty(T) - 1$, then $S \in \mathcal{T}_j^{|S|}$ where $j \leq LE(S)$. In particular, the Labeling Attack Strategy will force an uncovered edge in at most $LE(S)$ attacks.*

Proof: By Corollary 22 and the definition of $LE(S)$ there exists an exposed S-vulnerable edge $d = (u, v)$ where $L(d) = LE(S)$. Following the Labeling Attack Strategy, we attack this edge and let f be a defense to this attack. By definition, f shifts a guard from u to v; so, $|f(S) \cap (V(T_d) - \{v\})| = |S \cap V(T_d)| - 1 < |S \cap V(T_d)| \leq |I(T_d)|$. By Corollary

21 there exists an exposed $f(S)$-vulnerable directed edge d' in T_d such that d' is in the same direction as d and $L(d') < L(d)$. By the definition of life expectancy and the choice of d, $LE(f(S)) \leq L(d') < LE(S)$. By induction, $f(S) \in \mathcal{T}_t^{|S|}$ where $t \leq LE(f(S))$. It follows that $S \in \mathcal{T}_j^{|S|}$ where $j \leq L(d') + 1 \leq L(d) = LE(S)$. □

Lemma 23 shows that the Labeling Attack Strategy provides an upper bound on the life expectancy of a set of size $\alpha_m^\infty(T) - 1$. Our next task is to provide a defense that forces the bound. We begin by describing the guard placements that will be generated by our defense.

A subset S of $V(T)$ is a *standard defensive position* (an *SDP*) if $|S| = |I(T)|$ and $|S - I(T)| \leq 1$ (i.e., S contains at most one leaf). We let S_0 be the standard defensive position $I(T)$. Otherwise, $|S - I(T)| = 1$ and there exists a single leaf $s \in S - I(T)$ and a single internal vertex $v \in I(T) - S$. We call v the *central vertex* of S. Among the various tail-graphs of v, we define the *guarded – leaf component* of S, denoted $GLC(S)$, to be the one that contains s. Note that $GLC(S)$ has a guard on each of its internal vertices as well as on the leaf s and that S_0 has no guarded-leaf component. A simple but effective response to attacks on edges in the guarded-leaf component is described in the proof of the following observation.

Observation 24 *If e is an edge in $GLC(S)$ where S is an SDP, then there is a response to an attack on e which results in an SDP S' such that either $S' = S_0$ or $GLC(S') = GLC(S)$.*

Proof: Let x and y be the end vertices of e. If x and y are both in S, we swap them and $S' = S$. Otherwise, without loss of generality $x \in S$ and $y \notin S$. By the definition of $GLC(S)$, either $y = v$ or y is a leaf. In either case, the path P in T from s to y is contained in $GLC(S)$ and goes through x to y. Obtain S_{k+1} by shifting the guards along P leaving all other guards fixed. If $y = v$ then $S' = S_0$, and if y is a leaf then $GLC(S') = GLC(S)$. □

We use this observation in proving the following lemma.

Lemma 25 *If the initial guard set is S_0, then there is a defense strategy such that after any k attacks the guard set will be a standard defensive position S with $LE(S) + k \geq LE(S_0)$.*

Proof: For $k = 0$, $LE(S_0) + 0 \geq LE(S_0)$. Suppose after $k \geq 0$ attacks the guard set is an SDP, S_k, with $LE(S_k) + k \geq LE(S_0)$. Further suppose an attack is made on the directed edge $d = (x, y)$ where $x \in S_k$ and $y \notin S_k$. Let P_d be a non-extendable L(d)-labeled path $d_m, d_{m-1}, ..., d_1$ with $d_1 = d$ and $d_m = (x_m, x_{m-1})$.

If $S_k = S_0$, then y is a leaf of T so $L(d) \geq LE(S_0)$. Obtain S_{k+1} by shifting guards along P_d so x_m becomes the central vertex of S_{k+1}. This defense puts the vertex x_{m-1} into the guarded-leaf component of S_{k+1}, so, (x_{m-1}, x_m) is not S-vulnerable. It follows that $Exp(S_{k+1}) \subset Fan(P_d) \cup Exp(S_k)$. By Observation 18, $LE(S_{k+1}) \geq min(L(d) - 1, LE(S_k)) \geq LE(S_k) - 1$. Hence, $LE(S_{k+1}) + k + 1 \geq LE(S_k) - 1 + k + 1 \geq LE(S_0) + k \geq LE(S_0)$. If $k \geq 1$ then the inequality is strict, which corresponds to the intuitive notion that an attack that allows the defense to return to S_0 cannot be optimal.

We now assume $S_k \neq S_0$. By the definition of SDP, either $y = v$ or y is a leaf. Let t be the vertex adjacent to the guarded leaf s. Note that $L(t, s) \geq LE(S_0)$ since s is a leaf. Let $P_{s,y}$ be the path in T from s to y. There are three cases.

Case 1. The edge (x, y) is in $GLC(S_k)$. Obtain S_{k+1} by using the response from Observation 24. If $S_{k+1} = S_0$, then $LE(S_{k+1}) + k + 1 = LE(S_0) + k + 1 > LE(S_0)$. If $GLC(S_{k+1}) = GLC(S_k)$, then $Exp(S_{k+1}) = Exp(S_k)$ which implies $LE(S_{k+1}) = LE(S_k)$ and $LE(S_{k+1}) + k + 1 > LE(S_k) + k \geq LE(S_0)$. Note that both instances produce a strict inequality which confirms the notion that optimal attacks will be on edges which are not in $GLC(S_k)$.

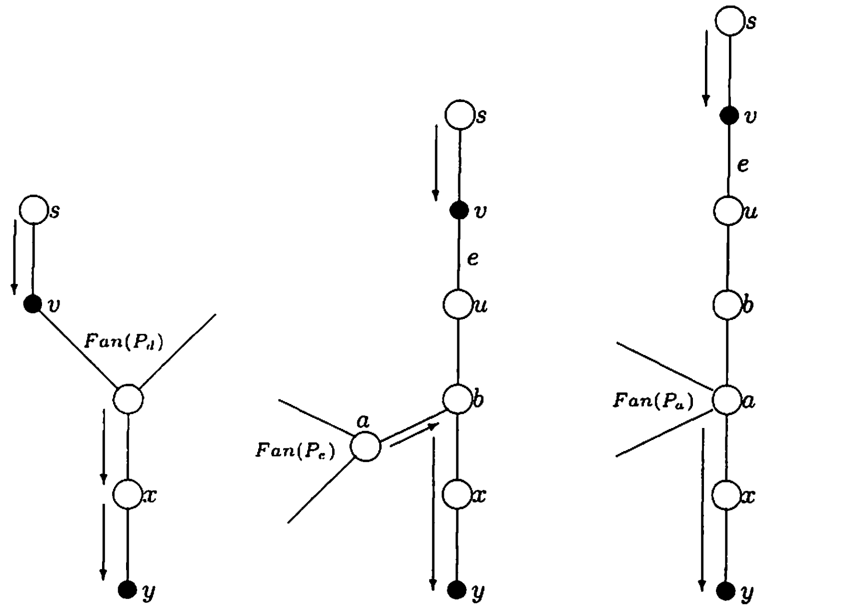
Case 2. The edge (x, y) is not in $GLC(S_k)$ and $y = v$. Obtain an SDP S_{k+1} by shifting guards along P_d. The central vertex of S_{k+1} is x_m and x_{m-1} is in the new guarded-leaf component, so, (x_{m-1}, x_m) is in the guarded-leaf component of S_{k+1} and so is not S_{k+1}-vulnerable. This implies $Exp(S_{k+1}) \subset Fan(P_d) \cup Exp(S_k)$. By Observation 18, $LE(S_{k+1}) \geq min(L(d) - 1, LE(S_k)) \geq LE(S_k) - 1$. Hence $LE(S_{k+1}) + k + 1 \geq LE(S_k) - 1 + k + 1 \geq LE(S_0)$.

Case 3. The edge (x, y) is not in $GLC(S_k)$ and $y \neq v$ which implies y is a leaf of T. If $v \notin V(P_d)$, obtain S_{k+1} by shifting guards along the path from s to v and along the path P_d toward y. As in the previous case x_m is the central vertex of S_{k+1} and (x_{m-1}, x_m) is not S_{k+1}-vulnerable, so $Exp(S_{k+1}) \subseteq Exp(S_k) \cup Fan(P_d) \cup \{(t, s)\}$ (see Figure 3 (a)).

Now suppose $v \in V(P_d)$. Let $e = (u, v)$ be the directed edge such that $y \in T_e$. Note that u is not in the guarded-leaf component of S_k which implies $e \in Exp(S)$ and $L(e) \geq LE(S_k)$. Let $P_{a,e}$ be a non-extendable $L(e)$-labeled path $e_l, e_{l-1}, ..., e_1 = e$ with $e = e_1$ and $e_l = (a, b)$. Let $P_{a,y}$ be the unique path in T_e from a to y. Note that $v \notin V(P_{a,y})$ since v is a leaf of T_e. Obtain S_{k+1} by shifting guards along the path from s to v and along $P_{a,y}$ from a to y. This makes a the central vertex of S_{k+1}. If $a \notin V(P_d)$, then b is in the guarded-leaf component of S_{k+1}, so (b, a) is not S_{k+1}-vulnerable and $Fan(P_{a,y}) = Fan(P_{a,e})$ (see Figure 3 (b)). If $a \in V(P_d)$, then $(b, a) \in E(P_d)$ and $Fan(P_{a,y}) \subseteq Fan(P_{a,e}) \cup \{(b, a)\}$ (see Figure 3 (c)). In all cases, $Exp(S_{k+1}) \subseteq E(P_d) \cup Fan(P_d) \cup Fan(P_{a,e}) \cup Exp(S_k)$.

Hence, $LE(S_{k+1}) \geq min(\{L(d), L(d)-1, L(e)-1, LE(S_k)\}) \geq LE(S_k)-1$, and $LE(S_{k+1}) + k + 1 \geq LE(S_k) + k \geq LE(S_0)$. □

The Labeling Defense Strategy is the strategy given in the Proof of Lemma 25.



$v \notin V(P_d)$ (a)

$v \in V(P_d)$
$a \notin V(P_d)$ (b)

$v \in V(P_d)$
$a \in V(P_d) \Rightarrow b \in v(P_d)$ (c)

Figure 3: Case 3, (x, y) exposed and $y \neq v$.

Corollary 26 *For any attack strategy, the number of attacks needed to uncover an edge against the Labeling Defense Strategy is greater than or equal to $LE(S_0)$; that is, $ds^{|I(T)|}(T) \geq LE(S_0)$.*

Proof: The attack strategy must terminate with an SDP, S, that has an uncovered edge. If $I(T) = \emptyset$ then $S = \emptyset$ and no attacks are needed. Otherwise, none of the leaves are adjacent and the uncovered edge must have a leaf and the central vertex as its endpoints. Consequently, the life expectancy of S is 0 and the result follows from Lemma 25. □

We can now compute the death spiral number for a tree when the size of the guard set equals the number of internal vertices.

Theorem 27 *For any tree T, $ds^{|I(T)|}(T) = LE(S_0)$.*

Proof: By Corollary 26, $ds^{|I(T)|}(T) \geq LE(S_0)$. Let $d = (u, v)$ be a directed edge such that v is a leaf and $L(d) = LE(S_0)$ and let S be a subset of $V(T)$ such that $|S| \leq |I(T)|$. Since $T_d = T$, d is S-vulnerable; hence, by Lemma 23, $S \in \mathcal{T}_j^{|S|}$ where $j \leq L(d) = LE(S_0)$. □

Given a vertex cover S, an attack/defense pair (A, D) is *consistent for* S if there is a path P which contains the edge $D(S^i(D, A), A(S^i(D, A)))$ for all $i \geq 0$ and the guards that respond to each attack all shift in the same direction along P. If A is the attack strategy described in the proof of Lemma 23 and D is the defense strategy described in Lemma 25, then (A, D) is consistent. This leads to the following corollary.

Corollary 28 *For any tree T there exists a consistent, $|I(T)|$-optimal attack/defense pair.*

It is not true that every optimal attack/defense strategy pair is consistent as can be seen from Figure 4. Note that the given guard set is in $\mathcal{T}_2^3$ and that attacking the edge $\{a, b\}$ followed by the edge $\{d, e\}$ is a consistent optimal attack. However, attacking the edge $\{a, b\}$, which forces the guards on b and d to move, followed by the edge $\{e, g\}$ is also optimal but is not consistent.

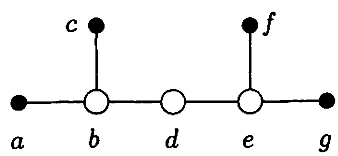


Figure 4: Example showing inconsistent edge attacks can be optimal

5 Questions

1. What are precise values of $ds^k(T)$ for a tree T when $k < I(T)$?
2. What are precise values of $ds^k(G)$ for general graphs or classes of graphs?
3. What are upper and lower bounds for $ds^k(G)$ for general graphs or classes of graphs?
4. If X is mortal, does there always exist a consistent optimal attack/defense strategy for general graphs?

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