Some Bounds on the Size of DI-Pathological Graphs

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Abstract

A DI-pathological graph is a graph in which every minimum dominating set intersects every maximal independent set. DI-patholigcal graphs are related to the Inverse Domination Conjecture; hence, it is useful to characterize properties of them. One characterization question is how large or small a graph can be relative to the domination number. Two useful characterizations of size seem relevant, namely the number of vertices and the number of edges. In this paper, we provide two results related to this question. In terms of the number of vertices, we show that every connected, DI-pathological graph has at least $2\gamma(G)+4$ vertices except $K_{3,3}$, $K_{3,4}$, and six graphs on nine vertices and show that our lower bound is best possible. We then show that with one exception, every connected, DI-pathological graph with no isolated vertices has at least $2\gamma(G)+5$ edges and show that our lower bound is best possible.

1 Introduction

Let $G = (V(G), E(G))$ be a graph. If $\{x, y\} \in E(G)$, we write $x \sim y$. An independent set of vertices is a set $I \subset V(G)$ such that if $v_1, v_2 \in I$, then $\{v_1, v_2\} \notin E(G)$. An independent set I is said to be maximal independent if $I \subsetneq J \subset V(G)$ implies that J is not independent. An independent set I is said to be a maximum independent set if $|I| \geq |J|$ for any independent set J of G. The cardinality of a maximum independent set is denoted by $\alpha(G)$. If W is a set of vertices, then we define $N_G(W) = \{x \in V(G) \mid x \sim y \text{ for some } y \in W\}$ and $N_G[W] = N_G(W) \cup W$. A set D is said to be dominating if $N[D] = V(G)$. A dominating set is said to be minimum dominating if no other dominating set has smaller size. We denote the cardinality of a minimum dominating set by $\gamma(G)$ or just γ if the underlying graph is not important. If D is a dominating set in G and $d \in D$, let $P_D(d) = \{v \in V(G) \setminus D \mid v \notin N_G(D \setminus \{d\})\}$ and $P_D[d] = \{d\} \cup P_D(d)$. The vertices of $P_D(d)$ are called private neighbors of d.

When G has no isolated vertices (so in particular G is not K_1), we define $\gamma'(G) = \min(\{|B| \mid B \subset V(G) \setminus D, D$ minimum dominating in G, B dominating in $G\})$, and call $\gamma'(G)$ the inverse domination number. One area where inverse dominating sets are of interest is in computer science, where resource centers (such as large databases) can correspond to the nodes of dominating sets. In this situation, an inverse dominating set can serve as a reliability safeguard. The inverse domination number was first introduced by Kulli and Sigarkanti [6] who showed that the quantity is indeed well defined. They also claimed that $\gamma'(G) \leq \alpha(G)$; however, it was quickly realized that their proof was flawed. After observing that the problem is nontrivial, Domke, Dunbar, and Markus stated the above claim as a conjecture (the Inverse Domination Conjecture) in [1]. Some work has been done on the subject, but the results are relatively sparse. In [5], the conjecture is proved for all graphs G with $\gamma(G) \leq 4$. In [2], the conjecture is proved for several families of graphs. Note that if there is some minimum dominating set D and maximal independent set I that are disjoint in a graph G, then $\gamma'(G) \leq |I| \leq \alpha(G)$, which would prove the conjecture for G. Hence, if the conjecture is false, the counterexample would be found in a graph in which every minimum dominating set intersects every maximal independent set. Such graphs are called DI-pathological.

Since DI-pathological graphs are the only graphs which can be counterexamples to the Inverse Domination Conjecture, it is useful to characterize some properties of DI-pathological graphs. In [5], Johnson, Prier, and Walsh showed that if $\gamma(G) = 1$ and G has no isolated vertices, then G is not DI-pathological. In the same paper, they showed that if $\gamma(G) = 2$ and G has no isolated vertices, then G is DI-pathological if and only if $G \cong K_{m,n}$ with $m, n \geq 3$. In [4], Johnson and Prier showed that if $\gamma(G) = 3$, G is DI-pathological, and G is connected, then $|V(G)| \geq 9$. They further showed that $|E(G)| \geq 10$ for all graphs G with $\gamma(G) = 3$ that are connected, DI-pathological, and have $|V(G)| = 9$. In the same paper, they conjectured that if G is connected and DI-pathological with $\gamma(G) \geq 3$, then $|V(G)| \geq 3\gamma(G)$, basing the conjecture on a natural extension of the vertex and edge minimal graph they found for $\gamma(G) = 3$. In this paper, we disprove this conjecture showing that we can find a connected, DI-pathological graph on $2\gamma(G) + 4$ vertices for all $\gamma(G) \geq 4$ ($K_{3,3}$ has domination number two and $6 = 2\gamma(K_{3,3}) + 2$ vertices, and the graph Prier and Johnson found has $\gamma(G) = 3$ and $|V(G)| = 9 = 2\gamma(G) + 3$). We also show that this result is best possible. Finally, we look at the natural question of finding a lower bound on the number of edges in a connected, DI-pathological graph, proving that with one exception, every DI-pathological graph has at least $2\gamma(G)+5$ edges and show this result is best possible by finding a graph with $2\gamma(G)+5$ edges for each $\gamma(G) \geq 2$.

2 Introductory Lemmas

The proof of our two main theorems (see Sections 3 and 4) will proceed by contradiction using minimal counterexamples (with respect to the domination number). In this section, we prove a few introductory lemmas that will provide the basis for the proofs of the two main theorems. The first lemma allows us to take a DI-pathological graph and form a smaller DI-pathological graph under certain conditions.

Lemma 1. *Suppose G is a DI-pathological graph with some minimum dominating set D with a vertex $d \in D$ that has exactly one private neighbor, say v_d. Then $G \setminus P_D[d]$ has no isolated vertices, has domination number $\gamma(G)-1$, and is DI-pathological.*

Proof. We first show that $G \setminus P_D[d]$ has no isolated vertices. Suppose for a contradiction that $G \setminus P_D[d]$ has an isolated vertex v. Since v is not isolated in G, it is either adjacent to d or v_d. If v is adjacent to d, then v cannot be in the dominating set (since otherwise $D \setminus \{v\}$ is still dominating and so D is not minimum). So v would then be a private neighbor of d, a contradiction. So v cannot be adjacent to d and hence must be adjacent to v_d. Since v_d is a private neighbor of d, v cannot be in D. But then no vertex in D is adjacent to v, a contradiction with D being dominating.

Next note that $G \setminus P_D[d]$ has domination number $\gamma(G)-1$ since $D \setminus \{d\}$ is a dominating set of size $\gamma(G)-1$ and if there is a dominating set of size at most $\gamma(G)-2$, then that set along with d is a dominating set of size at most $\gamma(G)-1$ in G.

We now show that $G \setminus P_D[d]$ is DI-pathological. Suppose by way of contradiction that $G \setminus P_D[d]$ is not DI-pathological and let D' and I' be a minimum dominating set and maximal independent set of $G \setminus P_D[d]$ respectively such that $D' \cap I' = \varnothing$. Note that both $D' \cup \{d\}$ and $D' \cup \{v_d\}$ are dominating sets in G.

If some vertex in I' is adjacent to d, take $\widehat{D} = D' \cup \{d\}$ as the dominating set of G, and let $\widehat{I} = I' \cup \{v_d\}$ if $I' \cup \{v_d\}$ is independent; otherwise let $\widehat{I} = I'$. Then $\widehat{I}$ is a maximal independent set that is not adjacent to the minimum dominating set $\widehat{D}$, contradicting our assumption that G is DI-pathological.

If no vertex in I' is adjacent to d, take $\widehat{D} = D' \cup \{v_d\}$ and take $\widehat{I} = I' \cup \{d\}$. Then $\widehat{D}$ and $\widehat{I}$ are a minimum dominating set and a maximal independent set respectively which do not intersect, contradicting our assumption that G is DI-pathological. □

The above proof can be modified slightly to show that the lemma still holds if d has no private neighbors instead of one. However, it is more useful for us to delete two vertices (d and its private neighbor) than just one, and as the next lemma shows, if we can find a dominating set containing a vertex with no private neighbors, then we can find a dominating set containing a vertex with one private neighbor under certain mild conditions.

Lemma 2. *Let G be a graph with no isolated vertices and fewer than $3\gamma(G)$ vertices. Then there exists a dominating set D such that there exists $d \in D$ with precisely one private neighbor.*

Proof. Let D' be some minimum dominating set of G. If every vertex of D' has at least two private neighbors, then G has at least $3\gamma(G)$ vertices, a contradiction. So there is some vertex in D', say d', with zero or one private neighbor(s). If d' has one private neighbor, we are done (set $D = D'$ and $d = d'$). So suppose d' has no private neighbors. If d' is adjacent to any other vertex in D, then $D \backslash \{d'\}$ is a smaller dominating set, a contradiction. Let $d \in V(G) \setminus D'$ be adjacent to d' (such a d is guaranteed to exist since G has no isolated vertices and d' is not adjacent to any vertices in D'). Let $D = (D' \setminus \{d'\}) \cup \{d\}$. Then d' is adjacent to d and every vertex in $V(G) \setminus D$ other than d' is adjacent to some vertex in $D' \setminus \{d'\} \subset D$ since d' had no private neighbors. So D is a dominating set and since $|D| = |D'|$, D is a minimum dominating set. Further, d' is not adjacent to D' so the only vertex it is adjacent to in D is d. So d' is a private neighbor of d. Finally, d has no other private neighbors since every vertex in $V(G) \setminus D$ other than d' is adjacent to some vertex in $D' \setminus \{d'\} = D \setminus \{d\}$. So d has exactly one private neighbor as claimed. □

3 Vertex Minimality

In this section, for each $\gamma \geq 2$ we find a lower bound on the number of vertices in any connected, DI-pathological graph with domination number γ. Further, we provide an example showing that this lower bound is sharp. We will start by stating known results for $\gamma(G) = 2$ and $\gamma(G) = 3$ and then use a known result for $\gamma(G) = 3$ to classify all connected, DI-pathological graphs on nine vertices. We then show that for $\gamma \geq 4$, any connected, DI-pathological graph with domination number γ has at least $2\gamma + 4$ vertices by using contradiction, specifically by using minimal counterexamples. Although it may be added for emphasis, we will assume that any given graph G is connected (and hence has no isolated vertices since $\gamma(G) \geq 2$).

For $\gamma(G) = 2$, Johnson, Prier, and Walsh showed in [5] that G is DI-pathological if and only if $G \cong K_{m,n}$ with $m, n \geq 3$. Then clearly, the unique DI-pathological graph with the minimum number of vertices is $K_{3,3}$ with $2\gamma + 2$ vertices. The next two results are due to Prier and Johnson in [4]. From the first result, we immediately get that $|V(G)| \geq 9 = 2\gamma(G) + 3$ when G is connected, DI-pathological, and $\gamma(G) = 3$. They further find a connected, DI-pathological graph with $\gamma(G) = 3$ that has 9 vertices; this graph appears as A_1 in Figure 1. We will make further use of the next two lemmas to classify the set of all connected, DI-pathological graphs on nine vertices with $\gamma(G) = 3$.

Lemma 3. [4] *Let G be a connected, DI-pathological graph with $\gamma(G) = 3$, and let D be a minimum dominating set in G. Then for all $x \in D$,* $|P_D(x)| \geq 2$.

Lemma 4. [4] *If D is a minimum dominating set in G, and $J \subseteq V(G) \setminus D$ is an independent set such that $D \subseteq N(J)$, then G is not DI-pathological.*

In [4], Prier and Johnson showed that the graph A_1 in Figure 1 is DI-pathological. Lemmas 6, 7, 8, 9, and 10 will show that the graphs A_2, A_3, A_4, A_5, and A_6 in Figure 1 are DI-pathological respectively. We will then show in Lemma 11 that there are no other connected, DI-pathological graphs on nine vertices with $\gamma(G) = 3$.

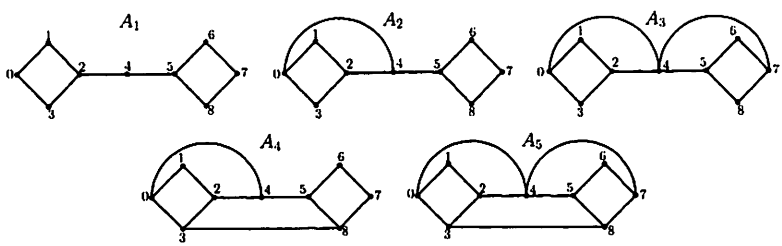

Figure 1: All of the connected, DI-pathological graphs on nine vertices with $\gamma(G) = 3$

Lemma 5. *Let A_2, A_3, A_4, A_5, and A_6 be the graphs denoted in Figure 1. Then for $2 \leq i \leq 6$, $\gamma(A_i) = 3$. Moreover, in A_2 and A_3, v_4 must be in the dominating set.*

Proof. Since the maximum degree of the vertices in A_2 and A_4 is three, it is impossible to dominate nine vertices with a set of size two. In A_3, A_5, and A_6 there is only one vertex, namely v_4 in A_3 and A_5 and v_3 in A_6, of degree at least four; hence v_4 must be in any dominating set of size two in A_3 and

A_5, and then v_1 and v_6 must be dominated by a single vertex, which is not possible. In A_6, v_3 must be in any dominating set of size two, and then v_1 and v_7 must be dominated by a single vertex which is not possible. So there is no dominating set of size two in A_3, A_5, or A_6. So for $2 \leq i \leq 6$, $\gamma(A_i) \geq 3$, and since $\{v_0, v_4, v_7\}$ dominates A_i, $\gamma(A_i) = 3$.

Now if v_4 is not in the dominating set in A_2 or A_3, then two vertices must be picked from each of $\{v_0, v_1, v_2, v_3\}$ and $\{v_5, v_6, v_7, v_8\}$ in order to dominate the graph. This is a contradiction, however, since $\gamma(A_2) = \gamma(A_3) = 3$. □

Lemma 6. *Graph A_2 in Figure 1 is DI-pathological.*

Proof. By Lemma 5, $\gamma(A_2) = 3$ and v_4 must be in any minimum dominating set. It is then clear that one vertex must dominate both v_1 and v_3, so v_0 or v_2 must be in the dominating set. Likewise, one vertex must dominate v_6, v_7, and v_8, so v_7 must be in the dominating set. Hence, $\{v_0, v_4, v_7\}$ and $\{v_2, v_4, v_7\}$ are the only two possible minimum dominating sets.

Let $D = \{v_t, v_4, v_7\}$ be a minimum dominating set where $t \in \{0, 2\}$. Since a maximal independent set is dominating, if we wish to pick a maximal independent set I disjoint from D, we must pick either v_6 or v_8 to dominate v_7 and pick either v_1 or v_3 to dominate v_t. Then every neighbor of v_4 is dominated by I but v_4 is not. So v_4 must be in I to make it maximal, so I and D cannot be disjoint. Hence A_2 is DI-pathological. □

Lemma 7. *Graph A_3 in Figure 1 is DI-pathological.*

Proof. By Lemma 5, $\gamma(A_3) = 3$ and v_4 must be in any minimum dominating set. It is then clear that one vertex must dominate both v_1 and v_3, so v_0 or v_2 must be in the dominating set. Likewise, one vertex must dominate v_6 and v_8, so v_5 or v_7 must be in the dominating set. Hence, $\{v_0, v_4, v_7\}, \{v_2, v_4, v_7\}, \{v_0, v_4, v_5\}$, and $\{v_2, v_4, v_5\}$ are the only four possible minimum dominating sets.

Let $D = \{v_t, v_4, v_s\}$ be a minimum dominating set where $t \in \{0, 2\}$ and $s \in \{5, 7\}$. Since a maximal independent set is dominating, if we wish to pick a maximal independent set I disjoint from D, we must pick either v_6 or v_8 to dominate v_s and pick either v_1 or v_3 to dominate v_t. Then every neighbor of v_4 is dominated by I but v_4 is not. So v_4 must be in I to make it maximal, so I and D cannot be disjoint. Hence A_3 is DI-pathological. □

Lemma 8. *Graph A_4 in Figure 1 is DI-pathological.*

Proof. By Lemma 5, $\gamma(A_4) = 3$. We will consider two cases, split by whether v_4 is in the minimum dominating set or not.

Case 1: Suppose v_4 is in the minimum dominating set. First notice that v_1, v_3, v_6, v_7 and v_8 still have to be dominated by the other two vertices in the dominating set. Neither v_1 nor v_6 is dominated by either $\{v_4, v_3\}$ or $\{v_4, v_8\}$. Moreover, v_1 and v_6 are not adjacent and do not share a common neighbor; hence we cannot pick v_3 or v_8 to be in the dominating set. Thus, v_6, v_7, and v_8 must be dominated by a single vertex, the only candidate being v_7, and v_1 and v_3 must be dominated by a single vertex, the two options being v_0 and v_2. Thus the only two minimum dominating sets containing v_4 are $\{v_0, v_4, v_7\}$ and $\{v_2, v_4, v_7\}$.

Let $D = \{v_t, v_4, v_7\}$ be a minimum dominating set where $t \in \{0, 2\}$. Since a maximal independent set is dominating, if we wish to pick a maximal independent set I disjoint from D, we must pick either v_6 or v_8 to dominate v_7 and pick either v_1 or v_3 to dominate v_t (of course we cannot pick both v_3 and v_8). Then every neighbor of v_4 is dominated by I but v_4 is not. So v_4 must be in I to make it maximal, so I and D cannot be disjoint if v_4 is in D.

Case 2: Suppose v_4 is not in the minimum dominating set. Clearly, in this case, the minimum dominating set must contain either one vertex from $\{v_0, v_1, v_2, v_3\}$ and two vertices from $\{v_5, v_6, v_7, v_8\}$ or vice versa.

Case 2.1: Suppose that the dominating set contains exactly one vertex from $\{v_0, v_1, v_2, v_3\}$. Then the dominating set must contain a vertex from the set $\{v_5, v_6, v_7, v_8\}$ that dominates a vertex in $\{v_0, v_1, v_2, v_3\}$, so it must contain v_8. We still need one vertex to dominate $\{v_0, v_1, v_2\}$, so we must use v_1. Then we need the last vertex of the dominating set to dominate v_4 and v_6, so we must use v_5.

Case 2.2: Now suppose the dominating set contains two vertices from $\{v_0, v_1, v_2, v_3\}$. Then the dominating set must contain a vertex from the set $\{v_0, v_1, v_2, v_3\}$ that dominates a vertex in $\{v_5, v_6, v_7, v_8\}$, so it must contain v_3. We still need one vertex to dominate $\{v_5, v_6, v_7\}$, so we must use v_6. Then we need the last vertex of the dominating set to dominate v_1 and v_4, so we must use either v_0 or v_2.

Hence the only minimum dominating sets in Case 2 are $\{v_1, v_5, v_8\}$, $\{v_2, v_3, v_6\}$, and $\{v_0, v_3, v_6\}$.

Suppose $D = \{v_1, v_5, v_8\}$ is the minimum dominating set. Since a maximal independent set is dominating, if we wish to pick a maximal independent set I disjoint from D, we must pick either v_0 or v_2 to dominate v_1, and hence v_3 cannot be in I. Then v_7 must be in I to dominate v_8. However, then every neighbor of v_5 is dominated by I but v_5 is not. So v_5 must be in I to make it maximal, so I and D cannot be disjoint.

Finally, let $D = \{v_t, v_3, v_6\}$ be a minimum dominating set where $t \in \{0, 2\}$. Since a maximal independent set is dominating, if we wish to pick a maximal independent set I disjoint from D, we must pick either v_1 or v_4 to dominate v_t (we cannot pick v_3 since it is in D). Since both v_0 and v_2 are adjacent to v_1 and v_4, neither can be in I so v_8 must be in I to dominate v_3. Then every neighbor of v_6 is dominated by I but v_6 is not. So v_6 must be in I to make it maximal, so I and D cannot be disjoint.

Combining all of this we get that A_4 is not DI-pathological. □

Lemma 9. *Graph A_5 in Figure 1 is DI-pathological.*

Proof. By Lemma 5, $\gamma(A_5) = 3$. We will consider two cases, split by whether v_4 is in the minimum dominating set or not.

Case 1: Suppose v_4 is in the minimum dominating set. First notice that v_1, v_3, v_6, and v_8 still have to be dominated by the other two vertices in the dominating set. Neither v_1 nor v_6 is dominated by either $\{v_4, v_3\}$ or $\{v_4, v_8\}$. Moreover, v_1 and v_6 are not adjacent and do not share a common neighbor; hence we cannot pick v_3 or v_8 to be in the dominating set. Thus, v_6 and v_8 must be dominated by a single vertex, the candidates being v_5 and v_7. Also v_1 and v_3 must be dominated by a single vertex, the two options being v_0 and v_2. Thus the only four minimum dominating sets containing v_4 are $\{v_0, v_4, v_7\}, \{v_2, v_4, v_7\}, \{v_0, v_4, v_5\}$, and $\{v_2, v_4, v_5\}$.

Let $D = \{v_t, v_4, v_s\}$ be a minimum dominating set where $t \in \{0, 2\}$ and $s \in \{5, 7\}$. Since a maximal independent set is dominating, if we wish to pick a maximal independent set I disjoint from D, we must pick either v_6 or v_8 to dominate v_s and pick either v_1 or v_3 to dominate v_t (of course we cannot pick both v_3 and v_8). Then every neighbor of v_4 is dominated by I but v_4 is not. So v_4 must be in I to make it maximal, so I and D cannot be disjoint if v_4 is in D.

Case 2: Suppose v_4 is not in the minimum dominating set. Clearly, in this case, the minimum dominating set must contain either one vertex from $\{v_0, v_1, v_2, v_3\}$ and two vertices from $\{v_5, v_6, v_7, v_8\}$ or vice versa.

Case 2.1: Suppose that the dominating set contains exactly one vertex from $\{v_0, v_1, v_2, v_3\}$. Then the dominating set must contain a vertex from the set $\{v_5, v_6, v_7, v_8\}$ that dominates a vertex in $\{v_0, v_1, v_2, v_3\}$ so it must contain v_8. We still need one vertex to dominate $\{v_0, v_1, v_2\}$, so we must use v_1. Then we need the last vertex of the dominating set to dominate v_4 and v_6, so we must use v_5 or v_7.

Case 2.2: Now suppose the dominating set contains two vertices from $\{v_0, v_1, v_2, v_3\}$. Then the dominating set must contain a vertex from the set $\{v_0, v_1, v_2, v_3\}$ that dominates a vertex in $\{v_5, v_6, v_7, v_8\}$, so it must contain v_3. We still need one vertex to dominate $\{v_5, v_6, v_7\}$, so we must use v_6. Then we need the last vertex of the dominating set to dominate v_1 and v_4, so we must use either v_0 or v_2.

Hence the only minimum dominating sets in Case 2 are $\{v_1, v_5, v_8\}$, $\{v_1, v_7, v_8\}$, $\{v_2, v_3, v_6\}$, and $\{v_0, v_3, v_6\}$.

Let $D = \{v_1, v_t, v_8\}$ be a minimum dominating set where $t \in \{5, 7\}$. Since a maximal independent set is dominating, if we wish to pick a maximal independent set I disjoint from D, we must pick either v_0 or v_2 to dominate

v_1, and hence v_3 cannot be in I. Then v_5 or v_7 (whichever is not v_t) must be in I to dominate v_8. Then v_4 and v_6 cannot be in I. However, then every neighbor of v_t is dominated by I but v_t is not. So v_t must be in I to make it maximal, so I and D cannot be disjoint.

Finally, let $D = \{v_t, v_3, v_6\}$ be a minimum dominating set where $t \in \{0, 2\}$. Since a maximal independent set is dominating, if we wish to pick a maximal independent set I disjoint of D, we must pick either v_1 or v_4 to dominate v_t (we cannot pick v_3 since it is in D). Since both v_0 and v_2 are adjacent to v_1 and v_4, neither can be in I so v_8 must be in I to dominate v_3. Then every neighbor of v_6 is dominated by I but v_6 is not. So v_6 must be in I to make it maximal, so I and D cannot be disjoint.

Combining all of this we get that A_5 is not DI-pathological. □

Lemma 10. *Graph A_6 in Figure 1 is DI-pathological.*

Proof. By Lemma 5, $\gamma(A_6) = 3$. We will consider two cases, split by whether v_4 is in the minimum dominating set or not.

Case 1: Suppose v_4 is in the minimum dominating set. First notice that v_1, v_3, v_6, v_7 and v_8 still have to be dominated by the other two vertices in the dominating set. Neither v_1 nor v_7 is dominated by $\{v_4, v_3\}$. Moreover, v_1 and v_7 are not adjacent and do not share a common neighbor; hence we cannot pick v_3 to be in the dominating set. Likewise neither v_1 nor v_6 is dominated by $\{v_4, v_8\}$ and neither v_1 nor v_8 is dominated by $\{v_4, v_6\}$. Moreover v_1 and v_6 are not adjacent and do not share a common neighbor and the same holds for v_1 and v_8; hence we cannot pick v_6 or v_8 to be in the dominating set. Thus, v_7 must be in the dominating set, and v_1 and v_3 must be dominated by a single vertex, the two options being v_0 and v_2. Thus the only two minimum dominating sets containing v_4 are $\{v_0, v_4, v_7\}$ and $\{v_2, v_4, v_7\}$.

Let $D = \{v_t, v_4, v_7\}$ be a minimum dominating set where $t \in \{0, 2\}$. Since a maximal independent set is dominating, if we wish to pick a maximal independent set I disjoint from D, we must pick either v_6 or v_8 to dominate v_7 and pick v_1 to dominate v_t (v_3 is adjacent to both v_6 and v_8). Then every neighbor of v_4 is dominated by I but v_4 is not. So v_4 must be in I to make it maximal, so I and D cannot be disjoint if v_4 is in D.

Case 2: Suppose v_4 is not in the minimum dominating set. Clearly, in this case, the minimum dominating set must contain either one vertex from $\{v_0, v_1, v_2, v_3\}$ and two vertices from $\{v_5, v_6, v_7, v_8\}$ or vice versa.

Case 2.1: Suppose that the dominating set contains exactly one vertex from $\{v_0, v_1, v_2, v_3\}$. Then the dominating set must contain a vertex from the set $\{v_5, v_6, v_7, v_8\}$ that dominates a vertex in $\{v_0, v_1, v_2, v_3\}$, so it must contain v_6 or v_8. We still need one vertex to dominate $\{v_0, v_1, v_2\}$, so we must use v_1. Then we need the last vertex of the dominating set to dominate v_4 and v_6 or v_4 and v_8, so we must use v_5.

Case 2.2: Now suppose the dominating set contains two vertices from $\{v_0, v_1, v_2, v_3\}$. Then the dominating set must contain a vertex from the set $\{v_0, v_1, v_2, v_3\}$ that dominates a vertex in $\{v_5, v_6, v_7, v_8\}$, so it must contain v_3. We still need one vertex to dominate v_5 and v_7, so we must use v_6 or v_8. Then we need the last vertex of the dominating set to dominate v_1 and v_4, so we must use either v_0 or v_2.

Hence the only minimum dominating sets in Case 2 are $\{v_1, v_5, v_6\}$, $\{v_1, v_5, v_8\}$, $\{v_2, v_3, v_6\}$, $\{v_2, v_3, v_8\}$, $\{v_0, v_3, v_6\}$, and $\{v_0, v_3, v_8\}$.

Let $D = \{v_1, v_5, v_t\}$ be a minimum dominating set where $t \in \{6, 8\}$. Since a maximal independent set is dominating, if we wish to pick a maximal independent set I disjoint from D, we must pick either v_0 or v_2 to dominate v_1 and hence v_3 cannot be in I. Then v_7 must be in I to dominate v_t. However, then every neighbor of v_5 is dominated by I but v_5 is not. So v_5 must be in I to make it maximal, so I and D cannot be disjoint.

Finally, let $D = \{v_t, v_3, v_s\}$ be a minimum dominating set where $t \in \{0, 2\}$ and $s \in \{6, 8\}$. Since a maximal independent set is dominating, if we wish to pick a maximal independent set I disjoint from D, we must pick either v_1 or v_4 to dominate v_t (we cannot pick v_3 since it is in D). Since both v_0 and v_2 are adjacent to v_1 and v_4, neither can be in I, so v_6 or v_8 (whichever is not v_s) must be in I to dominate v_3. Then every neighbor of v_s is dominated by I but v_s is not. So v_s must be in I to make it maximal, so I and D cannot be disjoint.

Combining all of this we get that A_6 is not DI-pathological. □

We will now show that the graphs in Figure 1 are the only connected, DI-pathological graphs on nine vertices with $\gamma(G) = 3$.

Lemma 11. *The only connected, DI-pathological graphs G on nine vertices with $\gamma(G) = 3$ are those in Figure 1.*

Proof. We will first show that any connected graph on nine vertices with $\gamma(G) = 3$ that is DI-pathological must have A_1 in Figure 1 as a subgraph. Then we will show that the six graphs in Figure 1 are the only connected graphs on nine vertices that can be DI-pathological.

Suppose that G is a connected, DI-pathological graph on nine vertices with $\gamma(G) = 3$. Let $D = \{d_1, d_2, d_3\}$ be a minimum dominating set of G. By Lemma 3 each d_i has at least two private neighbors, and since G has nine vertices, each d_i must have exactly two private neighbors. Let $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, and $C = \{c_1, c_2\}$ be the set of private neighbors of d_1, d_2, and d_3 respectively. By Lemma 4 there cannot be an independent set of vertices in $A \cup B \cup C$ such that at least one vertex is from each of A, B, and C.

Suppose that each vertex in A is adjacent to each vertex in B. If $d_j \sim d_3$ for some $j \in \{1, 2\}$, then $\{a_1, b_1, d_3\}$ is a minimum dominating set where a_1

or b_1 (when $j = 1$ or $j = 2$ respectively) has at most one private neighbor contradicting Lemma 3. So $d_2 \not\sim d_3$ and $d_1 \not\sim d_3$. If a vertex in C is adjacent to a vertex in A or B, say $c_1 \sim a_1$, then $\{a_1, b_1, d_3\}$ is a minimum dominating set where d_3 has at most one private neighbor, contradicting Lemma 3. So no vertex in C is adjacent to a vertex in A or B and $d_3 \not\sim d_1$ and $d_3 \not\sim d_2$, so G is not connected. So it is not possible for each vertex in A to be adjacent to each vertex in B. Similarly, it is not possible for each vertex in A to be adjacent to each vertex in C or for each vertex in B to be adjacent to each vertex in C.

By Lemma 4, there does not exist an independent set of vertices in $A \cup B \cup C$ such that at least one vertex is from each of the sets A, B, and C. The only way a_1 would not form an independent set with a vertex in B and a vertex in C is if one of the following scenarios occurred: a_1 is adjacent to both vertices in B or C or $a_1 \sim b_i$, $a_1 \sim c_j$, and $b_k \sim c_\ell$ where $\{i, k\} = \{j, \ell\} = \{1, 2\}$. An analogous result holds for any vertex in $A \cup B \cup C$. Also, $a_1 \not\sim a_2$ since otherwise for $j \in \{1, 2\}$, $\{a_j, d_2, d_3\}$ is a minimum dominating set and either some vertex has at most one private neighbor (if a_j is adjacent to some vertex in $B \cup C$ or d_1 is adjacent to d_2 or d_3) or G is not connected. Likewise $b_1 \not\sim b_2$ and $c_1 \not\sim c_2$. We now show A_1 is a subgraph of G in two cases, split by whether or not at least one vertex in $A \cup B \cup C$ is adjacent to both vertices in one of the other two sets. (We really show that G is either not DI-pathological, a contradiction with our assumption, or that A_1 is a subgraph of G.)

Case 1: Suppose no vertex in $A \cup B \cup C$ is adjacent to both vertices in either of the other two sets. If some vertex, say a_1, is not adjacent to either vertex in a different set, say B, then since in this case a_1 is not adjacent to some vertex $c \in C$ and c is not adjacent to some vertex $b \in B$, then $\{a_1, b, c\}$ is an independent set which dominates $\{d_1, d_2, d_3\}$ so G is not DI-pathological by Lemma 4, a contradiction. So each vertex in $A \cup B \cup C$ is adjacent to at least one vertex in each other set and since we assumed each vertex is not adjacent to both vertices in another set in this case, each vertex in $A \cup B \cup C$ must be adjacent to exactly one vertex in each other set. Using this fact and recalling that vertices in the same set are not adjacent, $G[A \cup B \cup C]$ is a 2-regular simple graph on six vertices. So this graph is either a C_6 or two disjoint C_3s. Since in this case no vertex is adjacent to two vertices in the same set, we can assume the C_6 looks like $(a_1, b_1, c_1, a_2, b_2, c_2)$ and then $\{a_1, c_1, b_2\}$ forms an independent set which dominates D, and hence G is not DI-pathological by Lemma 4, a contradiction. In the case where $G[A \cup B \cup C]$ is two disjoint C_3s, we can assume the C_3s are (a_1, b_1, c_1) and (a_2, b_2, c_2). But then $\{a_1, b_2, d_3\}$ is a minimum dominating set where d_3 has zero private neighbors, contradicting Lemma 3.

Case 2: Now suppose some vertex in $A \cup B \cup C$ is adjacent to both vertices in another set. Without loss of generality suppose $a_1 \sim b_1$ and $a_1 \sim b_2$. If $a_2 \sim c_1$, and $a_2 \sim c_2$ then A_1 would be a subgraph of G (with $v_0 = d_2, v_1 = b_1, v_2 = a_1, v_3 = b_2, v_4 = d_1, v_5 = a_2, v_6 = c_1, v_7 = d_3, v_8 = c_2$). So suppose $a_2 \not\sim c_1$. Since it is not possible for each vertex of A to be adjacent to each vertex of B, and since $a_1 \sim b_1$ and $a_1 \sim b_2$, then either $a_2 \not\sim b_1$ or $a_2 \not\sim b_2$, so say $a_2 \not\sim b_1$. Then by Lemma 4, $b_1 \sim c_1$ since otherwise $\{a_2, b_1, c_1\}$ would be an independent set which dominates D. Since a_2 and b_1 are not adjacent, one of these two must be adjacent to c_2 by Lemma 4, and since a_2 and c_1 are not adjacent, one of these must be adjacent to b_2. So one of the following must be true: (a) $a_2 \sim b_2$ and $b_1 \sim c_2$, (b) $a_2 \sim b_2$ and $a_2 \sim c_2$, (c) $b_2 \sim c_1$ and $b_1 \sim c_2$, and (d) $a_2 \sim c_2$ and $b_2 \sim c_1$.

For (a), b_1 is adjacent to both vertices in C and b_2 is adjacent to both vertices in A, so A_1 is a subgraph of G. For (b), $\{a_2, b_1, d_3\}$ is a minimum dominating set and d_3 has zero private neighbors, contradicting Lemma 3. For (c), $\{d_1, b_1, c_1\}$ is a minimum dominating set and d_1 has at most one private neighbor, contradicting Lemma 3. For (d), if $d_1 \sim d_2$ then $\{d_1, c_1, c_2\}$ is a minimum dominating set and c_2 has zero private neighbors, contradicting Lemma 3. If $d_2 \sim d_3$ then $\{d_3, a_1, a_2\}$ is a minimum dominating set and a_2 has zero private neighbors, contradicting Lemma 3. If $d_1 \sim d_3$ then A_1 is a subgraph of G (with $v_0 = a_2, v_1 = c_2, v_2 = d_3, v_3 = d_1, v_4 = c_1, v_5 = b_2, v_6 = a_1, v_7 = b_1, v_8 = d_2$). Otherwise $\{d_1, d_2, d_3\}$ is an independent set which dominates the mimimum dominating set $\{c_1, a_2, b_1\}$, so by Lemma 4, G is not DI-pathological, a contradiction. Hence if G is DI-pathological then A_1 must be a subgraph of G.

We shall now investigate which edges we could add to A_1 while keeping the graph DI-pathological. For convenience, relabel the vertices of A_1 by letting $v_i = i$ for each $0 \leq i \leq 8$. Since $\{0, 4, 7\}$ is a minimum dominating set, we cannot add any of the following edges (since otherwise at least one of the vertices 0, 4, and 7 will have at most one private neighbor so the graph will not be DI-pathological by Lemma 3): $\{0,2\}$, $\{0,5\}$, $\{0,6\}$, $\{0,8\}$, $\{1,4\}$, $\{3,4\}$, $\{4,6\}$, $\{4,8\}$, $\{1,7\}$, $\{2,7\}$, $\{3,7\}$, and $\{5,7\}$. If $\{0,7\}$ is an edge in G, then $\{0,4,5\}$ is a minimum dominating set in which vertex 4 has at most one private neighbor, contradicting Lemma 3. A similar argument can be used to eliminate several other edges from being added to A_1. The following table contains each edge we cannot add using this argument:

Edge	Minimum dominating set	Vertex with < 2 private neighbors
$\{0,7\}$	$\{0,4,5\}$	4
$\{2,5\}$	$\{2,3,7\}$	3
$\{1,3\}$	$\{3,4,7\}$	4
$\{6,8\}$	$\{0,4,8\}$	4
$\{2,6\}$	$\{0,2,8\}$	0
$\{2,8\}$	$\{0,2,6\}$	0
$\{1,5\}$	$\{3,5,7\}$	7
$\{3,5\}$	$\{1,5,7\}$	7

At this point, the only possible edges left that are not in A_1 and have not been forbidden are $\{0,4\}$, $\{4,7\}$, $\{1,6\}$, $\{1,8\}$, $\{3,6\}$, and $\{3,8\}$. Clearly, adding the edge $\{0,4\}$ results in a graph isomorphic to the graph formed by adding the edge $\{4,7\}$. Likewise, adding any single edge from $\{\{1,6\}$, $\{1,8\}$, $\{3,6\}$, $\{3,8\}\}$ results in the same graph regardless of the edge we pick. If both $\{1,6\}$ and $\{3,8\}$ are added as edges then $\{3,4,6\}$ is a minimum dominating set where vertex 4 has zero private neighbors, contradicting Lemma 3. Similarly, it is not possible to add both of the edges $\{1,8\}$ and $\{3,6\}$. If we add the edge $\{3,6\}$ without adding either of $\{0,4\}$ or $\{4,7\}$ then $\{2,3,8\}$ is a minimum dominating set and $\{0,4,7\}$ is a maximal independent set ensuring we do not get a DI-pathological graph. Similarly, we cannot add any of the edges $\{1,6\},\{1,8\}$, and $\{3,8\}$ without adding at least one of $\{0,4\}$ and $\{4,7\}$. We now have just a few possibilities left. We could add just the edge $\{0,4\}$. This is the graph A_2. We could add just the edges $\{0,4\}$ and $\{4,7\}$. This is the graph A_3. We could add the edges $\{3,8\}$ and $\{0,4\}$. This is the graph A_4. We could add the edges $\{3,8\},\{0,4\}$, and $\{4,7\}$. This is the graph A_5. We could add the edges $\{3,8\},\{3,6\}$, and $\{0,4\}$. This is the graph A_6. We could add the edges $\{3,8\},\{3,6\}$, and $\{4,7\}$. In this case, $\{1,3,4\}$ is a minimum dominating set in which vertex 1 has zero private neighbors. Finally, we could add the edges $\{3,8\},\{3,6\},\{0,4\}$ and $\{4,7\}$. Again, $\{1,3,4\}$ is a minimum dominating set in which vertex 1 has zero private neighbors. Hence A_1 through A_6 are the only connected, DI-pathological graphs on nine vertices with domination number three. □

The next lemma will show that the graph in Figure 2 is DI-pathological on $2\gamma(G)+4$ vertices. We will later show that if G is a connected, DI-pathological graph with $\gamma(G) \geq 4$, then $|V(G)| \geq 2\gamma(G)+4$ and this example shows that bound is tight.

Lemma 12. *The graph in Figure 2 is DI-pathological.*

Proof. Let G be the graph depicted in Figure 2. Vertex v_9 must be chosen as part of the dominating set, or else both v_{10} and v_{11} would have to be part

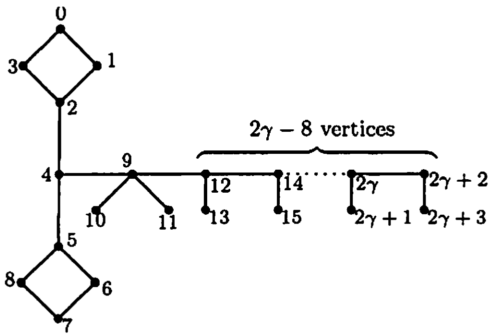

Figure 2: A DI-pathological graph with the smallest number of vertices

of the dominating set, which certainly would increase the size. In addition, v_4 must be part of the dominating set, for if it were not, two vertices would be needed to dominate $\{v_0, v_1, v_2, v_3\}$ and two vertices would be needed to dominate $\{v_5, v_6, v_7, v_8\}$ which would increase the size of the dominating set since $\{v_0, v_4, v_7\}$ dominates v_0 through v_8. We then need two vertices to dominate $\{v_0, v_1, v_3, v_6, v_7, v_8\}$ and it is clear that our only option is to pick both v_0 and v_7. Since v_9 must be an element in the minimum dominating set (and hence cannot be in the independent set) and $G[\{v_0, v_1, \ldots, v_8\}]$ is DI-pathological (it is isomorphic to A_1 in Figure 1), it follows that there are no maximal independent sets disjoint from any of the minimum dominating sets. So G is DI-pathological. □

It is clear that for $\gamma(G) \geq 4$, the graph in Figure 2 has $2\gamma(G) + 4$ vertices, and so there are connected, DI-pathological graphs on $2\gamma(G) + 4$ vertices for all $\gamma(G) \geq 4$. We have shown that $K_{3,3}$ and $K_{3,4}$ are the only connected, DI-pathological graphs with $\gamma(G) = 2$ and at most $2\gamma(G) + 3$ vertices and that the graphs A_1 through A_6 are the only connected, DI-pathological graphs with $\gamma(G) = 3$ and at most $2\gamma(G) + 3$ vertices. We will now show that every connected, DI-pathological graph G with domination number $\gamma(G) \geq 4$ has at least $2\gamma(G) + 4$ vertices. First, we prove a lemma on how the connected, DI-pathological graphs with domination number two or three are affected when we add two vertices, keep a connected graph, and increase the domination number by one.

Lemma 13. *Let $G \in \{K_{3,3}, K_{3,4}, A_1, A_2, A_3, A_4, A_5, A_6\}$ and let H be any connected graph formed from G by adding the two vertices d and v_d along with the edge $\{d, v_d\}$ such that no new edges are added between two vertices in $V(G)$. If $\gamma(H) = \gamma(G) + 1$, then H is not DI-pathological.*

Proof. First suppose that $G \in \{K_{3,3}, K_{3,4}\}$ and let $R = \{r_1, r_2, r_3\}$ and S be the partitions of G. Since H is connected, d or v_d is adjacent to G, so say d is adjacent to $x \in R \cup S$. Let X be the set of vertices in the same partition as x that are not adjacent to d. Note that d cannot be adjacent to every vertex in R since then $\{d, r_1\}$ is a dominating set of size two, contradicting the fact that $\gamma(H) = \gamma(G) + 1 = 3$. Similarly, d cannot be adjacent to every vertex in S. So X is nonempty. Hence if $X \subset R$ then X dominates S and if $X \subset S$ then X dominates R. Let y be any vertex in the partition that x is not in. Then $\{v_d, x, y\}$ is a minimum dominating set disjoint from the maximal independent set $X \cup \{d\}$ (note that $x \notin X$ since $x \sim d$), so H is not DI-pathological.

Next suppose $G \in \{A_1, A_2, A_3, A_4, A_5, A_6\}$ and let the vertices in G be denoted by the subscripts of the vertices in Figure 1. By assumption d or v_d, say d, must be adjacent to some vertex in G. We will first show that d is not adjacent to vertex 2 or vertex 5 and then we will show that d is not adjacent to vertices 1, 3, 6, or 8 by using Lemma 1. Suppose d is adjacent to vertex 2. Let $D_1 = \{0, 5, 6, d\}$. If $d \sim 7$ then $\{0, 5, d\}$ is a smaller dominating set, a contradiction. Notice that D_1 is a minimum dominating set where vertex 6 has a single private neighbor (since $d \not\sim 7$), namely 7. So by Lemma 1, $G \backslash P_{D_1}[6]$ is DI-pathological. Since $G \backslash P_{D_1}[6]$ has nine vertices, is connected, and $\gamma(G \setminus P_{D_1}[6]) = 3$, it follows that $G \setminus P_{D_1}[6] \in \{A_1, A_2, A_3, A_4, A_5\}$. But since vertex 2 has degree four, $G \setminus P_{D_1}[6]$ is not isomorphic to A_1, A_2, or A_4. Notice that vertex 2 is adjacent to vertices $1, 3$, and 4. Moreover, each of these aforementioned vertices are adjacent to vertex 0; however, the vertex of degree four in A_3 and A_5 does not have a set of three neighbors that share a common neighbor (besides the vertex of degree four itself). So d cannot be adjacent to vertex 2. A similar argument will show that d cannot be adjacent to 5. Since the argument can be made even if v_d is adjacent to vertex 2 or vertex 5, it follows that $v_d \not\sim 2$ and $v_d \not\sim 5$. Now suppose that d is adjacent to vertex 1 (a similar argument can be made for vertex 3, 6, and 8). Let $D_3 = \{3, 5, 6, d\}$. Notice that D_3 is a minimum dominating set where vertex 6 has a single private neighbor, namely 7. So by Lemma 1, $G \backslash P_{D_3}[6]$ is DI-pathological. Assume by way of contradiction that $v_d \sim 0$. Then $(v_d, 0, 3, 2, 1, d)$ creates a C_6; however, no connected DI-pathological graph on nine vertices with $\gamma(G) = 3$ contains a C_6, a contradiction. So, $v_d \not\sim 0$. Similarly, assume by way of contradiction that $d \sim 4$. Then $(d, 1, 0, 3, 2, 4)$ is a C_6, contradicting the fact that no connected DI-pathological graph on nine vertices with $\gamma(G) = 3$ contains a C_6. Hence, $d \not\sim 4$. Assume by way of contradiction that $d \sim 3$. Let $D_4 = \{1, 4, 7, d\}$. Notice that D_4 is a minimum dominating set where vertex 0 is not adjacent to d and v_d and so it is the only private neighbor of vertex 1. Then by Lemma 1, $G \setminus P_{D_4}[1]$ is DI-pathological. If $v_d \sim 4$, then we would have a contradiction since every 5-cycle shares an edge (at the very least a vertex) with every C_4 in A_1-A_5 and $(d, 3, 2, 4, v_d)$ is a 5-cycle disjoint from the 4-cycle $(5, 8, 7, 6)$ in $G \setminus P_{D_4}[1]$. Since $d \sim 3$, $v_d \not\sim 3$ (this would form a 3-cycle in $G\setminus P_{D_4}[1]$, and v_d is adjacent to 6, 7, or 8, $G\setminus P_{D_4}[1]$ cannot be isomorphic to A_1, A_2, or A_3 since $G \setminus P_{D_4}[1]$ contains a cycle of length at least 7. But then $G \setminus P_{D_4}[1]$ cannot be isomorphic to A_4 or A_5 since vertex 2, 3, 4, and 5 all have degree 2, so $d \not\sim 3$. Let $D_2 = \{0, 4, 7, d\}$. Notice that D_2 is a minimum dominating set where vertex 0 has one private neighbor, namely 3. So by Lemma 1 $G \setminus P_{D_2}[0]$ is DI-pathological. Assume by way of contradiction that $v_d \sim 4$. Then there is a disjoint C_5 and C_4 in $G \setminus P_{D_2}[0]$. However, in A_4 or A_5 (the only connected DI-pathological graphs on nine vertices with $\gamma(G) = 3$ that have a C_5), $\{v_3, v_8\}$ must be an edge in every C_5 of A_4 and A_5. Moreover, v_4 must be in every C_5, since it is the only vertex that has neighbors adjacent to both v_3 and v_8. However, both $A_4 \setminus \{v_4, v_3, v_8\}$ and $A_5 \setminus \{v_4, v_3, v_8\}$ are disconnected with three vertices in each component, so it is not possible to create a C_4 that is disjoint from the C_5 regardless of the other two vertices in the C_5, a contradiction. Hence, $v_d \not\sim 4$. Notice that vertex 2, 4, and 5 are all degree two and form a path between themselves. The only DI-pathological graphs on nine vertices with $\gamma = 3$ and a path of length two of vertices with degree two are A_1 and A_2. Since it is impossible to form either of these graphs with the remaining adjacencies in $G \setminus P_{D_4}[1]$ (d can only be adjacent to vertex 6 or 8, and v_d can only be adjacent to 6, 7, or 8), $G\setminus P_{D_4}[1]$ is not isomorphic to A_1 or A_2, so d cannot be adjacent to vertex 3. We now return to the minimum dominating set D_3. Either d or v_d must be adjacent to vertex 8 since there are no pendant vertices in A_1 through A_5. Since there are no C_6s in A_1 through A_5, $d \not\sim 8$, so $v_d \sim 8$. Since $G \setminus P_{D_3}[6]$ now contains a C_7, $G \setminus P_{D_3}[6]$ is not isomorphic to A_1, A_2, or A_3. Since d and v_d are not adjacent to vertices 0, 2, 4, and 5, there are four vertices with degree two in $G \setminus P_{D_3}[6]$, a contradiction since A_4 and A_5 have at most three vertices with degree two.

We can now assume both d and v_d are not adjacent to any of the vertices $1, 2, 3, 5, 6$, and 8. So d or v_d, say d, is adjacent to $0, 4$, or 7. If d is adjacent to vertex 4, let $D = \{0, 4, 7, v_d\}$ and let I contain $\{1, 6, d\}$. Then I is independent and dominates D so by Lemma 4, H is not DI-pathological. If d is adjacent to 0, let $D = \{0, 4, 7, v_d\}$ and let I contain $\{2, 6, d\}$. Then I is independent and dominates D so by Lemma 4, H is not DI-pathological. If d is adjacent to vertex 7, let $D = \{0, 4, 7, v_d\}$ and let I contain $\{1, 5, d\}$. Then I is independent and dominates D so by Lemma 4, H is not DI-pathological. □

The following result will be useful in the proof of Theorem 15, as it allows us to classify what graphs with $|V(G)| = 2\gamma(G)$ and no isolated vertices look like.

Theorem 14. [7] *For a graph G with even order n and no isolated vertices, the domination number equals $n/2$ if and only if the components of G are the cycle C_4 or the corona $J \circ K_1$, for any connected graph J (where $J \circ K_1$ is the graph formed from J by adding $|V(J)|$ new vertices and $|V(J)|$ new edges, the new edges forming a 1-factor with each new edge having exactly one vertex in $V(J)$).*

Now we will bring all of this together to classify the smallest DI-pathological graph in regards to the number of vertices.

Theorem 15. *Let G be a connected, DI-pathological graph with domination number $\gamma(G)$ where $\gamma(G) \geq 4$. Then $|V(G)| \geq 2\gamma(G) + 4$.*

Proof. Suppose for a contradiction that there is some connected, DI-pathological graph G with $\gamma(G) \geq 4$ and $|V(G)| < 2\gamma(G) + 4$ and pick such a G with as small a domination number as possible (still at least four). Since $\gamma(G) \geq 4$ and $|V(G)| < 2\gamma(G) + 4$, we have that $V(G) < 3\gamma(G)$. So by Lemma 2, there exists a minimum dominating set D and a vertex $d \in D$ such that d has one private neighbor, say v_d. Then by Lemma 1, $G \setminus P_D[d]$ is a DI-pathological graph (with no isolated vertices), and so one of the components in $G \setminus P_D[d]$ must be DI-pathological. Let t be the number of components in $G \setminus P_D[d]$; call the components $H_0, \ldots, H_{t-1}$ with H_0 being DI-pathological. Suppose that $|V(H_0)| \geq 2\gamma(H_0) + 4$. Since there are no isolated vertices, for $1 \leq i \leq t-1$, $|V(H_i)| \geq 2\gamma(H_i)$. Note that $\sum_{i=0}^{t-1} \gamma(H_i) = \gamma(G) - 1$. Then $V(G) = \cup_{i=0}^{t-1} V(H_i) \cup P_D[d]$, so

$$|V(G)| \geq \left(\sum_{i=0}^{t-1} 2\gamma(H_i) \right) + 4 + 2 = 2\gamma(G) - 2 + 4 + 2 = 2\gamma(G) + 4,$$

a contradiction.

So $|V(H_0)| \leq 2\gamma(H_0) + 3$. By picking a smallest counterexample, we must have $\gamma(H_0) \leq 3$ (H_0 is connected since it is a component and $\gamma(H_0) \leq \gamma(G)$). By Lemma 11 and by the results in [5], H_0 is one of the following graphs: $K_{3,3}$, $K_{3,4}$, A_1, A_2, A_3, A_4, A_5, and A_6. Let G' be the subgraph induced by $V(G) \setminus (P_D[d] \cup V(H_0))$ and let G^* be the induced subgraph on $V(H_0 \cup P_D[d])$. Since G is connected, it is clear that G^* must be connected; further, $\gamma(G^*) = \gamma(H_0) + 1$ (else G has a dominating set of size $\gamma(G) - 1$). By Lemma 13, there are a minimum dominating set D^* and maximal independent set I^* in G^* that are disjoint. Since d and v_d are adjacent, it is not possible for both to be in I^*, so say $v_d \notin I^*$.

Suppose $H_0 \in \{K_{3,4}, A_1, A_2, A_3, A_4, A_5, A_6\}$. $\gamma(G') = \gamma(G) - \gamma(H_0) - 1$. Further, $|V(G')| \leq 2\gamma(G) + 3 - (2\gamma(H_0) + 3 + 2) = 2\gamma(G) - 2\gamma(H_0) - 2$ and $|V(G')| \geq 2\gamma(G') = 2\gamma(G) - 2\gamma(H_0) - 2$, so $|V(G')| = 2\gamma(G) - 2\gamma(H_0) - 2$. In what follows, note that if we can ever dominate a component H_i with fewer than $\gamma(H_i)$ vertices by using d or v_d, then we can form a dominating set with at most $\gamma(G) - 1$ vertices, which would be a contradiction. By Theorem 14, G' is a collection of components that are C_4 or $J \circ K_1$ where J is a connected graph.

If d is adjacent to b_1 where $b_1 \in V(H_i)$ where $H_i = C_4 = (b_1, b_2, b_3, b_4)$, then we can dominate the C_4 with d and b_3 so d cannot be adjacent to any vertex of a component that is a C_4. If $J = K_1$, then d cannot be adjacent to both vertices in $K_1 \circ K_1$ since otherwise d dominates this component. If $J \neq K_1$, d cannot be adjacent to the vertices in $V(J \circ K_1) \setminus V(J)$ since otherwise we can dominate this component with $(V(J) \setminus \{j\}) \cup \{d\}$ where $j \in V(J)$ is the vertex whose neighbor in $V(J \circ K_1) \setminus V(J)$ is adjacent to d (j must be adjacent to some other vertex in $V(J)$ since J is connected).

Form D' and I' component by component as follows: If $H_i = C_4$, let D' contain two nonadjacent vertices and I' contain the other two nonadjacent vertices. If $H_i = K_1 \circ K_1$ (i.e. K_2), let I' contain a vertex not adjacent to d and D' contain the other vertex. If $H_i = J \circ K_1$ for $J \neq K_1$, let $D' = V(J)$ and let $I' = V(J \circ K_1) \setminus V(J)$. In any case, d is not adjacent to I' so $I' \cup I^*$ is independent. Then $D' \cup D^*$ is a minimum dominating set disjoint from the maximal independent set $I' \cup I^*$, so G is not DI-pathological.

Finally, suppose $H_0 = K_{3,3}$. Then G' contains $2\gamma(G)-5 = 2\gamma(G')+1$ or $2\gamma(G)-6 = 2\gamma(G')$ vertices when $|V(G)| = 2\gamma(G)+3$ or $|V(G)| = 2\gamma(G)+2$ respectively. Let $\{r_1, r_2, r_3\}$ and $\{s_1, s_2, s_3\}$ be the partition of vertices in the $K_{3,3}$ into independent sets. Since $V(G') \leq 2\gamma(G') + 1$, it is clear that G' is not DI-pathological. So let D' and I' be a minimum dominating set and maximal independent set in G' such that $D' \cap I' = \varnothing$. Since $P_D[d]$ must be adjacent to $K_{3,3}$, we may suppose v_d is adjacent to some vertex, say s_1, in $K_{3,3}$. If $v_d \sim s_2$ then $\{v_d, s_3\} \cup D'$ is a dominating set of size $\gamma(G) - 1$, a contradiction. Similarly, $v_d \not\sim s_3$. If v_d is not adjacent to I' then $I' \cup \{v_d, s_2, s_3\}$ is a maximal independent set disjoint from the minimum dominating set $D = \{r_1, s_1, d\} \cup D'$. If v_d is adjacent to I' then let $D = \{v_d, s_2, s_3\} \cup D'$ and let $I = I' \cup \{r_1, r_2, r_3\}$. Then I is an independent set that dominates the minimum dominating set D so G is not DI-pathological by Lemma 4, a contradiction. □

4 Edge Minimality

Having found the minimum number of vertices in a connected, DI-pathological graph with domination number γ for all positive integers γ, we turn to a related question, namely the question of finding the minimum number of edges in a connected, DI-pathological graph with domination number γ. In the introduction we noted that if $\gamma(G) = 2$ and G is DI-pathological, then $G \cong K_{m,n}$ with $m, n \geq 3$ (see [4]), so the minimum number of edges in such a graph is $9 = 2\gamma(G) + 5$. We start with the following theorem and corollary, which will show that trees (excluding isolated vertices) are not DI-pathological.

Theorem 16. [3] *If T is a tree other than K_1 and D is a minimum dominating set in T containing at most one leaf, then there is a maximal independent set for T contained in $V(T) \setminus D$.*

Corollary 17. *If T is a tree other than K_1, then T is not DI-pathological.*

Proof. Let T be a tree. $T = K_2$ is clearly not DI-pathological. If $T \neq K_2$, pick D to be a minimum dominating set in T that contains as few leaves as possible. If D contains any leaf, say a, let b be any neighbor of a. Since $T \neq K_2$, b is not a leaf. Futher, $(D \setminus \{a\}) \cup \{b\}$ is a minimum dominating set with fewer leaves than D, a contradiction. Hence D has no leaves and by Theorem 16, it is not DI-pathological. □

We will now proceed to find the smallest number of edges in a connected, DI-pathological graph with domination number γ for all $\gamma \geq 3$. We start by showing that the minimum number of edges in a connected, DI-pathological graph with domination number three is $10 = 2\gamma(G) + 4$, and show that only one graph satisfies this lower bound.

Theorem 18. *Suppose G is a connected, DI-pathological graph with $\gamma(G) = 3$. Then $E(G) \geq 10$ with equality if and only if $G = A_1$ in Figure 1.*

Proof. Let G be a connected, DI-pathological graph with $\gamma(G) = 3$. By Lemma 3, $|V(G)| \geq 9$. We know A_1 in Figure 1 satisfies $|E(A_1)| = 10$ and every other connected, DI-pathological graph on nine vertices has more than ten edges. If $|V(G)| \geq 11$ then $|E(G)| \geq 11$, since it is connected and trees are not DI-pathological. If $|V(G)| = 10$ then $|E(G)| \geq 10$ for the same reason. So $|E(G)| \geq 10$ as claimed. To show the second claim, we need only show that any connected, DI-pathological on ten vertices and ten edges is not DI-pathological. Let $D = \{d_1, d_2, d_3\}$ be a minimum dominating set in G. Each vertex in $V(G) \setminus D$ is adjacent to some vertex in D, so this uses at least seven edges. If there are only three edges left, then since there are seven vertices in $V(G) \setminus D$, some vertex, say x, cannot be adjacent to any other vertex in $V(G) \setminus D$, so x is only adjacent to a vertex in D, say d_1. Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ be private neighbors of d_2 and d_3 respectively (which are guaranteed to exist by Lemma 3). Again, if there are only three edges left, some vertex from B and some vertex from C cannot be adjacent. So say $b_1 \not\sim c_1$. Then $\{x, b_1, c_1\}$ is an independent set which dominates D so by Lemma 4, G is not DI-pathological, which proves the second claim. □

We now find a sharp bound for the minimum number of edges in a connected, DI-pathological graph G with $\gamma(G) \geq 4$.

Theorem 19. *Let G be a connected, DI-pathological graph with $\gamma(G) \geq 4$. Then $|E(G)| \geq 2\gamma(G) + 5$ and this bound is sharp.*

Proof. We showed in Lemma 12 that the graph in Figure 2 is DI-pathological and it clearly contains $2\gamma(G) + 5$ edges. Now suppose for a contradiction that there is a connected, DI-pathological G such that $|E(G)| \leq 2\gamma(G) + 4$ with $\gamma(G) \geq 4$ and pick such a graph with as small a domination number as possible (still greater than or equal to four). By Theorem 15, $|V(G)| \geq 2\gamma(G)+4$, and then, since G cannot be a tree, $|E(G)| \geq 2\gamma(G)+4$. So $|E(G)| = 2\gamma(G) + 4$.

Case 1: Suppose $\gamma(G) = 4$ (so $|E(G)| = 12$ and $|V(G)| = 12$) and each vertex in a minimum dominating set D has exactly two private neighbors.

Let $D = \{d_1, d_2, d_3, d_4\}$ be the minimum dominating set and let $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $C = \{c_1, c_2\}$, and $E = \{e_1, e_2\}$ be the set of private neighbors of d_1, d_2, d_3, and d_4 respectively. We will prove a claim that will be used to classify the possible degrees of the vertices in $A \cup B \cup C \cup E$.

Claim: There is at most one isolated vertex in $G[A \cup B \cup C \cup E]$.

We prove this claim by contradiction, so assume that there are at least two isolated vertices in $G[A \cup B \cup C \cup E]$. First suppose that two of the isolated vertices are in different sets, say one is e_1 and another is c_1. Then every vertex in A is adjacent to every vertex in B since otherwise there is an independent set, say $\{a_1, b_1, c_1, e_1\}$, that dominates D, so G is not DI-pathological by Lemma 4, a contradiction. But then all four remaining edges go from A to B and the graph is not connected, a contradiction. Now suppose the two isolated vertices are in the same set, say they are e_1 and e_2. Since G is connected, d_4 must be adjacent to some other vertex in D. This leaves three edges to be used in $G[A \cup B \cup C \cup E]$. Since $G[A \cup B \cup C \cup E]$ cannot have isolated vertices in different sets (we already proved this) and E has isolated vertices, each vertex in $A \cup B \cup C$ must be adjacent to some other vertex in $A \cup B \cup C$. Since there are only three edges left to be used, each vertex in $A \cup B \cup C$ must be adjacent to exactly one other vertex in $A \cup B \cup C$. So a_1 is not adjacent to either vertex in either B or C, say C, and is also not adjacent to some vertex in B, say b_2. Further, b_2 is adjacent to at most one vertex in C, so say it is not adjacent to c_1. Then $\{a_1, b_2, c_1, e_1\}$ is an independent set which dominates D, so G is not DI-pathological by Lemma 4, a contradiction. This proves the claim.

So, since there are only four edges left and there is at most one isolated vertex in $G[A \cup B \cup C \cup E]$, either every vertex in $G[A \cup B \cup C \cup E]$ has degree one or one vertex in $G[A \cup B \cup C \cup E]$ has degree two, say a_1, one has degree zero, and the rest have degree one. In either case, a_2 has degree at most one, so we can assume it is not adjacent to any vertex in C or E and that it is adjacent to at most one vertex in B; so say $a_2 \not\sim b_2$. Further, b_2 has degree at most one, so we can assume it is not adjacent to either vertex in E and that it is adjacent to at most one vertex in C; so say $b_2 \not\sim c_2$. Finally, c_2 has degree at most one, so it is adjacent to at most one vertex in E; so say $c_2 \not\sim e_2$. Then $\{a_2, b_2, c_2, e_2\}$ is an independent set which dominates D so G is not DI-pathological by Lemma 4, a contradiction.

Hence Case 1 is not a possibility for G.

Case 2: Otherwise, there is some minimum dominating set D with a vertex $d \in D$ such that d has one private neighbor v_d ($|V(G)| = 2\gamma(G)+4 < 3\gamma(G)$ if $\gamma(G) \geq 5$).

Then by Lemma 1, $G \setminus P_D[d]$ is a DI-pathological graph (with no isolated vertices), and so one of the components in $G \setminus P_D[d]$ must be DI-pathological. Let t be the number of components in $G \setminus P_D[d]$; call them $H_0, H_1, \ldots, H_{t-1}$ where H_0 is DI-pathological. Suppose $|E(H_0)| \geq 2\gamma(H_0) + 5$. Since the H_is are components (and hence connected) and are not isolated vertices, $|V(H_i)| \geq 2\gamma(H_i)$, and hence $|E(G)| \geq 2\gamma(H_i) - 1$. There must be at least one edge from d or v_d to each component (since G was connected). Further, d and v_d are adjacent. So $|E(G)| \geq \sum_{i=0}^{t-1} |E(H_i)| + t - 1 \geq 2\gamma(H_0) + 5 + \sum_{i=1}^{t-1} \gamma(H_i) - (t-1) + t + 1 = 2\sum_{i=0}^{t-1} \gamma(H_i) + 7 = 2\gamma(G) - 2 + 7 = 2\gamma(G) + 5$, a contradiction. So the DI-pathological component H_0 must have $2\gamma(H_0) + 4$ edges and hence must be A_1 (for otherwise $\gamma(H_0) \geq 4$ but $\gamma(H_0) < \gamma(G)$ and so H_1 is a smaller counterexample than G). Further, for $1 \leq i \leq t-1$, $\gamma(H_i)$ must have exactly $2\gamma(H_i) - 1$ edges (and $2\gamma(H_i)$ vertices) and there must be precisely one edge from one of d or v_d to each of the t components (else we could repeat the argument on the edge count of G to get $|E(G)| \geq 2\gamma(G) + 5$). By Lemma 13, the subgraph in G induced by the vertex set $V(H_1) \cup \{d, v_d\} = V(A_1) \cup \{d, v_d\}$ is not DI-pathological, so let D' be a minimum dominating set and I' be a maximal independent set in this induced subgraph such that $D' \cap I' = \varnothing$. For $1 \leq i \leq t-1$, H_i is a tree and hence is bipartite with partition $D_i \cup E_i$. Since there is precisely one edge from either d or v_d to H_i, we can assume that the edge is incident to D_i. Then $D^* = D' \cup \{D_i \mid 1 \leq i \leq t-1\}$ is a minimum dominating set in G and $I^* = I' \cup \{E_i \mid 1 \leq i \leq t-1\}$ is a maximal independent set in G with $D^* \cap I^* = \varnothing$. □

5 Final Remarks

Two natural questions are left unanswered by our results so far. The first question is what happens to the vertex and edge bounds if we drop the condition that our graphs must be connected. However, it seems this is a far less interesting question as one can easily show that there are graphs G with $|V(G)| = \gamma(G)$ and $|E(G)| = 0$ for all $\gamma(G) \geq 1$ if we allow isolated vertices and that there are graphs G with $|V(G)| = 2\gamma(G)+2$ and $|E(G)| = \gamma(G)+7$ for all $\gamma(G) \geq 2$ if we prohibit isolated vertices but allow the graph to be disconnected (the graph consisting of $K_{3,3}$ and independent edges meets both bounds). Further, it easy to show in both cases that the bounds are best possible.

For our second question, note that the same graph was sharp for both the vertex and edge bound. In light of there being six DI-pathological graphs on nine vertices, it does not seem likely that the graph in Figure 2 is the only DI-pathological graph on $2\gamma(G) + 4$ vertices. The graph in Figure 3 confirms this suspicion.

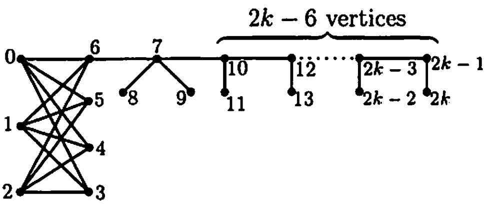

Figure 3: Another DI-pathological graph with the smallest number of vertices

However, one might wonder if the graph in Figure 2 is the only graph that is sharp for the edge bound. This is not true for $\gamma(G) = 4$ or $\gamma(G) = 5$ as evidenced in the figure 4; however, these graphs do not generalize in the same way as the graph in Figure 2, so it is remains open whether the graph in Figure 2 is the only sharp graph for the edge bound for higher γ.

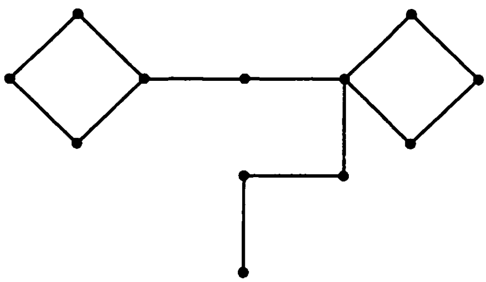

Figure 4: Two DI-pathological graphs with $\gamma = k$ (the graph on the left has $\gamma = 4$ and the graph on the right has $\gamma = 5$)

References

[1] Gayla S. Domke, Jean E. Dunbar, and Lisa R. Markus. The inverse domination number of a graph. *Ars Combin.*, 72:149–160, 2004.

[2] Allan Frendrup, Michael A. Henning, Bert Randerath, and Preben Dahl Vestergaard. On a conjecture about inverse domination in graphs. *Ars Combin.*, 97A:129–143, 2010.

[3] Michael A. Henning, Christian Löwenstein, and Dieter Rautenbach. An independent dominating set in the complement of a minimum dominating set of a tree. *Appl. Math. Lett.*, 23(1):79–81, 2010.

[4] P. D. Johnson, Jr. and D. R. Prier. Graphs such that all minimum dominating sets intersect all maximally independent sets. *Util. Math.*, 89:211–224, 2012.

[5] P. D. Johnson, Jr., D. R. Prier, and M. Walsh. On a problem of Domke, Dunbar, Haynes, Hedetniemi, and Markus concerning the inverse domination number. *AKCE Int. J. Graphs Comb.*, 7(2):217–222, 2010.

[6] V. R. Kulli and S. C. Sigarkanti. Inverse domination in graphs. *Nat. Acad. Sci. Lett.*, 14(12):473–475, 1991.

[7] T. Tamizh Chelvam and G. S. Grace Prema. Equality of domination and inverse domination numbers. *Ars Combin.*, 95:103–111, 2010.