Flanking Numbers and Arankings of Cyclic Graphs

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Abstract

Given a graph G a k-ranking is a labeling of the vertices such that any path connecting two vertices with the same label contains a vertex with a larger label. A k-ranking is minimal if and only if reducing any label violates the ranking property. The arank number of a graph $\psi_r(G)$, is the maximum k such that G has a minimal k-ranking. The arank number of a cycle was first investigated by Kostyuk and Narayan. They determined precise arank numbers for most cycles, and determined the arank number within 1 for all other cases. In this paper we introduce a new concept called the flanking number, which is used to solve all open cases. We prove that $\psi_r(C_n) = \lfloor \log_2(n+1) \rfloor + \lfloor \log_2(\frac{n+2}{3}) \rfloor + 1$ for all $n > 6$ which completely solves the problem that has been open since 2003.

1 Introduction

Given a graph G, a labeling $f : V(G) \to \{1, 2, ..., k\}$ is a *k-ranking* if and only if $f(u) = f(v)$ implies that every $u - v$ path contains a vertex w such that $f(w) > f(u)$. The rank number of a graph $\chi_r(G)$, is the smallest k such that G has a k-ranking. This problem has been well studied [1]. In 1996 Ghoshal, Laskar, and Pillone introduced the idea of minimal rankings [2]. A k-ranking f is *minimal* if for all $u \in V(G)$, a function g satisfying $g(v) = f(v)$ when $u \neq v$ and $g(u) < f(u)$, is not a ranking [2]. Then the *arank number of a graph* $\psi_r(G)$ is largest k such that G has a minimal k-ranking. Rankings with $\psi_r(G)$ labels will be referred to as *arankings*.

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The arank number of a cycle was first investigated by Kostyuk and Narayan [7]. They determined precise arank numbers for most cycles, and determined the arank number within 1 for all other cycles. In particular, they determined the following results for $m \geq 4$. These are shown in Table 1.

n	$\psi_r(C_n)$
$2^m + 2^{m-2} \leq n \leq 2^m + 2^{m-1} - 3$	$2m - 1$ or $2m$
$2^m - 2^{m-2} - 2 \leq n \leq 2^m - 2^{m-2} - 1$	$2m - 2$
$2^m - 2^{m-3} \leq n \leq 2^m - 2$	$2m - 2$ or $2m - 1$
$2^m - 2^{m-2} - 2 \leq n \leq 2^m - 2^{m-3} - 1$	$2m - 2$
$2^m - 1 \leq n \leq 2^m + 2^{m-2} - 1$	$2m - 1$

Table 1. Known arank numbers for cycles

The next open cases $\psi_r(C_{14}) = 6$, and $\psi_r(C_{20}) = \psi_r(C_{21}) = 7$ were solved by Kaplan [6]. The next unsolved case is to determine whether $\psi_r(C_{30}) = 8$ or 9. It is known that a minimal 8-ranking of C_{30} can be constructed, but it is not known whether a minimal 9-ranking of C_{30} exists. In this paper we introduce the concept of a *flanking number of a ranking* which can be used to determine if a minimal ranking of one graph can be extended to a create a minimal ranking of a larger graph. In Section 5, we use the idea of the flanking number to investigate all possible ways for constructing a minimal 9-ranking of C_{30}. All of these ways will involve extending a minimal 7-ranking of C_{15}. We carefully exhaust all possibilities and show that any 9-ranking of C_{30} is not minimal, leading to the conclusion that $\psi_r(C_{30}) = 8$. Later we apply the same ideas to all unsolved cases, showing that $\psi_r(C_n) = \lfloor \log_2(n+1) \rfloor + \lfloor \log_2(\frac{n+2}{3}) \rfloor + 1$ for all $n > 6$.

2 Preliminaries

Given a path P_n with vertices $v_1, ..., v_n$ and a ranking f, we will use the notation $\langle f(v_1), ..., f(v_n) \rangle$ to represent the labels of the vertices. For a cycle, C_n, we use the notation $\langle f(v_1), ..., f(v_n), - \rangle$ to represent the labels of each vertex in C_n.

We restate a series of known results, starting with result of Ghoshal, Laskar, and Pillone.

Theorem 1 (Ghoshal, Laskar, and Pillone [2]) *Let S_i denote the set of vertices labeled i in a ranking. Then in any minimal k-ranking $|S_1| \geq |S_2| \geq \cdots \geq |S_k|$.*

Building on this result, Kostyuk and Narayan obtained the following two theorems.

Theorem 2 (Kostyuk and Narayan [7]) *For any minimal ranking of* C_n, $|S_1 \cup S_2| \geq \frac{n}{2}$.

Theorem 3 (Kostyuk and Narayan [7]) *Let* $m \leq n$. *Then* $\psi_r(C_m) \leq \psi_r(C_n)$.

We restate a definition of Ghoshal, Laskar, and Pillone [2].

Definition 4 *For a graph* G *and a set* $S \subseteq V(G)$, *the* ***reduction*** *of* G, *denoted* $G^\flat_S$, *is a subgraph of* G *induced by* $V - S$ *with an extra edge* uv *in* $E(G^\flat_S)$ *if there exists a* $u - v$ *path in* G *with all internal vertices belonging to* S.

Unless otherwise stated, we will let the set $S = S_1$. In this article, a reduction of G will imply removal of all vertices with label 1 and the label of each remaining vertex is decreased by 1 to produce the ranking of $G^\flat_S$. For example, given C_7, where the labels are $\langle 1,2,3,2,1,4,5,-\rangle$, then the reduction, $(C_7)^\flat_S$, is $\langle 1,2,1,3,4,-\rangle$.

Lemma 5 *Let* G *be a graph and let* f *be a minimal* $\psi_r(G)$*-ranking of* G. *Then a reduction of* G *yields a minimal* $\psi_r(G^\flat_S)$*-ranking of* $G^\flat_S$.

Definition 6 *Given a graph* G, *an* ***expansion*** *of* G *is a graph* $G^\#$ *such that* $(G^\#)^\flat_S = G$.

Unless stated otherwise, an expansion is created by raising the label of vertices by one and then inserting a set of new vertices with a label of 1. To insert a vertex w, we subdivide an edge by removing edge uv and adding edges uw and vw. For example if $G = C_4$ is labeled with $\langle 1,2,1,3-\rangle$, then each of the following is an expansion of C_4 : $\langle 1,2,3,1,2,1,4,-\rangle$, $\langle 1,2,1,3,1,2,1,4,-\rangle$, or $\langle 1,2,1,3,2,1,4,-\rangle$. We note that the reductions are unique, but there may be multiple expansions.

Kostyuk and Narayan [7] presented the next lemma which involves inserting vertices labeled 1 so that the expansion is a minimal ranking. For example, if $G = C_5$ and we have the minimal ranking $\langle 1,2,1,3,4-\rangle$, we can insert a vertex labeled 1 between the vertices with label 3 and 4 to get C_6 with the minimal ranking $\langle 1,2,1,3,1,4-\rangle$.

Lemma 7 *Let f be a minimal k-ranking of G with adjacent vertices u and v where $f(u) > 1, f(v) > 1$, and $f(u) \neq f(v)$. Let $G^{\#}$ be the graph created by subdividing (u, v) and inserting a vertex w between u and v. Then let the ranking $f^{\#}$ of $G^{\#}$ be defined so that $f^{\#}(w) = 1$ and $f^{\#}(x) = f(x)$ for all $x \neq w$. Then $f^{\#}$ is a minimal k-ranking of $G^{\#}$.*

Kostyuk and Narayan [7] also presented the following:

Lemma 8 *Let f be a minimal k-ranking of G. A graph G' is created by subdividing edges of G and adding a set of vertices S that dominates G'. Then the labeling f' where $f'(x) = f(x) + 1$ for all $x \in V(G)$ and $f'(x) = 1$ for all $x \notin V(G)$ is a minimal $(k+1)$-ranking of G'.*

This lemma gives insight on how we can expand a graph G with a minimal k-ranking obtain G' with a minimal $(k+1)$-ranking. For example consider $G = C_4$ with the minimal ranking $\langle 1, 2, 1, 3- \rangle$. We expand this cycle by raising the labels first to get $\langle 2, 3, 2, 4- \rangle$ and then insert dominating set of vertices with label 1 to get $\langle 2, 1, 3, 2, 1, 4- \rangle$. It is also possible to produce $\langle 1, 2, 3, 1, 2, 4- \rangle$.

In the next section, we will present a refinement of this lemma that makes it clear which vertices need to be dominated.

3 Flanking Numbers

To help us construct cycles with arankings, we introduce the concept of a *flanking number.* The idea is to start with a ranking f and increase all of its labels by 1, producing the ranking f^+. Then we check to see which vertices in the new ranking has a label that can be reduced to 1 while respecting the definition of ranking. The flanking number of a vertex will be 1 if its label cannot drop to label of 1 and still maintain rankings, and 0 otherwise. Note that the words "drop" and "reduce" are used interchangeably here.

We formally state this in the following definition.

Definition 9 *Let (G, f) denote a graph G with a minimal ranking f, and let (G, f^+) be the ranking where $f^+(v) = f(v) + 1$ for every $v \in V(G)$. The flanking number of a vertex is a function $\zeta(v) \longrightarrow \{0, 1\}$ where $\zeta(v) = 0$ if the label of v can drop to 1 in f^+ such that the resulting graph is still a ranking, and 1 otherwise. As a shortcut, we can use $\zeta(V(G))$ as a function that assigns a flanking number to each vertex v in G.*

Example 10 *Let $G = P_5$ with the ranking $f = \langle 1, 2, 1, 3, 1 \rangle$. Then $f^+ = \langle 2, 3, 2, 4, 2 \rangle$. Then $\zeta(V(G)) = (0, 1, 0, 1, 0)$.*

The next theorem establishes a sufficient and necessary condition to ensure that the resulting graph is minimal after subdividing edges with a set of vertices labeled 1.

Theorem 11 *Let G be a graph with a minimal ranking f. Suppose that v is a vertex in G with $\zeta(v) = 0$. If $(G^{\#}, f^{+})$ is an expansion of G, and v is not adjacent to any subdividing vertex with label 1, then the resulting ranking of $G^{\#}$ is not minimal.*

Proof. Suppose $\zeta(v) = 0$ in (G, f^{+}). Consider $(G^{\#}, f^{+})$ and suppose that there is no subdividing vertex adjacent to v. Then we observe that all vertices adjacent to v will have labels greater than 1. If we change the label of $f(v)$ by dropping it to 1, then we need to check to see whether the definition of ranking holds or not. Since the paths from v to any new subdividing vertices must go through adjacent vertices, which has a label of at least 2 (recall, G has raised labels). Since the label of subdividing edges is 1 and $f(v) = 1$, and the fact that the path contains adjacent vertices with the labels of at least 2, the definition of rankings is not violated after dropping $f(v)$ to 1, therefore the ranking of $G^{\#}$ is not minimal. ■

In other words, a set of subdividing vertices must dominate the set of all vertices with flanking number of 0 in G or the resulting ranking of $G^{\#}$ is not minimal.

The next proposition will give us a method for constructing a minimal ranking. The main idea is, if vertex v has flanking number 0, then it can drop to label 1 in G^{+}. However, if we subdivide two edges adjacent to v with new vertices with labels of 1 in G^{+} to produce $G^{\#}$, then v is *flanked* by the new vertices, and v cannot drop to a label of 1 in any of the subsequent expansions. In the rest of the paper, v is *flanked* if in an expansion of a cycle, $G^{\#}$, there exists a path $u - w$ that goes through v such that the label of v is larger than the labels of vertices u and w, and all other labels of vertices on the path is smaller than u and w. v is automatically flanked soon as two adjacent subdividing vertices *flanks* it in an expansion of G. The vertex v is automatically flanked as soon as two adjacent subdividing vertices flank it in an expansion of G.

Proposition 12 *Let G be a graph with a minimal ranking f and vertex v with $\zeta(v) = 0$. If an expansion of G has a minimal ranking f^{+}, then the flanking number for a given v in $G^{\#}$ is:*

a) 0 if only one edge adjacent to v is subdivided.

b) 1 if at least two edges adjacent to v are subdivided.

Proof. First, consider that if v in $G^{\#}$ does not have any edge adjacent to it that is not subdivided, then Theorem 11 implies that the rankings of $G^{\#}$ is not minimal. Therefore in order to ensure that f^{+} is a minimal ranking, then v must have at least one adjacent edge that is subdivided. Suppose that we have only one edge adjacent to v was subdivided in the expansion of G. Then the subdividing vertex, w will have a label of 1, and v will have label that is one greater than the label it had in G. Now to check the flanking number of v, we raise all labels of $G^{\#}$, including the label of w, then we allow the label of v to drop to 1. Following the proof for Theorem 11, we only consider the relationship between v and w. Since the label of w is not equal to the label of v and $\zeta(v) = 0$ in G, therefore $\zeta(v) = 0$ in $G^{\#}$.

In the second case, let the two vertices, w_1 and w_2 subdivide the edges adjacent to v during the expansion of G. Next, let us check the flanking number of v by raising all the labels of $G^{\#}$, which implies the labels of vertices w_1 and w_2 will both be 2, then dropping the label of v to 1. This time, we need to look at the relationship between w_1, w_2, and v. Notice that there exists a path from w_1 to w_2 through v such that the label of w_1 is equal to w_2 but the label of v is smaller than the label of w_1, violating the definition of rankings. Therefore, in this case, the flanking number of v in a minimal ranking of $G^{\#}$ is 1. Note that the proof in the second case can be extended to the cases with more than 2 subdividing vertices. ■

Proposition 13 *In a cycle G containing a vertex v where $\zeta(v) = 1$, then in any expansions of G with a minimal ranking, the flanking number of v remains* 1.

Proof. If $\zeta(v) = 0$, then the label of v cannot drop to 1 in G without violating the definition of ranking. This means there exists a path from vertex w to x going through v where the label of x is equal to the label of w and the label of v is greater than both labels. Now if we expand the cyclic graph G, then all new vertices subdividing will have labels smaller than vertices v, w, x, and the labels of vertices w, x will still be equal to each other and smaller than the label of v. So there still exists a path from w to x in $G^{\#}$ through v. Therefore, the label of v cannot drop to 1, and therefore $\zeta(v) = 1$ in $G^{\#}$. ■

This proposition sheds light on the stability of flanking number. That is, once a vertex has a flanking number of 1, it will always remain flanked by the same vertices in any future expansions, and thus will never become a vertex that can drop its label to 1 in subsequent expansions. The application of this idea becomes evident later with the flanking partition structure. Next we introduce tools for identifying the flanking number for any vertex of a given graph G:

Proposition 14 *Given two adjacent vertices v and u on a path or a cycle G with a minimal ranking f, if the label of v is smaller than the label of u, then $\zeta(v) = 0$.*

Proof. Since the label of v is smaller than the label of u in (G, f) and (G, f^+), v has no bearing on whether dropping the label to 1 in f^+ would yield a path from u to any other vertices in G that would yield us a non-ranking. ■

In the above sense, u does not *flank* v. But v *flanked* u in the sense that if there are two vertices adjacent to u with the same label, but less than the label of u, then both vertices flanks that label such that the label u cannot drop to 1 without violating the definition of rankings.

Corollary 15 *Adjacent vertices on a cycle or a path cannot both have a non-zero flanking number.*

The above propositions yield tools for constructing arankings of cycles. For example we will construct C_7 with arankings and show that C_7 has a unique aranking (up to permutation of labels with the largest rankings). This will be our starting point for further constructions of arankings for open cases.

Working from the aranking of C_3 to get to C_7, there are two possible strategies for domination of all zero flanking numbers, namely two or three edge subdivisions. These produce four possibilities:

- $\langle 1, 2, 1, 3, 4- \rangle$
- $\langle 1, 2, 3, 1, 4- \rangle$
- $\langle 2, 1, 3, 1, 4- \rangle$
- $\langle 1, 2, 1, 3, 1, 4- \rangle$

The last case involving C_6 is different from all other cases, since $\zeta(C_6) = (0, 1, 0, 1, 0, 1-)$, and requires three additional vertices to dominate all vertices in C_6. Since we would get a minimum of 9 vertices in the next minimal expansion, this is not the way to reach a minimal aranking of C_7. In the three other cases, they differ only by the position of the top three labels. Without loss of generality we choose any of the three cases and demonstrate that we can get to C_7 by using two subdividing vertices. Taking

$\langle 1,2,1,3,4-\rangle$, we have $\zeta(C_5) = (0,1,0,0,0-)$. Given that C_5 has five vertices, we only can use two vertices to subdivide edges in order to produce C_7. There is a unique way of dominating four vertices with flanking number of 0 with two subdividing vertices by inserting a subdividing vertex between label 1 and 3, and label 4 and 1, so we get $\langle 1,2,3,2,1,4,5-\rangle$ for C_7.

For C_4 we have the labels $\langle 1,2,1,3-\rangle$ with $\zeta(C_4) = (0,1,0,0-)$. By Theorem 2, at least half of the vertices of C_7 must be labeled 1 or 2. Then it is clear that we cannot expand an aranking of C_4 into an aranking of C_7. Hence we are left with one arankings of C_7, which is unique up to the permutation of the top three labels. We will use this unique aranking of C_7 in the future constructions of arankings of C_{15} and some open cases.

We summarize two observations in the following lemmas. The first follows the nature of flanking numbers of newly inserted vertices, and the second involves the monotonicity property of the number of vertices with flanking numbers of zero:

Lemma 16 *Given a cycle G, if a vertex v with label 1 subdivides an edge of G^+ to produce an expansion of the graph, $G^\#$ with a minimal ranking, then $\zeta(v) = 0$.*

Proof. Since vertex v has a label of 1, the lowest possible labeling in a ranking of any graph, and using Proposition 14, we see that the label of v is smaller than the labels of adjacent vertices, and therefore $\zeta(v) = 0$. ■

Lemma 17 *Let G be a cycle with a minimal ranking, then the number of vertices with flanking number of 0 in G is equal or less than the number of vertices with flanking number of 0 in $G^\#$.*

Proof. By Lemma 16, all inserted vertices will have flanking number 0. By Proposition 12, we see that for each vertex in G with flanking number 0, we need to insert at least one vertex adjacent to each vertex to produce $G^\#$ with a minimal ranking. There are two possible types of insertions – if v in G has only one adjacent subdividing vertex in $G^\#$, then v will have a flanking number of 0, increasing the number of vertices with flanking number of 0 by 1 for each vertices in this case. Now, if there are two vertices subdividing edges adjacent to v, then the flanking number of v will become 1, but both vertex inserted will have flanking number of 0, with a gain of 1 or 0 vertices with flanking number of 0. So therefore, the number of vertices with flanking number of 0 in G is always equal to or less than the number of vertices with flanking number of 0 in $G^\#$. ■

Lemma 17 provides tools for extending minimal rankings of graphs to other minimal rankings of graphs. When a vertex with label 1 is inserted, the number of vertices with flanking number of 0 either stays the same or increases. It increases only when vertices are inserted next to a vertex that is not already dominated by another vertex. It stays the same only when the insertion of 1 is next to a vertex that is already dominated by another vertex.

Corollary 18 *Given a cycle G with a minimal ranking, the number of vertex insertions needed to dominate all vertices with flanking number of* 0 *in G on a cycle is less than or equal to the number of vertices insertions needed to dominate all vertices with flanking number of* 0 *in $G^{\#}$*

We can simplify our analysis by focusing at only components of cycle G with minimal rankings at a time, where we only look at vertices with flanking number of 0. That is, we *partition* G into a set of path by removing all vertices with flanking number of 1. We say that each partition is independent from other partitions, and we can analyze on how best expand each partition in order to get arankings for each partition.

Proposition 19 *In any cycle, vertices with flanking number* 0 *can be partitioned by removing vertices with non-zero flanking number.*

Proof. Given a cycle G with two nonadjacent vertices, $u, v \in V(G)$, and $\zeta(u, v) = (1, 1)$, we observe that u and v will never have flanking number 0 in any expansions of G. This follows by Proposition 13. Suppose we insert a vertex w into G^{+}, and we consider two possible paths from u to v, one that contains w and one that does not. Now observe that the label of w is smaller than the labels of both u and v, so it has no hope of flanking a vertex that is on the path uv without w. This implies that we can focus on each path segment between each vertex with flanking number 1 independently. ■

Next we will investigate the construction of arankings of larger cycles, and generating all possible arankings for C_{2^k-1}.

Corollary 20 *The number of vertices with flanking number* 0 *will always remain the same or increase in each partition.*

Proof. This corollary follows directly from Proposition 19 and Corollary 18, showing that inside each partition, the number of vertices that need to be dominated by inserted vertices with label 1 will always increase or remain the same. ■

Extending Theorem 11, and applying Lemma 19, we can use vertices with flanking number 1 to subdivide all vertices with flanking number 0 into sets of vertices that must be dominated by vertices inserted into G^+. This allows us to look at each partition individually:

Lemma 21 *Given a cycle G, if there exist m consecutive vertices with flanking number 0, then the expansion of the subgraph induced by m consecutive vertices must have at least $\frac{m}{2}$ insertions of vertices with label 1 in order to dominate all m vertices and achieve a minimal ranking for the expanded graph.*

Proof. If any vertex with flanking number 0 is not dominated by a vertex insertion during the expansion, then by Theorem 11 the graph $G^\#$ is not minimal. Also, since each vertex insertion is capable of dominating two vertices in the m consecutive vertices at the same time in a cycle, then the number of insertions that will dominate all vertices is at least $\frac{m}{2}$. ∎

This proposition gives us the tools that allow us to count the number of vertices we need to insert in order to dominate all vertices with flanking number 0 in order to preserve the minimality of a ranking.

Theorem 22 *All minimal rankings of cycle C_n are constructed by expanding multiple times from C_3, C_4 where in each expansion, all vertices with flanking number of 0 is dominated by subdividing vertices with label 1.*

Proof. Given any cycle with a minimal ranking, observe that the largest two labels cannot be equal to each other, or the definition of ranking would be violated. Furthermore, also observe that the second largest and third largest labels cannot be equal to each other in a cycle or the ranking would be violated, since there exists a path from both vertices that does not go through a vertex with the largest label on a cycle. So we need to look at how many vertices have a label that is equal to the third largest label in a ranking of a cycle. Note that if there are no other vertex with label to third largest label, then reduction of the cycle to only three vertices would yield C_3. Now, if we assume that there is one other vertex, then we need to place it such that the path from a vertex with third largest label to other vertex with the same label must go through vertex with the largest label or a vertex with the second largest label, otherwise the definition of ranking would be violated. Reducing this cycle yield us C_4. Finally, if we try to put in one more vertex with the label equal to the third largest label, then there will be a path that links two vertices with the third largest label, violating the definition of rankings, so the resulting cycle is not minimal.

Therefore, we only can expand from C_3 and C_4 to generate all possible minimal rankings of C_n. ■

Using Theorem 22 we will be able to construct all possible cycles with minimal rankings, and consequently all arankings of cycles.

Note that with Theorem 2, we will only consider the expansions from C_3 when dealing with open cases for arankings.

4 Flanking Partition Structure

We begin with two propositions.

Proposition 23 *Given $2m + b, b \in \{0, 1\}$ consecutive vertices where each vertex has flanking number of 0 and $m + r + b$ (where $0 \leq r < m$) labels to insert in the graph during the expansion process, the maximum number of flankings that can occur is $2r + b$.*

Proof. We apply the pigeonhole principle for both the odd and the even case. First, observe that to optimize the number of flankings, we need to make sure that each subdividing vertex is shared by two vertices in the path. So in this case, we need to insert one vertex each for both first and last edge of consecutive vertices with flanking number of 0. Next, distribute $m + b - 2$ subdividing vertices among vertices so that every possible vertex in G with flanking number 0 is dominated. Note that in the $b = 1$ case, there will be one subdividing vertex that will flank a vertex in G. After distributing the vertices, we insert the remaining labels, and we get two flankings for each insertion. Therefore, the maximum number of flankings that can occur is $2r + b$. ■

Proposition 24 *Given a cycle or a path G with m vertices with flanking number of 0, if the insertion of r vertices yields s flankings. Then in an expansion of $G^{\#}$, the number of vertices that need to be dominated to ensure that $G^{\#^{\#}}$ has a minimal ranking is $m + r - s$.*

Proof. Recall that if two subdividing vertices are inserted adjacent to vertex v with flanking number 0, then v will gain flanking number of 1, removing it from the set of vertices with flanking number 0. If we flank s vertices, then they cannot be included in the set of vertices with flanking number 0. All of the newly inserted vertices will have flanking number 0, so we add r to the number of vertices with flanking number 0, leaving us with $m + r - s$. ■

The combination of the above two propositions and Proposition 19 will be useful in our approach.

Definition 25 *The flanking partition structure is a set of consecutive vertices with flanking number* 0 *separated by a vertex with flanking number* 1*. Let us establish that* $[m_1, m_2, ..., m_k]$ *notation on a cycle where we have* m_1 *consecutive vertices with flanking number* 0*, ending with one vertex with flanking number* 1*, then* m_2 *consecutive vertices with flanking number* 0*, then one vertex with flanking number* 1*, and so on.*

For example, consider the following arankings of C_{15} with its respective flanking numbers:

$$\langle 5,4,1,2,3,2,1,6,7,1,2,3,2,1,4-\rangle \rightarrow (1,0,0,0,1,0,0,0,0,0,0,1,0,0,0-).$$

The partition structure of the above example is $[3,6,3]_p$. Next, we will use the idea of partition structure to establish all possible arankings of C_{15}.

5 Arank of C_{30}

Now that we have all the tools along with Theorem 22, Theorem 2, and the language of partition structures, we will start looking at the arank number of C_{30}. We will show that there is no way of constructing 9-rankings of C_{30} using arankings of C_{15}. We will first take all seven arankings of C_{15} (up to rotation in cycles and permutation of the top three labels) and attempt to construct a 9-ranking of C_{30} and observe why it is not possible in each case. First, we transform each aranking of C_{15} into a partition structure as follows:

$$\begin{aligned}
&\text{a) } \langle 5,1,4,2,1,3,6,1,2,7,1,3,2,1,4-\rangle \rightarrow [14]_p\\
&\text{b) } \langle 5,4,1,2,1,3,6,1,2,7,1,3,2,1,4-\rangle \rightarrow [2,11]_p\\
&\text{c) } \langle 5,4,1,2,3,1,6,1,2,7,1,3,2,1,4-\rangle \rightarrow [5,8]_p\\
&\text{d) } \langle 5,4,1,2,3,1,6,2,1,7,1,3,2,1,4-\rangle \rightarrow [8,5]_p\\
&\text{e) } \langle 5,4,1,2,3,1,6,2,1,7,3,1,2,1,4-\rangle \rightarrow [11,2]_p\\
&\text{f) } \langle 5,4,1,2,3,1,6,2,1,7,3,1,2,4,1-\rangle \rightarrow [14]_p\\
&\text{g) } \langle 5,4,1,2,3,2,1,6,7,1,2,3,2,1,4-\rangle \rightarrow [3,6,3]_p
\end{aligned}$$

With these constructions, we can now move on to prove the following proposition:

Proposition 26 $\psi_r(C_{30}) = 8$.

Proof. Consider C_{30}. By Theorem 2, at least half of the vertices of C_{30} must be labeled 1 or 2. So, we only can use cycles with 15 or fewer vertices. Since $\psi_r(C_{14}) = 6$ and $\psi_r(C_{15}) = 7$, $\psi_r(C_{30}) = 8$ or 9. We can find an 8-ranking of C_{30} easily, but we need to see if it is possible to extend a 7-ranking of C_{15} to a minimal 9-ranking of C_{30}.

Note that with 15 vertices to insert into C_{15} in a series of two expansions, we will need to determine how to divide up vertices into two groups of vertex insertions. First, observe that for each ranking, we would need to insert at least 7 vertices in order to dominate a set of vertices with flanking number of 0. Finally, by Theorem 1, we only can insert 7 subdividing vertices in the first expansion of C_{15}, and 8 subdividing vertices in the expansion of expansion of C_{15} to get C_{30}

Now, starting with cases (a) and (f), with partition structure of $[14]_p$. Observe that there is only one way of dominating fourteen vertices with flanking number of 0 using 7 subdividing vertices. This will give us a partition structure $[21]_p$. With this structure, we will need at least eleven subdividing vertices needed to dominate the vertices with flanking number of 0 in $C_{15}^{\#}$ in this case. Therefore, we cannot obtain a 9-ranking from these two cases.

The next cases, (b) and (e) have a partition structure of $[2, 11]_p$. The only possible way of inserting 7 subdividing vertices would be putting one vertex in the partition with length 2, and 6 vertices in the partition with length 11. We can expect at most one flanking in the partition with length 11 and no flanking in the partition with length 2, so the total number of vertices with flanking number of 0 in the expansion of C_{15} is 3 in component and 16 in other component. Therefore, in the next expansion, it would require at least 2+8 vertex insertions to dominate all vertices with flanking number of 0 to ensure the resulting cycle to have a minimal ranking. So we eliminate (b) and (e). Similar arguments can be made for (c) and (d).

Finally, we are left with the last case, (g), which has three partitions with lengths 3, 6, and 3. It is clear that we need two vertex insertions to dominate the partitions with length 3, and three vertex insertions to dominate the partition of length 6. This gives us seven vertices for the first expansion. However, since there is a unique way to dominate the partition of length 6, and in the expansion of the C_{15}, we can see that the number of vertices with flanking number 0 in that particular partition increases to 9, which implies that it needs five vertices to dominate it completely. Since the two other partitions already need two vertices to dominate each, more than 8 vertices are needed to reach C_{30}. Hence, $\psi_r(C_{30}) = 8$. ■

Notice that in the previous proof, it is possible to simply use the idea of flanking numbers to construct a graph, and also to help motivate the search for the arank number of a cycle. It is a good exercise to use the ideas above, along with the seven cases delineated for C_{15} to demonstrate that there are only seven possible cases of C_{31} that can be reached from C_{15} (and all of these are derived from case (g), and cannot come from any other cases).

Our work thus far makes it easier for us to understand how to focus our efforts on each partition, as long as we focus on each partition independently. It is easier to focus on the number of vertices within a partition than focusing on each vertex. How we focus on each partition will become clearer with each proof of the exhaustion lemmas (named as we will exhaust all possible labelings for insertions) below. Before we move on to the exhaustion lemmas, we will define the notation of "insertions" within a graph. For example if one partition has three consecutive vertices with flanking number 0, then we denote this as:

$$0-0-0$$

Now suppose we wish to insert two vertices. Both insertions will occur between two 0's or at an end, and will be denoted with an asterisk. All possible insertions are as follows:

$$\begin{array}{c} *-0-0-*-0 \\ 0-*-0-*-0 \\ 0-*-0-0-* \end{array}$$

Note that after inserting two vertices in first and second, we get a partition structure of $[5]_p$, but for the middle case, we get a partition structure of $[2,2]_p$. Before going further with this, we will need to make another observation: in order to get arank number and arankings of a cycle of a given length, we need to find a construction such that it uses the fewest vertex insertions over two expansions from one cycle with rankings to other cycle in order to determine arankings for open cases. However, while we are trying to find the fewest vertices, we are also limited by the number of vertices with flanking number of 0, as it determines how many vertices we would need to insert to ensure that the ranking is minimal in subsequent expansions of a cycle.

Before proving the general case, we need to first study the basic partitions such as $[3]_p$ and few others. Note that we already inserted two vertices to produce three possible cases with the following partition structures, $[5]_p$,

$[2,2]_p$, and $[5]_p$. Note that in the first and last partition structures, we need three vertex insertions to dominate the vertices with flanking number of 0. Compare to the middle one, which needs only two vertex insertions, yielding partition structure of $[3,3]_p$! So therefore, if we have four vertices to insert over two expansions of $[3]_p$ (which resembles a path), then the only possible partition structure would be $[3,3]_p$.

Formalizing the above into a lemma:

Lemma 27 *Given the partition structure of* $[3]_p$*, the fewest number of vertex insertions over two expansions is* 4*. The partition structure after the first expansion is is* $[2,2]_p$*, and the partition structure after two expansions of* $[3]_p$ *is* $[3,3]_p$.

Lemma 28 *Given the partition structure of* $[6]_p$*. The fewest number of vertex insertions over the series of two expansions of* $[6]_p$ *is* 8*, yielding seven possible partition structures:*

$[14]_p$, $[11,2]_p$, $[8,5]_p$, $[5,8]_p$, $[2,11]_p$, $[14]_p$, *and* $[3,6,3]_p$

Proof. The case with three vertices inserted into the partition size of 6 is trivial. As each vertex insertion must dominate two vertices, and since no vertices were flanked, we are left with partition structure of $[9]_p$, which will require 5 vertices to dominate in the subsequent expansions, which will yield 6 possible partition structures as follows:

$(*,0,0,*,0,0,*,0,0,*,0,0,*,0) \rightarrow [14]_p$
$(0,*,0,*,0,0,*,0,0,*,0,0,*,0) \rightarrow [2,11]_p$
$(0,*,0,0,*,0,*,0,0,*,0,0,*,0) \rightarrow [5,8]_p$
$(0,*,0,0,*,0,0,*,0,*,0,0,*,0) \rightarrow [8,5]_p$
$(0,*,0,0,*,0,0,*,0,0,*,0,*,0) \rightarrow [11,2]_p$
$(0,*,0,0,*,0,0,*,0,0,*,0,0,*) \rightarrow [14]_p$

Next, we seek to insert an additional four vertices, and we will check all possible cases, and look at the number of vertices needed to ensure the subsequent expansion would yield a cycle with a minimal ranking. All possible insertions of four vertices into $[6]_p$ yields:

$(*,0,*,0,0,*,0,0,*,0) \rightarrow [1,8]_p$
$(*,0,0,*,0,*,0,0,*,0) \rightarrow [4,5]_p$
$(*,0,0,*,0,0,*,0,*,0) \rightarrow [7,2]_p$
$(*,0,0,*,0,0,*,0,0,*) \rightarrow [10]_p$
$(0,*,0,*,0,*,0,0,*,0) \rightarrow [2,1,5]_p$

$(0,*,0,*,0,0,*,0,*,0) \rightarrow [2,4,2]_p$
$(0,*,0,*,0,0,*,0,0,*) \rightarrow [2,7]_p$
$(0,*,0,0,*,0,*,0,*,0) \rightarrow [5,1,2]_p$
$(0,*,0,0,*,0,*,0,0,*) \rightarrow [5,4]_p$
$(0,*,0,0,*,0,0,*,0,*) \rightarrow [8,1]_p$

Note that for all partition structures except for $[2,4,2]_p$, an insertion of five vertices is needed to dominate all vertices for the next expansion. With $[2,4,2]_p$, we only need four vertex insertions, and this insertion is unique, yielding the partition structure of $[3,6,3]_p$ ■

Lemma 29 *Given $[14]_p$, the fewest vertex insertions through two expansions to ensure minimal ranking for a given partition requires a total of* 18 *vertex insertions.*

Proof. We begin with $[14]_p$, which requires seven subdividing vertices. Using Proposition 24 and Proposition 23, we get $[21]_p$, which implies that we need at least eleven vertex insertions to dominate the partition structure, getting 18 insertions over two expansions. If we attempt to insert eight vertices into $[14]_p$, then we will have $14+8-2=20$ vertices with flanking number of 0. This implies we will need at least ten additional vertex insertions, giving us a total of 18 vertices. Since we have a total of 18 vertices, we would consider 9 vertex insertions in the first expansion, but by Theorem 1, we would need at least 9 vertex insertions in the next expansion. Therefore, the number of fewest vertex insertions over two expansions of $[14]_p$ is 18. ■

Lemma 30 *Given $[11,2]_p$, the fewest vertex insertions through two expansions to ensure minimal ranking for a given partition requires a total of* 17 *vertex insertions*

Proof. First, we attempt to insert the fewest amount of vertices for both partitions, which is 7 vertices – 6 vertices for partition $[11]_p$, and one vertex for partition $[2]_p$. By Propositions 23 and 24, we see that we have $11+6-1=16$ vertices with flanking number of 0 for larger partition, and 3 vertices for other partition. Now, to ensure minimal ranking in the subsequent expansion, we need at least $8+2=10$ more vertex insertions, giving us a total of 17.

Now, we consider inserting 8 vertices, and observe that we have two ways of doing so: insert 7 vertices into $[11]_p$ and 1 into $[2]_p$ or insert 6 vertices into $[11]_p$ and 2 into $[2]_p$. In the first case, we get $11+7-3=15$

vertices with flanking number of 0 in the larger partition, and in other partition, $2+1=3$ vertices with flanking number of 0. This implies that we will need at least 8 vertices in first partition, and 2 vertices in other partition, with the total of 10 vertex insertions for the second expansion to ensure that the ranking would remain minimal. This gives us 18 vertex insertions in this case, which is greater than the above 17 insertions.

In the second case, we get at least $11+6-1=16$ vertices with flanking number of 0 in the larger partition, and in the other partition, at least $2+2-1=3$ vertices with flanking number of 0. This implies we need $8+2=10$ additional vertex insertions, giving us a total of 18 vertex insertions. Since 17 vertex insertions are still the smallest possible amount of vertex insertions over two expansions, we do not need to consider 9 vertex insertions in the first expansion. ∎

Lemma 31 *Given* $[5,8]_p$, *the fewest vertex insertions through two expansions to ensure minimal ranking for a given partition requires a total of* 17 *vertex insertions*

Proof. There is a unique way of dividing seven vertex insertions between the partitions in this partition structure $[5,8]_p$. This is done by inserting three vertices into $[5]_p$ and four vertices into $[8]_p$. This results in $[5+3-1]_p$ and $[12]_p$, which implies that we need $4+6$ vertices to dominate all vertices, giving us the total of 17 vertices over two expansions.

Now we look at inserting eight vertices, and notice that there are two possible ways of dividing eight vertex insertions between the partitions in $[5,8]_p$. One way is to insert four vertices in $[5]_p$, and four vertices in $[8]_p$, and the other way is to insert three vertices into $[5]_p$ and five vertices into $[8]_p$. In the first case, we get $[5+4-3,12]_p$ which implies that we need to insert at least nine vertices in the next expansion, giving the total of 17 vertex insertions over two expansions. But in the second case, we get $[5+3-1]_p$, $[8+5-2]_p$ which implies we need at least $4+6$ vertices to dominate all vertices in the next expansion, giving us the total of 18 vertex insertions over two expansions of $[5,8]_p$. ∎

We conclude this section with the following lemma.

Lemma 32 *Given a set of partitions of vertices with flanking number* 0 *in a cycle, the number of vertices with nonzero flanking number is equal to the number of partitions given. The number of vertices of a graph can be determined by adding the sizes of all partitions and the number of partitions in a graph.*

Proof. Each partition on a cycle is separated by a vertex with non-zero flanking number. Thus, the number of vertices with non-zero flanking number in the flanking partition structure is determined by counting the number of partitions. Therefore, the number of vertices in a graph is determined by adding the lengths of all partitions and the number of partitions in a graph. ■

6 The arank number of a cycle

Finally we next investigate the remaining open cases involving arank numbers of cycles. Before jumping into next two theorems, let us define a special type partition structure, where we have repeated partitions:

Definition 33 *If we have a partition of length k repeated r times, then we write the following partition structure as:* $[k_r]_p$.

Theorem 34 *Suppose that* $\psi_r(C_{2^m-2}) < \psi_r(C_{2^m-1})$ *and the following partition structures of* C_{2^m-1} *for* $m > 3$ *are given by:*
$[14, 3_{(2^{m-2}-4)}]_p$, $[2, 11, 3_{(2^{m-2}-4)}]_p$, $[5, 8, 3_{(2^{m-2}-4)}]_p$,$[3, 6, 3, 3_{(2^{m-2}-4)}]_p$.
Then $\psi_r(C_{2^m+2^{m-1}-3}) < \psi_r(C_{2^m+2^{m-1}-2})$.

Proof. Since $\psi_r(C_{2^m-2}) < \psi_r(C_{2^m-1})$, we will begin with $\psi_r(C_{2^m-1})$. We seek to dominate all $2^{m-2} - 4$ partitions of size 3. Since we need to insert two vertices for each of partition of size 3, we need to insert at least $2^{m-1} - 8$ vertices. Finally, we consider $[14]_p$, $[2, 11]_p$, $[5, 8]_p$, and $[3, 6, 3]_p$ and we see that we need to insert at least seven vertices for each, giving us a total of at least $2^{m-1} - 1$ vertices that need to be inserted into C_{2^m-1}. Expanding yields a minimal ranking of $C_{2^m+2^{m-1}-2}$. Thus, it is impossible to reach $C_{2^m+2^{m-1}-3}$ from C_{2^m-1} and expect to maintain a minimal ranking. Therefore, $\psi_r(C_{2^m+2^{m-1}-3}) < \psi_r(C_{2^m+2^{m-1}-2})$ for all $m > 3$. ■

The above theorem is an important component of the next theorem, which has two parts.

Theorem 35 *a) For every* $m > 3$, C_{2^m-1} *has seven arankings (up to permutation of the top 3 labels), and the partitions are as follows:* $[14, 3_{(2^{m-2}-4)}]_p$ *(two cases),* $[2, 11, 3_{(2^{m-2}-4)}]_p$ *(two cases),* $[5, 8, 3_{(2^{m-2}-4)}]_p$ *(two cases),* $[3, 6, 3, 3_{(2^{m-2}-4)}]_p$ *(one case)*

b) For all $m > 3$, $\psi_r(C_{2^m-2}) < \psi_r(C_{2^m-1})$.

Proof. The proof follows by induction on m. For the base cases we recall that $\psi_r(C_{14}) < \psi_r(C_{15})$, and that $\psi_r(C_{30}) < \psi_r(C_{31})$, and that for both C_{15} and C_{31}, there are exactly seven arankings up to permutation of the top three labels. We assume that both of the above statements are true for m, and seek to extend to $m+1$ by induction.

Observe that for each partition structure, we have $[3_{(2^{m-2}-4}]_p$. By Lemma 27, we observe that we need 4 vertices for each partition of length 3 over two expansions, yielding $2^{m-1}-8$ partitions with length 3, with the total of $2^m - 16$ vertex insertions over two consecutive expansions.

With this in mind, we look at other partitions. For $[14]_p$, by Lemma 29, we see that the number of the fewest vertex insertions over two expansions is 18, giving us the total of 2^m+2 insertions over two expansions, yielding minimal rankings for $C_{2^m-1+2^m+2} = C_{2^{m+1}+1}$, which does not suffice, since we want arankings for $2^{m+1}-1$.

Now we look at $[11,2]_p$, and again, by Lemma 30, we see that we need 17 vertex insertions along with insertions for partitions of length 3, which yields arankings for $C_{2^{m+1}}$. Similarly for $[5,8]_p$, using Lemma 31. Both of those partition structures also do not work.

Finally, we look the last partition structure, $[3,6,3]$, and apply Lemma 27 for partitions of length 3. We need 8 vertex insertions to produce $[3_4]_p$, and apply Lemma 28 for $[6]_p$. We need 8 vertex insertions over two expansions to produce the follow partition structures of:

$[14]_p$, $[11,2]_p$, $[8,5]_p$, $[5,8]_p$, $[2,11]_p$, $[14]_p$, and $[3,6,3]_p$

And yielding arankings for $C_{2^{m+1}-1}$ with the following partition structure:

$[14, 3_{(2^{m-1}-4)}]_p$ (two cases);

$[2, 11, 3_{(2^{m-1}-4)}]_p$ (two cases);

$[5, 8, 3_{(2^{m-1}-4)}]_p$ (two cases); and

$[3, 6, 3, 3_{(2^{m-1}-4)}]_p$ (one case).

Note that we cannot construct $C_{2^{m+1}-2}$ by expanding the arankings of C_{2^m-1}, since the minimum number of vertices needed to dominate every vertex with flanking number of 0 in C_{2^m-1} over two expansions is 2^m in order to maintain a minimal ranking. But we just shown that it is possible to generate arankings for $C_{2^{m+1}-1}$ by expanding twice on $C_2^m - 1$ by inserting fewest vertices possible over two expansions, thus, showing that it is impossible to reach $C_{2^{m+1}-2}$ by expanding arankings of $C_2^m - 1$ twice. So $\psi_r(C_{2^{m+1}-2}) < \psi_r(C_{2^{m+1}-1})$, completing the proof by induction. ∎

Corollary 36 *For all $n > 6$, $\psi_r(C_n) = \lfloor \log_2(n+1) \rfloor + \lfloor \log_2\left(\frac{n+2}{3}\right) + 1 \rfloor$.*

We conclude with a surprising corollary stating that even as $m \geq 4$ increases the number of arankings of C_{2^m-1} stays constant.

Corollary 37 *For any $n \geq 4$ there are exactly seven arankings of C_{2^m-1}.*

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