A Collection of Results on Saturation Numbers

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Abstract

A graph G is H-saturated if G does not contain H as a subgraph, but the addition of any edge between two nonadjacent vertices in G results in a copy of H in G. The saturation number $sat(n,H)$ is the smallest possible number of edges in a n-vertex H-saturated. The values of saturation numbers for small graphs and n are obtained computationally, and some general results for some specific path unions are also obtained.

1 Introduction

A graph G is H-saturated if G does not contain H as a subgraph, but the addition of any edge between two nonadjacent vertices in G results in a copy of H in G. The saturation number $sat(n,H)$ is the smallest possible number of edges in a n-vertex H-saturated graph. The saturation number is closely tied to the extremal number $ex(n,H)$, which is the *largest* possible number of edges in a n-vertex H-saturated graph. However, although the saturation and extremal numbers are closely related, their behavior can be dramatically different. For example, while the extremal number evaluated as a function of n is a nondecreasing function for any graph H, the saturation number can drop for certain values of H (and for the case of H=P_4, even drop infinitely often). In addition, while there are a considerable number of results on extremal numbers, the body of work on saturation numbers is far less extensive. While some results on the saturation numbers for certain classes of graphs, such as complete graphs, small cycles, and paths are known, most other graphs' saturation numbers are completely unknown. In this paper, some computational results on small saturation numbers and some results on general saturation numbers for specific path unions will be presented.

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We begin with results on small saturation numbers. Finding saturation numbers is a difficult problem, as currently the only known method for finding saturation numbers is direct search over a set of graphs. Direct constructions have been proposed for specific families of graphs, but as of yet no algorithm exists for exactly deriving a minimally saturated graph. An example strategy of direct generation, (by adding edges to a starting graph in such a way such that an undesired subgraph H is never formed) is not better than the method used in this paper, as to guarantee that the saturation number has been exactly found we must implicitly look at every graph in the set of H-free graphs. For the method explored in this paper, each graph requires at least one subgraph isomorphism test (to see if the graph already contains the target subgraph H), with the saturated graphs each requiring $\binom{n}{2}+1$ applications of the subgraph isomorphism algorithm. As the number of nonisomorphic graphs is extremely large even for small numbers of vertices, the problem of directly finding saturation numbers for arbitrary graphs is only feasible for graphs of a small number of vertices.

To make a computer program capable of finding saturation numbers, a set of graphs was required. For this, Brendan McKay's tool nauty was used; graphs of every order were generated, and with the help of some python scripts, the data was trimmed down to just the edge set data of a graph. Now, a tool that could operate on the graph sets and do the necessary subgraph isomorphism tests was required. For this, we used Aric A. Hagberg, Daniel A. Schult and Pieter J. Swart's python module networkx. However, networkx's implementation of VF2 only checks for induced subgraphs. This can be worked around by using line graphs. Except in the case of searching for claw or triangle saturation, line graphs allow one to check for general subgraph isomorphism with only induced subgraph isomorphism. A proof of this result is provided in the following lemma:

Lemma 1. *Assume A, B do not contain disjoint components isomorphic to C_3 or $K_{1,3}$. Then $A \subseteq B$ iff $L(A) \subseteq L(B)$ as an induced subgraph, where $L(X)$ represents the line graph of X.*

Proof. We know adding edges to a graph A does not change the original graph's underlying structure, so if $A \subseteq B$ then $L(A) \subseteq L(B)$. We also know that taking a subgraph of a line graph does not change the adjacency structure of the vertices in subgraph, so since A, B do not contain disconnected components isomorphic to C_3 or $K_{1,3}$, $A \subseteq B$ only if $L(A) \subseteq L(B)$. Therefore, $A \subseteq B$ iff $L(A) \subseteq L(B)$. We now can progress in the proof.

($\Leftarrow$) By the above result, we have the left arrow case already resolved.

($\Rightarrow$) Assume otherwise: $A \subseteq B$ but $L(A) \not\subseteq L(B)$ as an induced subgraph. We know by the above result that $A \subseteq B$ implies $L(A) \subseteq L(B)$, so we must have that the copy of $L(A)$ in $L(B)$ is not induced. So, there must be two vertices x, y in

L(A) such that $xy \in E(L(B))$ but $xy \notin E(L(A))$. This means that the *edges* x and y are incident in B but not incident in A. Define u to be the endpoint shared by both x and y in B. Since x and y are not incident in A, we must have $u \notin A$. But since x and y must have both endpoints in A if they are in A, we get that $x, y \notin A$; contradiction. This completes this direction. □

With these two tools and the above lemma, a Python script was written to find saturation numbers. All of the saturation numbers with $n \leq 9$ except for the cases $K_3, K_{1,3}, 2P_2$, and $3P_2$ in the following tables were verified computationally; these other results were either taken from existing papers or verified manually. The case $4P_2$ could also not be verified using the standard computer program as its line graph consists of four disjoint vertices, which networkx treats as an empty graph and therefore gives incorrect results. To get around this problem, a modified version of the saturation testing script was developed for kP_2-saturation which takes the complement of the line graph and searches for K_k.

2 Results for Small Saturation Numbers

In the following tables, the saturation number $sat(n,H)$ is found in the box corresponding to the n in the left column and the H in the top row. The saturation number's corresponding minimal graph is found in parentheses next to the actual number. The minimal graph is expressed in standard graph notation, except for the graphs denoted by Y_k, for some k. These graphs are nonstandard and appear in the back of this paper as an appendix. In addition, some graphs in this paper have defined names but no concise expression in terms of common graph theory notation. For clarity's sake, these graphs also appear in the appendix.

n \ H	P_2	P_3	K_3	$2P_2$	*pan*
2	0 ($\overline{K_2}$)				
3	0 ($\overline{K_3}$)	1 ($P_2 \cup K_1$)	2 (P_3)		
4	0 ($\overline{K_4}$)	2 ($2P_2$)	3 ($K_{1,3}$)	3 ($K_3 \cup K_1$)	3 ($K_{1,3}$)
5	0 ($\overline{K_5}$)	2 ($2P_2 \cup K_1$)	4 ($K_{1,4}$)	3 ($K_3 \cup \overline{K_2}$)	4 ($K_{1,4}$)
6	0 ($\overline{K_6}$)	3 ($3P_2$)	5 ($K_{1,5}$)	3 ($K_3 \cup \overline{K_3}$)	5 ($K_{1,5}$)
7	0 ($\overline{K_7}$)	3 ($3P_2 \cup K_1$)	6 ($K_{1,6}$)	3 ($K_3 \cup \overline{K_4}$)	6 ($K_{1,6}$)
8	0 ($\overline{K_8}$)	4 ($4P_2$)	7 ($K_{1,7}$)	3 ($K_3 \cup \overline{K_5}$)	7 ($K_{1,7}$)
9	0 ($\overline{K_9}$)	4 ($4P_2 \cup K_1$)	8 ($K_{1,8}$)	3 ($K_3 \cup \overline{K_6}$)	8 ($K_{1,8}$)
10	0 ($\overline{K_{10}}$)	5 ($5P_2$)	9 ($K_{1,9}$)	3 ($K_3 \cup \overline{K_7}$)	9 ($K_{1,9}$)
11	0 ($\overline{K_{11}}$)	5 ($5P_2 \cup K_1$)	10 ($K_{1,10}$)	3 ($K_3 \cup \overline{K_8}$)	10 ($K_{1,10}$)
12	0 ($\overline{K_{12}}$)	6 ($6P_2$)	11 ($K_{1,11}$)	3 ($K_3 \cup \overline{K_9}$)	11 ($K_{1,11}$)

The graph $2P_2$'s saturation number was taken from [3], while K_3's saturation number was taken from [5]. Except for the pan saturation number, none of the results in this table are new: P_2 is trivial, P_3 is found in [3].

$n \setminus H$	P_4	C_4	K_4	B_2	$K_{1,3}$
2					
3					
4	2 $(2P_2)$	4 (*pan*)	5 (B_2)	5 (*pan*)	3 $(K_3 \cup K_1)$
5	4 $(K_3 \cup P_2)$	5 (*bull*)	7 (B_3)	6 (F_2)	4 $(C_4 \cup K_1)$
6	3 $(3P_2)$	6 (Y_1)	9 (B_4)	7 (Y_6)	5 $(C_5 \cup K_1)$
7	5 $(K_3 \cup 2P_2)$	8 (Y_2)	11 (B_5)	9 (F_3)	6 $(C_6 \cup K_1)$
8	4 $(4P_2)$	9 (Y_3)	13 (B_6)	10 (Y_7)	7 $(C_7 \cup K_1)$
9	6 $(K_3 \cup 3P_2)$	11 (Y_4)	15 (B_7)	12 (F_4)	8 $(C_8 \cup K_1)$
10	5 $(5P_2)$	12 (Y_5)	17 (B_8)		9 $(C_9 \cup K_1)$
11	7 $(K_3 \cup 4P_2)$		19 (B_9)		10 $(C_{10} \cup K_1)$
12	6 $(6P_2)$		21 (B_{10})		11 $(C_{11} \cup K_1)$

K_4 is found in [5], and C_4 is found in [Ollmann], while $K_{1,3}$'s and P_4's saturation number were taken from [3].

$n \setminus H$	F_2	B_3	$K_{1,4}$	*chair*	C_5
2					
3					
4					
5	7 (Y_8)	7 (Y_8)	6 $(K_4 \cup K_1)$	4 $(K_{1,4})$	6 (F_2)
6	8 (Y_9)	9 (Y_{13})	7 $(K_4 \cup P_2)$	5 $(K_{1,5})$	8 (Y_{18})
7	9 (Y_{10})	11 (Y_{14})	9 $(K_4 \cup K_3)$	5 $(K_{1,4} \cup P_2)$	9 (F_3)
8	10 (Y_{11})	13 (Y_{15})	10 (Y_{17})	6 $(K_{1,5} \cup P_2)$	10 (Y_{19})
9	11 (Y_{12})	15 (Y_{16})	12 $(2K_4 \cup K_1)$	7 $(K_{1,6} \cup P_2)$	12 (F_4)
10					
11					
12					

The saturation numbers for $K_{1,4}$ can be found in [3]. As far as the authors know, no extremal results or exact results are known for any of the other graphs in this table. Extremal values for book and cycle saturation numbers are known and can be found in [7] and [6], respectively, but the values are known only for n greater than what is catalogued here.

n \ H	K_5	*gem*	*cricket*	*bull*	W_4
2					
3					
4					
5	9 ($B_{3,2}$)	6 (F_2)	4 ($K_{1,4}$)	5 (C_5)	8 (Y_{22})
6	12 ($B_{3,3}$)	9 (B_4)	5 ($K_{1,5}$)	7 ($K_4 \cup P_2$)	10 (Y_{23})
7	15 ($B_{3,4}$)	9 (F_3)	6 ($K_{1,6}$)	8 (Y_{20})	12 (Y_{24})
8	18 ($B_{3,5}$)	12 (B_5)	7 ($K_{1,7}$)	9 (Y_{21})	15 (Y_{25})
9	21 ($B_{3,6}$)	12 (F_4)	8 ($K_{1,8}$)	11 ($K_4 \cup C_5$)	
10					
11					
12					

The saturation numbers for K_5 can be found in [5]. The rest, as far as the authors can tell, are new. The saturation number $sat(9, W_4)$ was not computed, as the density of the graph meant that the minimal W_4-saturated graph could not be deduced efficiently with the method used here.

n \ H	*dart*	$K_{2,3}$	P_5	*kite*	*banner*
2					
3					
4					
5	6 (K_4)	7 (*gem*)	4 ($C_4 \cup K_1$)	6 ($K_4 \cup K_1$)	4 (C_4)
6	7 (Y_6)	9 (*triforce*)	5(Y_{40})	7 ($K_4 \cup P_2$)	6 (Y_1)
7	9 (F_3)	11 (Y_{14})	6(Y_{41})	9 (F_3)	8 (Y_{27})
8	10 (Y_7)	13 (Y_{15})	6 ($Y_{40} \cup K_1$)	11 (Y_{26})	9 (Y_3)
9	12 (F_4)	15 (Y_{16})	7($Y_{41} \cup K_1$)	12 ($2K_4 \cup K_1$)	11 (Y_{28})
10			8(Y_{42})		
11			9(Y_{43})		
12					

The graph P_5's saturation number can be found in [3]. The saturation numbers for $K_{2,3}$ are not known directly, but are bounded from above by $2n-3$ (the exact value of the number in this table), as demonstrated in [8].

n \ H	$co-banner$	$house$	$K_{1,5}$	$4P_2$	$2P_3$
2					
3					
4					
5	4 ($C_3 \cup P_2$)	5 (C_5)			
6	6 ($2C_3$)	8 (Y_{18})	10 (K_5)		6 (Y_1)
7	7 (Y_{29})	9 (F_3)	11 ($K_5 \cup P_2$)		6 ($Y_1 \cup K_1$)
8	7 ($2C_3 \cup P_2$)	10 (Y_{19})	13 (Y_{30})	13 ($K_5 \cup K_3$)	7 ($Y_1 \cup P_2$)
9	8 ($Y_{29} \cup P_2$)	12 (F_4)	15 (Y_{31})	9 ($3C_3$)	7 ($Y_1 \cup P_2 \cup K_1$)
10				9 ($3C_3 \cup K_1$)	8 ($Y_1 \cup 2P_2$)
11				9 ($3C_3 \cup \overline{K_2}$)	
12				9 ($3C_3 \cup \overline{K_3}$)	

The saturation numbers for $K_{1,5}$ can be found in [3]. The results for $4P_2$ and $2P_3$ (except for $sat(8,4P_2)$) were previously known and can be found in [3] and [9].Interestingly, $sat(8,4P_2)$ is 13, while $sat(9,4P_2)$ is 9.

n \ H	$3P_2$	$2P_4$	$3P_3$	$P_4 \cup P_3$	C_6	P_6
2						
3						
4						
5						
6	6 ($2C_3$)				9 (Y_{36})	6 (Y_1)
7	6($2C_3 \cup K_1$)			9 (Y_{11})	10 (Y_{37})	7 (Y_{29})
8	6($2C_3 \cup \overline{K_2}$)	10 (Y_{32})		10 (Y_{12})	11 (Y_{38})	7 ($Y_1 \cup P_2$)
9	6($2C_3 \cup \overline{K_3}$)	11 (Y_{33})	14 (Y_{34})	9 ($3K_3$)	12 (Y_{39})	8 ($Y_{29} \cup P_2$)
10	6($2C_3 \cup \overline{K_4}$)		11* (Y_{35})			
11	6($2C_3 \cup \overline{K_5}$)		11* ($Y_{35} \cup K_1$)			
12	6($2C_3 \cup \overline{K_6}$)		12 ($2Y_1$)			

The $3P_2$'s saturation number was taken from [3]. The saturation numbers for P_6 can be found in [3]. $3P_3$'s saturation number, for $n \geq 12$, can be found in [9]. Everything else in this table, as far as the authors know, is new. The * in the entries for $sat(10,3P_3)$ and $sat(11,3P_3)$ indicate that the exact value was not found, but an upper bound for the exact value was found by hand search.

3 The saturation number $sat(n+2, P_n \cup P_2)$

We now provide some more general results on saturation numbers for specific classes of graphs. We begin with a partial result on the saturation number $sat(n+2, P_n \cup P_2)$: namely that $sat(n+2, P_n \cup P_2)$ is within a constant of $3n/2$. Before we begin, we will need to provide some definitions.

Definition 1. *Let G be a graph on 4n vertices, which are grouped into four clusters of n vertices each. Call these clusters A_n, B_n, C_n, and D_n, and let x_k be the kth vertex in the cluster X. Draw edges between a_k and all of b_k, c_k, and d_k for every k, and form a cycle of length n among the set B_n. Finally, draw a cycle of length 2n which is traversed in the following pattern: $c_1c_2c_3...c_nd_1d_2d_3...d_nc_1$. Then G is the flower snark J_n.*

In addition, a flower snark J_k is odd if k itself is odd. We observe some basic properties of flower snarks: they are cubic nonplanar graphs which, for odd flower snarks, are hypohamiltonian. In addition, they possess the following property, which will be useful in deriving our result.

Lemma 2. *(Lemma 4, [10]) For any odd flower snark J_k of order at least 28, for any edge $e = xy$ in J_k and for any non-adjacent pair of vertices u and v, the graph $J_k \cup uv$ has a hamiltonian cycle containing e.*

We now need one final lemma before we start our result:

Lemma 3. *Any odd flower snark J_k contains a $C_{4k-2} \cup P_2$.*

Now, with the above lemmas, we can derive an upper bound of $sat(n+2, P_n \cup P_2)$.

Lemma 4. *Let J_r be an odd flower snark, and let $e = xy$ be a member of the C_{4r-2} in the $C_{4r-2} \cup P_2$ in the flower snark. Now let there be two sets of vertices v_1, v_2, v_3, and v_4 and w_1, w_2, ..., and w_s. Add the edges xv_1, xv_2, xv_3, xv_4, v_1v_2, and v_3v_4, and then make the subgraph induced by the vertices y, w_1, w_2, ..., and w_s complete. The resulting graph on $4r+4+s$ vertices is $P_{4r+2+s} \cup P_2$ saturated, and posesses $6r+6+\frac{s(s-1)}{2}$ edges.*

Proof. Observe that the graph described above cannot possibly contain a $P_{4r+4+s} \cup P_2$, as the standalone P_2 must then be two of the v_i. However, the remaining vertices of the graph cannot form a hamiltonian path among themselves, as that would imply the existence of a hamiltonian path in J_r from x to y, creating a hamiltonian cycle in a nonhamiltonian graph. Now, to show saturation, we will show that the addition of any edge to the graph creates a $P_{4r+4+s} \cup P_2$.

We first consider adding an edge from v_1, v_2 to v_3, v_4 This forms a P_4 from v_1, v_2 to v_3, v_4. Lemma 3 says that there exists a $C_{4r-2} \cup P_2$ inside theJ_r, implying there is a P_{4r-2} from x to y. This forms a P_{4r+2+s} starting from the four v_i, going to x, taking the P_{4r-2} to y, and then connecting through all of the w_i. Since the other two vertices of this graph are connected, this forms the desired subgraph.

We now consider adding an edge from a v_i to a w_j. Assume without loss of generality that $i = 1$. Since J_r is hypohamiltonian, the graph $J_r - x$ contains a hamiltonian path exiting y. Call the endpoint of this path z. Then, a P_{4r+4+s} is given by $xv_2v_1w_jw_{j+1}...w_sw_1w_2...w_{j-1}y...z$, and since the remaining two vertices v_3 and v_4 are connected, we have the desired subgraph in this graph.

If we add an edge internal to the J_r, the J_r now contains a hamiltonian cycle through xy, so a path of length $4r+4+s$ is given by $v_1v_2x...yw_1...w_s$. Since the other two vertices v_3 and v_4 are connected, we have the desired subgraph in this graph.

Now, consider adding an edge from x to w_i. Assume without loss of generality that $i = 1$. Now, since the J_r is hypohamiltonian, $J_r - x$ contains a hamiltonian path starting at y. Call the endpoint of this path z. Then, a P_{4r+4+s} is given by $v_1v_2xw_1...w_jy...z$. Since the other two vertices v_3 and v_4 are connected, we have the desired subgraph in this graph.

Consider adding an edge from w_i to some vertex z inside the J_r that is not x or y. Assume without loss of generality that $i = 1$. If zx is not an edge in J_r, there must be a hamiltonian path from z to x by J_r's hamiltonian cycle saturatedness. Then, $v_1v_2x...zw_1...w_s$ is a P_{4r+4+s} in the graph, and since v_3 and v_4 are connected we have the desired subgraph in this graph. Therefore, we can assume that zx is an edge of the J_r. Now, since J_r is hypohamiltonian, $J_r - y$ is hamiltonian. Since x has degree 2 inside $J_r - y$, it must be that the hamiltonian cycle of $J_r - y$ must use both edges connecting to x inside it, and therefore there must be a hamiltonian path inside $J_r - y$ connecting z and x. A P_{4r+4+s} is then given by $v_1v_2x...zw1...w_sy$, and since the two remaining vertices v_3 and v_4 are connected, we have the desired subgraph in this graph.

In the case where we add an edge connecting v_i to y assume without loss of generality that $i = 1$. We realize there must be a hamiltonian path inside $J_r - y$, so there must be a hamiltonian path inside this subgraph starting at x and ending at some vertex z. Then, a P_{4r+4+s} is given by $z...xv_2v_1yw_1...w_s$. Since the two remaining vertices v_3 and v_4 are connected, we have the desired subgraph in this graph.

Finally we consider the case where we add an edge connecting v_i to some vertex z inside the J_r that is not x or y. Assume without loss of generality that $i = 1$. If zy is not an edge in J_r, there must be a hamiltonian path from z to y by J_r's hamiltonian cycle saturatedness. Then $w_s...w_1y...zv_1v_2$ is a path of length $4r+4+s$ in the graph, and since the two remaining vertices v_3 and v_4 are connected, we have the desired subgraph in this graph in this case. Therefore, we can assume that zy is an edge of the J_r. Now since J_r is hypohamiltonian, $J_r - x$ is hamiltonian.

Since y has degree 2 inside $J_r - x$, it must be that the hamiltonian cycle of $J_r - x$ must use both edges connecting to y inside it, and therefore there mut be a hamiltonian path inside $J_r - x$ connecting z and y. A P_{4r+4+s} is then given by $xv_2v_1z...yw_1...w_s$. Since the two remaining vertices v_3 and v_4 are connected, we have the desired subgraph in this graph.

Since in every possible case adding an edge to this graph forms a $P_{4r+2+s} \cup P_2$ but the graph itself does not contain this graph, it is $P_{4r+2+s} \cup P_2$ saturated. □

In other words, the above lemma states that since s can be forced to be less than 8 as there exists a flower snark of any number of vertices of the form $8k+4$, $sat(n+2, P_n \cup P_2)$ is less than $3n/2$ plus some constant factor.

We now consider the lower bound of $sat(n+2, P_n) \cup P_2$. Let G be $P_n \cup P_2$ saturated on $n+2$ vertices. Now, assume G is disconnected. If G splits into two components of order h and $n-h+2$ and h is not equal to 2, the graph $K_h \cup K_{n-h+2}$ contains no $P_n \cup P_2$ and all graphs G, while having more edges than the upper bound. In the case where h=2, the component of size two must be a connected edge, as then the graph would be contained inside the $P_n \cup P_2$-free $K_n \cup \overline{K_2}$. This leaves a component of size n. Since this component is disconnected from the other component of size two, any edge internal to it must form a P_n inside of it; in essence, the component of size n must be hamiltonian path saturated. Returning to [10] once again, Dudek et al. show that if a graph on n vertices is hamiltonian path saturated, it must have at least $\frac{3(n-1)}{2} - 2$ edges, meaning that G must have at least $\frac{3(n-1)}{2} - 1$ edges– a constant difference from $3n/2$.

Thanks to the above result, we can consider only the case in which G is connected. In this case, we count degrees. Since every vertex in G is used in a $P_n \cup P_2$, there can be at most four vertices of degree 1, as if G had any more we could connect two vertices of degree two or greater and then we would be looking for a spanning subgraph with four vertices of degree 1 in a graph with at least 5. Now, we consider vertices of degree 2. We start with another lemma.

Lemma 5. *If G $P_n \cup P_2$-saturated on $n+2$ and if v is a vertex of degree 2, the neighbors of v must be connected.*

To prove this, we define u and w to be the vertices adjacent to v. Now, if w and u are not connected to each other, we add that edge to G. This new edge wu cannot be the P_2, as if it was v would have nothing to connect to as part of the P_n. So, this new edge must be part of the P_n. We notice that the P_k must end in either wuv or uwv, as if the edge uw was not immediately followed by v we would have used up both of the vertices it connects to and disconnected it from the rest of the graph. However, we could instead end the P_k with wvu in the former case or uvw in the latter, while keeping the rest of the $P_k \cup P_2$ from the original subgraph. This forms a $P_k \cup P_2$ in G + uw that does not use uw; meaning there is a $P_k \cup P_2$ in G; contradiction.

We now prove another lemma:

Lemma 6. *If v is a vertex of degree 2 in G, and if v has neighbors u and w, then either u or w has degree at least 4.*

To prove this, we consider what would happen if both u and w had degree 3, as the other case (where one vertex has degree 3 and the other has degree 2) follows by similar arguments. By the above lemma, we know u, v, and w must form a triangle. Since w and u have degree three, they must connect to exactly one point each. Call the point w connects to z, and call the point u connects to y. If y and z are the same point, connect v to this point, and it is trivially easy to see that adding this edge in cannot possibly form a $P_n \cup P_2$ without the original G containing it. So, we can assume y and z are not the same point. Now, add the edge vy. This edge cannot be a part of the P_2, as that would imply the existence of a P_n ending in zwu. Keeping the P_n constant but changing the last three vertices of it to zwv forms a P_n that skips the vertices y and u, which are connected. This forms a $P_k \cup P_2$ in G; contradiction. So, yv must be part of the P_n. Now, there are 4 possibilities for what the P_2 can be: it can be the edges zw, wu, z to some point other than w, or some edge between two completely different points. Now, a routine check in each of these cases eliminates all of them, forcing a contradiction.

Now, we consider what happens if two vertices of degree 2, u and v, connect to the same vertex. This breaks up into three cases: (a) u, v, and some other point form a triangle, (b), u and v have identical neighborhoods but do not themselves connect, and (c) u and v connect to exactly one point in common, but both connect to one other point each which are not the same. We settle these cases in order:

(a) If u and v form a triangle with some other point, called w, we define such a triangle for brevity's sake as a 'tag'. It is useful to see that w must have degree at least 3, as if it had a lower degree the triangle would be disconnected from the rest of the graph, reducing to the disconnected case analyzed above We note that G can contain at most three such tags, as we note a P_n can enter a tag but not exit it. This obervation means that if G contained more than three tags, we can add an edge between two vertices that are of degree 3 or greater (which must exist, as otherwise all of the vertices of degree 3 or greater would form a clique and therefore G would contain too many edges to be minimally saturated) and then realize that even if the P_2 uses both of the vertices of one tag and the P_n starts at one tag and ends at another, there would still be one tag left that cannot be used in a $P_n \cup P_2$. Therefore, there can only be at most 6 vertices of degree of this form.

(b) In this case, we define the points that u and v connect to as a and b. We begin by breaking this case into two subcases: case (b1), where the sum of degrees of the vertices a, b, u, and v is less than 12, and case (b2), where the same sum of degrees is greater than or equal to 12. We will show that case (b1) is impossible. We start by applying Lemma 5; this implies that a and b are connected to one another. Now, since G is connected and it contains these four vertices, one of tehe four must connect to the rest of the graph. Since u and v are limited to degree 2 by assumption, the extra edge must connect to a or b. Without loss of generality, we assume a is the vertex with the extra edge. However, while the degrees of u and v are still both 2, the degree of b is at least 3, while the degree of a is at least 4. This means that the sum of degrees of these four vertices is 11, meaning that in case (b1), these edges that connect to a, b, u, and v (which must all exist) are the only edgesthat connect to these vertices. However, if we connect u and v in case (b1), it is clear that that the resulting graph cannot contain a $P_n \cup P_2$ without the original G containing it as well. If the new edge uv formed a P_2 in G, there must be a P_n in G that ends in $...ab$. However, we could instead use the edge ub as the P_2 and modify the original P_n by making it end in $...av$ instead. This forms the required subgraph without needing the edge uv. If on the other hand the edge uv was part of a P_n, since all of the vertices of G must be used in the $P_n/cupP_2$, theP_n must end in $...auvb$, $...avub$, $...abvu$, or $...abuv$. In all of these cases, an alternative P_n can be found that uses all of the same vertices yet does not use the edge uv. This is a contradiction, and therefore case (b1) is impossible.

(c) We cannot force a nonexistence in this case, so we instead generalize it to a graph having $2k$ vertices of degree 2 instead of just two. This, with some applications of Lemma 5, reduces to G having a friendship graph F_k. We first realize that in the friendship graph F_k, one of the two outer vertices of every triangle inside of it must connect to something, as otherwise we would reduce to case (a). This means that the friendship graph has at least k vertices of degree 3 or greater, one vertex of degree $2k$, and at most k vertices of degree 2. Summing degrees gives the sum of degrees of the vertices of the friendship graph as $7k$, so the average degree of the friendship graph is $\frac{7k}{2k+1}$. This is greater than 3 for all k greater than 2, and since k must be greater than or equal to 2, this settles all cases except for $k = 2$. In this case, all that is required is to show that the friendship graph has one more edge connecting it to the rest of the graph. To do this, we let c be the center and have degree 4, a and b be the vertices of the friendship graph of degree 3 and let x and y be the remaining vertices of degree 2. Now, simply connecting the edge xy cannot create the desired subgraph in the graph, as routine checking will verify. Therefore, there must be at least one more edge connecting to the friendship graph, increasing the average degree above 3.

We can now find a lower bound for $sat(n+2, P_n \cup P_2)$. The argument is by summing of degrees. We start by looking at the friendship graphs of case (c). If the friendship graph in question is a F_k and has j tags, the sum of degrees of the vertices in this graph is at least $6k + k - j = 7k - j$, for k greater than 3. For k

equal to 2, we realize by the second part of (c) that the resulting F_2 must connect at least three times to the rest of the graph, giving it a sum of degrees equal to 15 on 5 vertices, giving it no net contribution to the difference in degree from $3n/2$. We next look at case (b2). Since the average degree of the vertices involved in case (b2) is at least 3 in every case, the addition of vertices of degree 2 as in case (b2) causes no deviation from the required average degree. We now sum over the rest of the graph. There can be at most 4 vertices of degree 1, and at most 6-$2j^+$ vertices of degree 2 of class (a), where j^+ is the sum of the js in the F_ks. We also know that if there are h vertices of degree two that do not share a neighbor with another vertex of degree 2, there are at least h vertices of degree 4 to cancel its effect of the degree sum by Lemma 6. So, since this counts all of the possible vertices of degree 2 or less, in the worst case where all of the other vertices have degree 3, the sum of degrees minus $3n$ is $\Sigma k - 3$ - 8, where $\Sigma k - 3$ is the sum of the ks in the friendship graphs which are greater than 3, minus 3. Minimizing this gives the sum of degrees minus $3n$ is at least -8, so if G is connected and $P_n \cup P_2$-saturated on $n+2$ vertices, G must have at least $\frac{3n}{2} - 4$ edges, a constant away from $\frac{3n}{2}$. This verifies the last remaining case, and so we can conclude with

Theorem 1. *$sat(n+2, P_n \cup P_2)$ is within a constant of $\frac{3n}{2}$.*

4 The saturation numbers $sat(n, rP_3 \cup kP_2)$

In this section, we will prove the following theorem:

Theorem 2. *For n large and $k \geq 3$, $sat(n, rP_3 \cup kP_2) = 3(k+r-1)$.*

We begin by proving the result for $r = 1$; that $sat(n, P_3 \cup kP_2) = 3k$ for sufficiently large n. We realize that $sat(n, P_3 \cup kP_2)$ must be less than or equal to $3k$, as the graph $kK_3 \cup \overline{K_{n-3k}}$ is $P_3 \cup kP_2$ saturated. Let G be $P_3 \cup kP_2$ saturated with fewer than $3k$ edges. Since G plus any edge must contain $P_3 \cup kP_2$, G itself must contain either $(k+1)P_2$ or $P_3 \cup (k-1)P_2$. If G does not contain $(k+1)P_2$, then since $P_3 \cup kP_2$ contains $(k+1)P_2$, G must also be $(k+1)P_2$ saturated. But for n sufficiently large, $sat(n, (k+1)P_2) = 3k$; a contradiction. So, G must contain $(k+1)P_2$. However, this implies that none of the vertices of G that are not in the $(k+1)P_2$ connect to the $(k+1)P_2$, as then G would contain a $P_3 \cup kP_2$. If G - $(k+1)P_2$ is the empty graph, G is contained in $K_{2k+2} \cup \overline{K_{n-2k-2}}$, which contains no $P_3 \cup kP_2$, and further has more than $3k$ edges; this implies that $G - (k+1)P_2$ contains an edge. Since adding an edge between $G - (k+1)P_2$ and $(k+1)P_2$ in G forms a $P_3 \cup kP_2$ regardless of G's other structure, the problem is equivalent to showing that $G - (k+1)P_2$ is P_3 saturated. But this means that $G - (k+1)P_2$ contains at least $\lfloor \frac{n-8}{2} \rfloor$ edges, so G contains at least $\lfloor \frac{n}{2} \rfloor$ edges. For n sufficiently large, this is greater than $3k$; contradiction. So $sat(n, P_3 \cup kP_2) = 3k$.

We now prove the result for r greater than 1. To begin, we need to prove that some G is minimally $rP_3 \cup kP_2$ saturated and contains no $(r-1)P_3 \cup (k+1)P_2$. By the above and by [9], we have this fact being true for $P_3 \cup kP_2 (k \geq 3)$ and for $rP_3 \cup 3P_2$. We now strong induct on both r and k simultaneously. Let x, y be such that every minimally $xP_3 \cup yP_2$ saturated G also contains $(x-1)P_3 \cup (y+1)P_2$. We know that since the graph $(x+y-1)K_3$ contains no $(x-1)P_3 \cup (y+1)P_2$, $sat(n, xP_3 \cup yP_2) < 3(x+y-1)$. Now consider $sat(n, (x+1)P_3 \cup (y-1)P_2)$. By the induction hypothesis we know there exists some graph H that is $(x+1)P_3 \cup (y-1)P_2$ saturated and contains no $xP_3 \cup yP_2$. So then, whatever H is, it must also be saturated in $xP_3 \cup yP_2$, since $(x+1)P_3 \cup (y-1)P_2$ contains $xP_3 \cup yP_2$. So then $sat(n, (x+1)P_3 \cup (y-1)P_2) \leq sat(n, xP_3 \cup yP_2) < 3(x+y-1)$. This process can be repeated until we get that $sat(n, (x+y-3)P_3 \cup 3P_2) \leq (...) \leq sat(n, (x+1)P_3 \cup (y-1)P_2) \leq sat(n, xP_3 \cup yP_2) < 3(x+y-1)$. But $sat(n, (x+y-3)P_3 \cup 3P_2)$ is $3(x+y-1)$, so $3(x+y-1) < 3(x+y-1)$, a contradiction.

So, since we now know that there exists some minimally $rP_3 \cup kP_2$ saturated graph G that contains no $(r-1)P_3 \cup (k+1)P_2$, we realize that since $rP_3 \cup kP_2$ contains $(r-1)P_3 \cup (k+1)P_2$, G must also be $(r-1)P_3 \cup (k+1)P_2$-saturated. This process can be repeated until we get that G must be $(P_3 \cup (k+r-1)P_2$-saturated. But then, by above, this saturation number is just $3(r+k-1)$, and we therefore get that $sat(n, rP_3 \cup kP_2) = 3(r+k-1)$.

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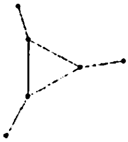
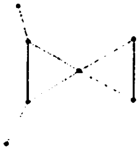
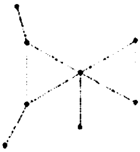
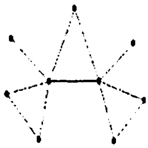
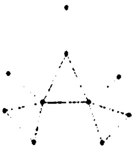
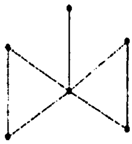
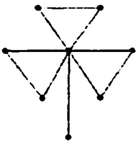
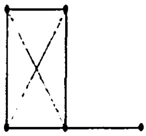
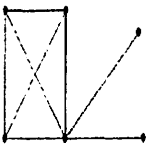
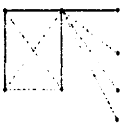
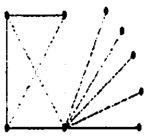
Y_1		Y_2	
Y_3		Y_4	
Y_5		Y_6	
Y_7		Y_8	
Y_9		Y_{10}	
Y_{11}		Y_{12}	

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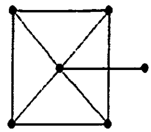
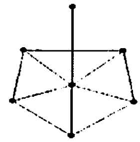
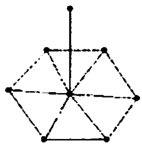
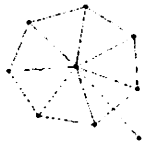
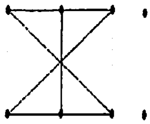
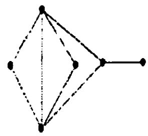
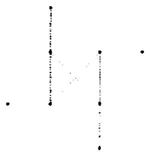
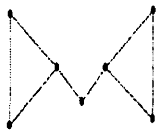
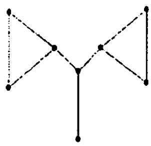
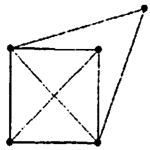
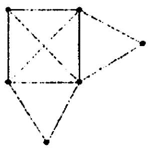
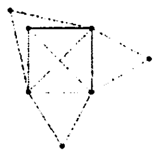
Y_{13}		Y_{14}	
Y_{15}		Y_{16}	
Y_{17}		Y_{18}	
Y_{19}		Y_{20}	
Y_{21}		Y_{22}	
Y_{23}		Y_{24}	

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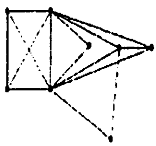
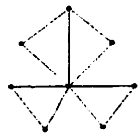
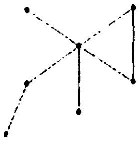
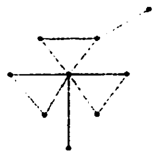
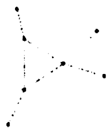
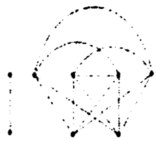
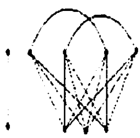
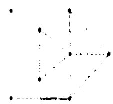
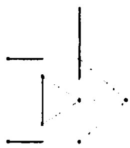
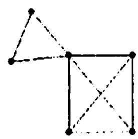
Y_{25}		Y_{26}	
Y_{27}		Y_{28}	
Y_{29}		Y_{30}	
Y_{31}		Y_{32}	
Y_{33}		Y_{34}	
Y_{35}		Y_{36}	

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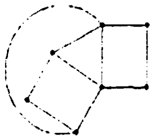
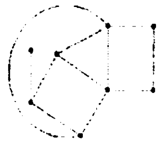
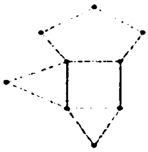
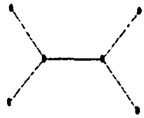
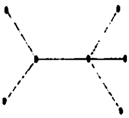
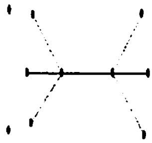
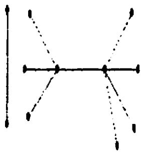
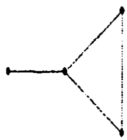
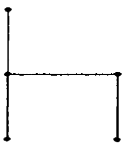
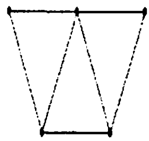
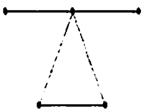
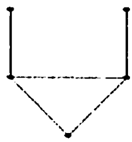
Y_{37}		Y_{38}	
Y_{39}		Y_{40}	
Y_{41}		Y_{42}	
Y_{43}		*pan*	
chair		*gem*	
cricket		*bull*	

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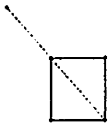
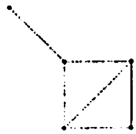
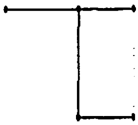
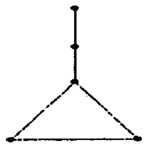
dart		*kite*	
banner		*co – banner*	
house