On the Equivalence of the Upper Open Irredundance and Fractional Upper Open Irredundance Numbers of a Graph

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Abstract

A set $S \subset V$ of vertices in a graph $G = (V, E)$ is called *open irredundant* if for every vertex $v \in S$ there exists a vertex $w \in V \setminus S$ such that w is adjacent to v but to no other vertex in S. The *upper open irredundance number* $OIR(G)$ equals the maximum cardinality of an open irredundant set in G. A real-valued function $g : V \to [0, 1]$ is called *open irredundant* if for every vertex $v \in V$, $g(v) > 0$ implies there exists a vertex w adjacent to v such that $g(N[w]) = 1$. An open irredundant function g is *maximal* if there does not exist an open irredundant function h such that $g \neq h$ and $g(v) \leq h(v)$, for every $v \in V$. The *fractional upper open irredundance number* equals $OIR_f(G) = sup\{|g| : g \text{ is an open irredundant function on } G\}$. In this paper we prove that for any graph G, $OIR(G) = OIR_f(G)$.

1 Introduction

Let $G = (V, E)$ be a graph of order $n = |V|$ and let $v \in V$ be an arbitrary vertex. The *open neighborhood* of v is the set $N(v) = \{u \in V | uv \in E\}$,

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while the *open neighborhood of a set* $S \subseteq V$ is the set $N(S) = \bigcup_{u \in S} N(u)$. Similarly, the *closed neighborhood* of a vertex v is the set $N[v] = N(v) \cup \{v\}$, and the *closed neighborhood of a set* $S \subseteq V$ is the set $N[S] = \bigcup_{u \in S} N[u]$.

A set $S \subset V$ of vertices is called *irredundant* if for every vertex $v \in S$,

$$N[v] - N[S - \{v\}] \neq \emptyset.$$

The *irredundance number* $ir(G)$ of a graph G equals the minimum cardinality of a maximal irredundant set S in G, while the *upper irredundance number* $IR(G)$ equals the maximum cardinality of an irredundant set in G. First defined by Cockayne et al. [2] in 1978, there are now more than 200 papers dealing with various aspects of irredundance in graphs. Noteworthy among these many papers are those by Finbow [6] and Cockayne and Finbow [3], which place irredundance in a very general context.

A set S is a *dominating set* of a graph $G = (V, E)$ if $N[S] = V$. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G, while the *upper domination number* $\Gamma(G)$ equals the maximum cardinality of a minimal dominating set in G.

A set S of vertices is *independent* if no two vertices in S are adjacent. The *independence number* $\beta_0(G)$ equals the maximum cardinality of an independent set in G, while the *independent domination number* $i(G)$ equals the minimum cardinality of a maximal independent set S in G.

The following inequality chain was first observed by Cockayne et al. [2].

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G).$$

Fractional analogs of dominating and irredundant sets have been defined as follows. Fractional domination was first introduced by Hedetniemi et al. in 1986 [9]. A function $g : V \rightarrow [0,1]$ is a *dominating function* if for every vertex $v \in V$, $g(N[v]) \geq 1$. The *weight* of a dominating function g is $g(V) = \sum_{v \in V} g(v)$. The *fractional domination number* $\gamma_f(G)$ of a graph G equals the minimum weight of a fractional dominating function g on G.

A dominating function g is *minimal* if for every dominating function h such that $g \neq h$, $g(v) \leq h(v)$, for every $v \in V$. The *upper fractional domination number* $\Gamma_f(G)$ equals the maximum weight of a minimal fractional dominating function on G.

It is easy to see that for every minimal dominating set S, the characteristic function $f : V \rightarrow \{0,1\}$, defined by $f(v) = 1$ if $v \in S$ and $f(v) = 0$ if $v \in V \setminus S$, is a minimal dominating function. Thus, for any graph G,

$$\gamma_f(G) \leq \gamma(G) \leq \Gamma(G) \leq \Gamma_f(G).$$

In [4] it was shown that for the Hajós graph G, $\gamma_f(G) < \gamma(G)$, and in [1] it was shown that there exist graphs G for which $\Gamma(G) < \Gamma_f(G)$.

A function $g : V \to [0,1]$ is an *irredundant function* if for every vertex $v \in V$, if $g(v) > 0$, then there exists a vertex $w \in N[v]$ for which $g(N[v]) = 1$. An irredundant function g is *maximal* if there does not exist an irredundant function h such that $g \neq h$, $g(v) \leq h(v)$, for every $v \in V$. The *fractional irredundance number* of a graph G equals $ir_f(G) = inf\{g(V) : g$ is a maximal irredundant function on $G\}$. The *upper fractional irredundance number* equals $IR_f(G) = sup\{g(V) : g$ is an irredundant function on $G\}$.

It is easy to see that for every maximal irredundant set S, the characteristic function $f : V \to \{0,1\}$, defined by $f(v) = 1$ if $v \in S$ and $f(v) = 0$ if $v \in V \setminus S$, is a maximal irredundant function. Thus, for any graph G,

$$ir_f(G) \leq ir(G) \leq IR(G) \leq IR_f(G).$$

In [7] it was pointed out that for the path P_7, $ir_f(P_7) < ir(P_7)$. But in [8] the following theorem was proved.

Theorem 1.1. *For any graph G, $IR(G) = IR_f(G)$.*

2 Open Irredundance and Fractional Open Irredundance in Graphs

In this paper we focus on open irredundant sets, first introduced by Farley and Schachum in 1983 [5], and their fractional analogs.

A set $S \subset V$ of vertices is called *open irredundant* if for every vertex $v \in S$, $N(v) - N[S - \{v\}] \neq \emptyset$. The *open irredundance number* $oir(G)$ of a graph G equals the minimum cardinality of a maximal open irredundant set S in G, while the *upper open irredundance number* $OIR(G)$ equals the maximum cardinality of an open irredundant set in G.

A fractional analog of open irredundant sets can be defined as follows.

A function $g : V \to [0,1]$ is *open irreducible* or *oiru* if for every $v \in V$ with $g(v) > 0$ there exists $w \in N(v)$ such that $g(N[w]) \leq 1$. In the special case that for every $v \in V$ with $g(v) > 0$ there exists $w \in N(v)$ such that $g(N[w]) = 1$ we say that g is *fractional open irredundant.* Finally, if g is a fractional open irredundant function such that $g : V \to \{0,1\}$ then g is *open irredundant.* Examples of each type of function are shown in Figures 1–3.

For $S \subset V$, we define $g(S) = \sum_{v \in S} g(v)$, and then define the *weight of a function* g to be $g(V)$. A *g-cover of y* is a closed neighborhood N which contains the vertex y but is not centered at y and $g(N) \leq 1$. So, if g is oiru, every $v \in V$ for which $g(v) > 0$ has a g-cover. An open irredundant function g is *maximal* if there does not exist an open irredundant function h such that $g \neq h$, $g(v) \leq h(v)$, for every $v \in V$.

Figure 1: An oiru function.

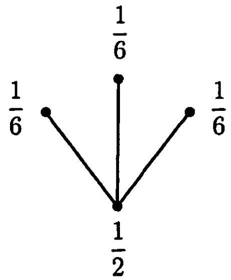
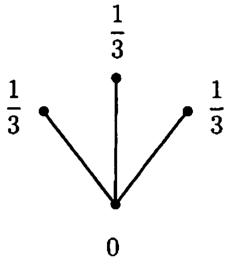


Figure 2: A fractional open irredundant function.



For a graph G, the *fractional lower open irredundance number* is

$$oir_f(G) = \inf\{g(V) : g \text{ is a maximal fractional open irredundant function}\},$$

the *fractional upper open irreducibility number* is

$$\text{OIRU}_f(G) = \sup\{g(V) : g \text{ is an oiru function}\},$$

and the *fractional upper open irredundance number* is

$$\text{OIR}_f(G) = \sup\{g(V) : g \text{ is a fractional open irredundant function}\}.$$

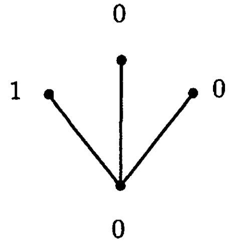
To simplify the notation in the remainder of the paper, let $W = \text{OIRU}_f$.

Note that since all open irredundant functions are fractional open irredundant and all fractional open irredundant functions are oiru, we immediately have

$$oir_f \le oir \le \text{OIR} \le \text{OIR}_f \le \text{OIRU}_f. \qquad (1)$$

There are graphs for which the strict inequality $oir_f < oir$ holds. For example, let G be the path P_4, shown in Figure 4. Each singleton set,

Figure 3: An open irredundant function.



$S = \{v_i\}$ is open irredundant, but not maximal, so $oir(G) \geq 2$. Consider the set $S = \{v_2, v_3\}$. It is open irredundant, since

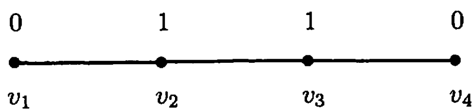
$N(v_2) - N[S - \{v_2\}] = \{v_1, v_3\} - \{v_2, v_3, v_4\} = \{v_1\} \neq \emptyset$ and similarly
$N(v_3) - N[S - \{v_3\}] = \{v_4\} \neq \emptyset$.

Furthermore, S is a maximal open irredundant set on G: $S' = S \cup \{v_1\}$ is not open irredundant since

$$N(v_2) - N[S' - \{v_2\}] = \{v_1, v_3\} - \{v_1, v_2, v_3, v_4\} = \emptyset$$

and by symmetry, $S'' = S \cup \{v_4\}$ is not open irredundant either. Thus oir(G) = 2.

Figure 4: An example of $oir(G) = 2$



Now, the function which is identically zero on the vertices of G will vacuously be fractional open irredundant, but not maximal. So, any maximal open irredundant function, g, must be non-zero on at least one vertex, and that vertex must have a neighbor, w such that $g(N[w]) = 1$, thus, the weight of g must be at least 1 and $oir_f(G) \geq 1$. For $\epsilon \in (0,1)$ define a function g_ϵ on the vertices of G as shown in Figure 5. Then v_1 has neighbor $w = v_2$ for which $g(N[w]) = 1$, v_3 has neighbor $w = v_2$ for which $g(N[w]) = 1$ and v_4 has neighbor $w = v_3$ for which $g(N[w]) = 1$. So g is fractional open irredundant. Since v_1 has only one neighbor, v_2, increasing the value of g on v_1, v_2 or v_3 would destroy open irredundance. Similarly, since v_4 has only one neighbor, v_3, increasing the value of v_4 will destroy open irredundance. So, g_ϵ is a maximal open irredundant function on G. Now, since $\{g_\epsilon : \epsilon \in (0,1)\} \subset \{ g : g$ is maximal open irredundant on $G\}$, we have

$$\begin{aligned} oir_f(G) &= \inf\{g(V) : g \text{ is maximal open irredundant on } G \\ &\leq \inf\{g_\epsilon : \epsilon \in (0,1)\} \\ &= \inf\{1+\epsilon : \epsilon \in (0,1)\} \\ &= 1. \end{aligned}$$

Thus $oir_f(G) = 1 < 2 = oir(G)$.

Figure 5: A family of functions g_ϵ on $G = P_4$

For the OIR $\leq$ OIR$_f$ $\leq$ OIRU$_f$ portion of Inequality 1, we will show that in fact, equality holds.

3 The Result

Lemma 3.1. *Every sequence (g_n) of functions of the form $g_n : V \to [0,1]$ has a subsequence (g_{n_t}) which converges pointwise to a function $g : V \to [0,1]$. That is*

$$\lim_{t\to\infty} g_{n_t}(v) = g(v), \text{ for all } v \in V.$$

Proof. Fix one $v_0 \in V$, and let (g_n) be a sequence of functions $g_n : V \to [0,1]$. Then since $g_n(v_0) \in [0,1]$ for each $n \in \mathbb{N}$, the sequence $(g_n(v_0))_{n=1}^{\infty}$ is bounded and by the Bolzano-Weierstrass Theorem has a subsequence (g_{n_s}) which converges to some value in the closed set $[0,1]$. We then define

$$g(v_0) = \lim_{s\to\infty} g_{n_s}(v_0)$$

Similarly, we may find a subsequence of (g_{n_s}) whose values at a second vertex, v_1, converge to some value in $[0,1]$, which we define to be $g(v_1)$. Repeating this process with each of the vertices in V and the resulting subsequence of functions from the previous step, will produce a subsequence (g_{n_t}) of (g_n) and define a function g for which

$$g(v) = \lim_{t\to\infty} g_{n_t}(v).$$

Definition 3.1. For a function $g : V \to [0,1]$, define

$$
\begin{aligned}
m_g &= \min\{g(w) : g(w) > 0\} \\
s_g &= \min\{1 - g(w) : g(w) < 1\} \\
n_g &= \min\{|1 - g(N)| : g(N) \neq 1 \text{ and } N \text{ is a neighborhood (open or closed) }\} \\
a_g &= \frac{1}{10}\min\{m_g, s_g, n_g\} \\
z_g &= \text{ the number of zeros of } g \text{ and} \\
u_g &= \text{ the number of vertices at which } g(v) = 1
\end{aligned}
$$

Lemma 3.2. *Let $(g_n)_{n=1}^{\infty}$ be a sequence of functions converging to a function g. There exists $k \in \mathbb{N}$ such that the following are true.*

a.) $n \geq k \Rightarrow \begin{cases} |g_n(v) - g(v)| < a_g, \ \forall v \in V. \\ |g_n(N) - g(N)| < a_g, \ \forall \text{ neighborhoods } N, \ \text{(open or closed.)} \end{cases}$

b.) If there exists $n \geq k$ such that $g_n(N) = 1$ then $g(N) = 1$.

c.) If there exists $n \geq k$ such that $g_n(N) \leq 1$ then $g(N) \leq 1$.

d.) If there exists $n \geq k$ such that $g_n(v) = 0$ then $g(v) = 0$.

e.) If $g(v) > 0$ then $g_n(v) > 0$ for all $n \geq k$.

f.) If $g(v) < 1$ then $g_n(v) < 1$ for all $n \geq k$.

Proof. Let M be one more than the maximum degree found in G. Note that $a_g > 0$, so by definition of convergence, for each $v \in V$ there exists n_v such that

$$|g_n(v) - g(v)| < a_g/M \text{ for all } n \geq n_v. \tag{2}$$

Define $k = \max\{n_v : v \in V\}$.

a.) Statement 2 implies the first inequality. Let $N = N[v_0]$ or $N(v_0)$ for any arbitrary vertex $v_0 \in V$ and let D be the degree of v_0. Then for $n \geq k$,

$$|g_n(N) - g(N)| \leq \sum_{v \in N} |g_n(v) - g(v)| < \sum_{v \in N} \frac{a_g}{M} \leq (D+1)\frac{a_g}{M} \leq a_g.$$

b.) To get a contradiction, suppose there exists $n \geq k$ such that $g_n(N) = 1$ on some neighborhood N and that $g(N) \neq 1$. Then by (a),

$$|g(N) - 1| = |g(N) - g_n(N)| < a_g,$$

which is impossible, since
$a_g \leq n_g = \min\{|g(N) - 1| : g(N) \neq 1 \text{ and } N \text{ is a neighborhood}\}$.

c.) Suppose there exists $n \geq k$ such that $g_n(N) \leq 1$ on some neighborhood N and that $g(N) > 1$. Then

$$|g(N) - 1| \leq |g(N) - g_n(N)| < a_g,$$

which, as before, is impossible.

d.) Suppose there exists $n \geq k$ such that $g_n(v) = 0$ for some $v \in V$ and that $g(v) > 0$. Then applying (a),

$$g(v) = |g(v) - g_n(v)| < a_g < m_g.$$

which is impossible, since $m_g = \min\{g(w) : g(w) > 0\}$.

e.) Suppose $g(v) > 0$, but that $g_n(v) = 0$ for some $n \geq k$, then

$$g(v) = |g_n(v) - g(v)| < a_g \leq m_g,$$

which again, is impossible.

f.) Suppose $g(v) < 1$, but that $g_n(v) = 1$ for some $n \geq k$, then

$$|1 - g(v)| = |g_n(v) - g(v)| < a_g \leq s_g,$$

which is impossible, since $s_g = \min\{|1 - g(v)| : v \in V\}$.

□

Proposition 3.3. *If* (g_n) *is a sequence of oiru functions which converges to a function* g*, then* g *is oiru.*

Proof. By Lemma 3.2 (d) and (e), there exists $k \in \mathbb{N}$ such that for $n \geq k$, the zeros of g_n and g are the same. Thus for any vertex v with $g(v) \neq 0$, we must have $g_n(v) \neq 0$ (for $n \geq k$), and since each g_n is oiru, v has a neighbor w such that $g_n(N[w]) \leq 1$. By Lemma 3.2(c), $g(N[w]) \leq 1$ as well, so g is oiru. □

Corollary 3.4. *The supremum,* $W(= OIRU_f)$ *is a maximum. That is, there exists an oiru function* g *such that* $g(V) = W$*.*

Proof. As W is a supremum, for every $n \in \mathbb{N}$ there exists $g_n \in \{g(V) : g \text{ is oiru}\}$ such that $W - g_n(V) < 1/n$, and thus $\lim_{n\to\infty} g_n(V) = W$. By Lemma 3.1, the sequence (g_n) has a subsequence (g_{n_t}) which converges to a function g. All of the functions g_{n_t} are oiru, so by Proposition 3.3, g is oiru as well. Finally, since every subsequence of a convergent sequence also converges to that same value, we have

$$g(V) = \sum_{v\in V} g(v) = \sum_{v\in V} \lim_{t\to\infty} g_{n_t}(v) = \lim_{t\to\infty} \sum_{v\in V} g_{n_t}(v) = \lim_{t\to\infty} g_{n_t}(V) = W. \tag{3}$$

□

We now define the set $G_W = \{g : g \text{ is oiru and } g(V) = W\}$. By the previous proposition, we have $G_W \neq \emptyset$, so we may also define

$$G_z = \{g \in G_W : z_g \geq z_f \text{ for all } f \in G_W\} \text{ and}$$
$$G_u = \{g \in G_z : u_g \geq u_f \text{ for all } f \in G_z\}.$$

(Recall that z_g is the number of zeros of g and u_g is the number of vertices, v, such that $g(v) = 1$.)

In addition, let $n_z = \max\{z_g : g \in G_W\}$. Thus, for every $g \in G_z$, $z_g = n_z$, the maximum number of zeros possible.

Finally, define $m = \inf\{m_g : g \in G_u\}$.

Proposition 3.5. *m is a minimum. That is, there exists $g \in G_u$ such that $m_g = m$.*

Proof. As m is an infimum, there must exist a sequence (g_n) of functions in G_u such that $\lim_{n\to\infty} m_{g_n} = m$. Now, for each g_n, there exists a $v_n \in V$ such that $m_{g_n} = g_n(v_n)$. Since V is finite, there must be at least one vertex, call it v_0, that appears infinitely many times in the sequence $(g_n(v_n))$. So, there exists a subsequence of the form $(g_{n_t}(v_0))$ which also converges to m. We now define

$$g = \lim_{t\to\infty} g_{n_t}.$$

First note that just as in Equation 3, $g(V) = W$, and by Lemma 3.2(d) and (e), g must have the same number of zeros as each of the functions g_n, so $z_g = n_z$. Similarly, by Lemma 3.2(f) (g), $u_g = u_{g_n}$. Thus, $g \in G_u$, and

$$g(v_0) = \lim_{t\to\infty} g_{n_t}(v_0) = m.$$

□

Since m is a minimum value, we may define the set $G_m = \{g \in G_u : m_g = m\}$ and know that it is non-empty.

Lemma 3.6. *Let $g \in G_m$ and $v \in V$ such that $g(v) = m$. For every $y \in V - \{v\}$ such that $g(y) > 0$, there exists a g-cover of y which does not include v, that is, there exists $w \in V$ such that $w \neq y$, $y \in N[w]$, $v \notin N[w]$, and $g(N[w]) \leq 1$.*

Proof. Let $y \in V - \{v\}$. If $g(y) = 1$, then since g is oiru, y has a g-cover N, and that cover cannot contain v since $g(v) = m > 0$ would cause $g(N) > 1$.

So, we need only consider the case of $0 < g(y) < 1$. To obtain a contradiction, assume that every g-cover of y contains v. Define a new function $\tilde{g} : V \to [0,1]$ by shifting an amount a_g from $g(v)$ to $g(y)$. That is, let

$$\tilde{g}(v) = g(v) - a_g$$
$$\tilde{g}(y) = g(y) + a_g$$

and $\tilde{g} = g$ at all other vertices. Since $g(v) = m = m_g$, from the definition of a_g, it follows that $z_{\tilde{g}} = z_g$ and $u_{\tilde{g}} = u_g$. Furthermore, since we simply shifted weight from one vertex to another, $g(V) = W$. However, $\tilde{g}(v) < m$, which was the minimum possible weight for an oiru function, thus $\tilde{g}$ must fail to be oiru. There must exist $p_y \in V$ such that $\tilde{g}(p_y) > 0$ but p_y has no $\tilde{g}$-cover.

Note that since g and $\tilde{g}$ share the same zeros and $\tilde{g}(p_y) > 0$, we know that $g(p_y) > 0$ and thus p_y has a g-cover. Any g-cover, N_p of p_y must contain y and not contain v, since it can only fail to be a $\tilde{g}$-cover if $\tilde{g}(N_p) > 1$. Note that N_p is not a g-cover of y then, since we assumed that all such covers also included v. The only way it can fail to be a g-cover is if y is the center of the neighborhood N_p. Therefore, we know that p_y and y are adjacent vertices and that $p_y \neq y$.

Now define one more function, $\breve{g} : V \to [0,1]$ as follows:

$$\breve{g}(y) = 0$$
$$\breve{g}(p_y) = g(p_y) + g(y)$$

and $\breve{g} = g$ at all other vertices. That is, shift all of the value at y to p_y. Now, $\breve{g}$ can't be oiru since it has one more zero than g, and g had the maximum possible for oiru functions. Thus, there exists some $t \in V$ such that $\breve{g}(t) > 0$ but t has no $\breve{g}$-cover. Now, $t \neq y$, since $\breve{g}(y) = 0$, so $g(t) > 0$. (The only other vertex where g and $\breve{g}$ fail to be equal is at p_y, and $g(p_y) > 0$.) So, t has a g-cover, N_t. Now N_t is not centered at t, so the only way it can fail to be a $\breve{g}$-cover is if $\breve{g}(N_t) > 1$. Since $g(N_t) \leq 1$, it must be the case that $p_y \in N_t$ and $y \notin N_t$. Recall that every g-cover of p_y must contain y, so N_t cannot be a g-cover for p_y, which means that p_y must be the center of N_t. We proved above that y is adjacent to p_y, so y must be in N_t, giving us a contradiction □

Proposition 3.7. *$OIRU_f = OIR_f = OIR$*

Proof. By the previous Proposition, $G_m \neq \emptyset$. If $m = 1$ then every $g \in G_m$ takes on values of either 0 or 1, thus they are all open irredundant, and the supremum of the weights of all open irredundant functions must be greater than their weight. That is, $\text{OIR} \geq W = \text{OIRU}_f$. Combining this with the inequality (1), gives the desired result.

We now show that in fact, $m = 1$ is the only possibility. Let $g \in G_m$ and $v \in V$ such that $g(v) = m$. By Lemma 3.6, each $y \in V - \{v\}$ with

$g(y) > 0$, has a g-cover, N_y, which does not contain v. For an arbitrary $\epsilon > 0$, define a new function, $h : V \to [0, 1]$ via

$$h(v) = m + \epsilon$$
$$h(y) = g(y), \text{for all } y \in V - \{v\}$$

Since $g(V) = W$, we have $h(V) = W + \epsilon$. However, the maximum weight for an oiru function is W, so h cannot be oiru. The only vertex whose value differs from g is v, so every $y \in V - \{v\}$ has an h-cover. To fail at being oiru, there must be no h-cover for v. This is only possible if for every $w \in N(v)$, $g(N[w]) > 1 - \epsilon$, and since ϵ is arbitrary, it must be the case that $g(N[w]) \geq 1$ for every $w \in N(v)$.

However, g is oiru, so for some $w_0 \in N(v)$ we must have $g(N[w_0]) = 1$. Let x be any vertex in $N[w_0] - \{v\}$, and define a new function $\tilde{h} : V \to [0, 1]$ via

$$\tilde{h}(v) = m + g(x)$$
$$\tilde{h}(x) = 0,$$

and $\tilde{h} = g$ at all other vertices. Then $\tilde{h}(V) = W$, and every $y \in V - \{v\}$ has an $\tilde{h}$-cover, since they each had a g-cover which did not include v, the only vertex whose value increased. Since we just shifted the value of $g(x)$ to v, $\tilde{h}(N[w_0]) = 1$, so v also has an $\tilde{h}$ cover. Thus, $\tilde{h}$ is oiru. Recall that g was oiru with the minimum number of zeros, so the only way $\tilde{h}$ can also be oiru is for $g(x) = 0$ for every $x \in N[w_0] - \{v\}$. Since $g(N[w_0]) = 1$, it must be the case that $m = g(v) = 1$. □

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