An Anti-Waring Theorem

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Abstract. It is proven that for all positive integers k, n, and r, every sufficiently large positive integer is the sum of r or more kth powers of distinct elements of $\{n, n+1, n+2, \dots\}$. The case $n = 1$ is the conjecture in the title of [1].

In 1770, Waring conjectured that for each positive integer k there exists a $g(k)$ such that every positive integer is a sum of $g(k)$ or fewer kth powers of positive integers. Hilbert proved this theorem in 1909, giving rise to Waring's problem, which asks, for each k, what is the smallest $g(k)$ such that the statement holds. For further details, see [3].

As a natural question arising from this problem, Johnson and Laughlin [1] proposed what they called an anti-Waring conjecture, which is the following: *If k and r are positive integers, then every sufficiently large positive integer is the sum of r or more distinct kth powers of positive integers.* When this holds for a pair k, r, let $N(k,r)$ denote the smallest positive integer such that each integer n greater than or equal to $N(k,r)$ is the sum of r or more kth powers of distinct positive integers. As noted in [1], it is easy to see that, for all r,

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$N(1,r) = 1+2+\cdots+r = \dfrac{r(r+1)}{2}$. It is also shown in [1] that $N(2,1) = N(2,2) = N(2,3) = 129$.

Johnson and Laughlin further posed the question of whether given any positive integers k, n, r, there exists an integer $N(k,n,r)$ such that every integer z greater than or equal to $N(k,n,r)$ can be written as a sum of r or more distinct elements from the set $\{m^k \mid m \in \mathbb{N}, m \geq n\}$. The aim of this paper is to prove both this statement and the anti-Waring conjecture to be true.

Definition Let S be a set of real numbers. S is said to be *complete* if all sufficiently large integers can be written as a sum of distinct elements of S.

Theorem 1 *If k and r are positive integers, then every sufficiently large positive integer is the sum of r or more distinct kth powers of positive integers.*

To prove this, we use the following theorem by Roth and Szekeres [2]:

Let $f(x) = \alpha_n x^n + \cdots + \alpha_1 x + \alpha_0$ with $n > 0, \alpha_n \neq 0$ be a polynomial which maps integers into integers. (Thus all the α_k are rational numbers.) Let $S(f)$ denote the set $\{f(j) \mid j = 1, 2, \ldots\}$. Then $S(f)$ is complete if and only if:

(1) $\alpha_n > 0$

(2) For any prime p, there exists an integer m such that p does not divide $f(m)$.

Proof of Theorem 1. For each k, we proceed by induction on r. For $r = 1$, apply the theorem of Roth and Szekeres with $f(x) = x^k$ to conclude that $N(k,1)$ exists.

Now suppose that r is greater than or equal to 1, and that $N(k,r)$ exists, say $N(k,r) = B$, for short. We aim to show that $N(k, r+1)$ exists, meaning that every sufficiently large integer m is the sum of $r+1$ or more kth powers of distinct positive integers.

Clearly $2a^k - (a+1)^k$ tends to infinity as a tends to infinity. Therefore, there is a positive integer A such that for all $a \geq A$, we have $2(a^k) > (a+1)^k + B$.

Now let m be any integer greater than or equal to $A^k + B$, so that $m - B \geq A^k$. Let a be the greatest integer such that $a^k \leq m - B$. By the definition of a, we have

$$a^k \leq m - B < (a+1)^k$$

Since $a \geq A$, we can combine these last two inequalities to get

$$a^k \leq m - B < (a+1)^k < 2(a^k) - B$$

Then $B \leq m - a^k$, so $m - a^k = s(1)^k + \cdots + s(t)^k$, where the $s(j)$ are distinct positive integers, and $t \geq r$. If any of the $s(j)$ were equal to a, then we would have that $m \geq 2a^k$, contradicting the inequality $m - B < 2a^k - B$, from above. Therefore, $m = a^k + [s(1)^k + + s(t)^k]$ is the sum of $r+1$ or more kth powers of distinct positive integers. Since m was an arbitrary integer greater than or equal to $A^k + B$, this finishes the inductive step. $N(k, r+1)$ exists and is no greater than $A^k + N(k, r)$.

We now address the second question of Johnson and Laughlin.

Theorem 2 *For any positive integers k, n, r, every sufficiently large integer can be expressed as a sum of r or more distinct elements of the set $\{m^k \mid m \in \mathbb{N}, m \geq n\}$.*

Proof of Theorem 2. The proof is very much like that of Theorem 1. For each pair k, n we proceed by induction on r. The case of $r = 1$ is disposed of by applying the theorem of Roth and Szekeres with $f(x) = (x + n - 1)^k$. From there the induction step is just as in the proof of Theorem 1, with the stronger induction hypothesis and with $N(k, n, r) = B$ playing the role played earlier by $N(k, r)$.

References

[1] Peter Johnson and Michael Laughlin, An anti-Waring conjecture and problem, *International Journal of Mathematics and Computer Science*, 6 (2011), no. 1, 21-26.

[2] K.F. Roth and G. Szekeres, Some asymptotic formulae in the theory of partitions, *Quarterly Journal of Mathematics*, 5 (1954), 241-259.

[3] Warings Problem, Wolfram MathWorld,
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