Construction of deterministic Compressed Sensing matrices based on error-correcting pooling designs

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Abstract Compressed sensing (CS), which is a rising technique of signal processing, successfully manages the huge expenditure of increasing the sampling rate as well as the intricate issues to our work. Hence, more and more attention has been paid to CS during recent years. In this paper, we construct a family of error-correcting pooling designs based on singular linear space over finite fields, which can be efficiently applied to signal processing in terms of CS.

Keywords: Compressed sensing, Pooling designs, disjunct and inclusive matrices, Singular linear space

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1. Introduction

In the wake of rapid development of information technology, CS[1−2] emerges as a new theory, which provides an effective way to cope with signals processing. Due to high efficiency and economizing resources, CS has attracted considerable attention. According to the Nyquist sampling theorem, it is well known that if we want to keep information from losing when uniformly sampling a signal, we must sample at least two times faster than its bandwidth. Acutually, not only does it become more and more difficult to satisfy the requirement in many applications, such as digital image, video cameras, medical scanners, radars and so on, but also wastes a mass of resources. However, CS breaks through traditional theory with remarkably reducing the time of signal processing and the cost of calculation. Consequently, it will lead to the coming of new era with respect to signal processing.

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Roughly speaking, CS can be generally described as two steps. Firstly, given a measurement matrix called (compressed) sensing matrix, which is served as collecting the information and simultaneously compressing a sparse signal. Secondly, recover the sparse signal with the measurement matrix by solving an optimization problem. It is apparent from the above-mentioned that sensing matrix plays an significant in the CS theorem. However, there are two kinds of sensing matrices. One is called random sensing matrices whose entries are randomly drawn from certain probability distributions, which concludes Gaussian matrices; Bernoulli matrices; Random partial orthogonal matrices[3−5] and so on. Another is named deterministic sensing matrices, whose properties are better than random sensing matrices. As a matter of fact, there are some defects about random sensing matrices[6]. First, as the random selections of entries, random sensing matrices demand a lot of storage space to store them. Second, there is a tedious calculation to computers. Adversely, the deterministic sensing matrices can get rid of those defects. Here we put our focus on the latter ones.

For a discrete sparse signal $x \in R^t$ which has $k \ll t$ nonzero entries. Define the support set of x as

$$S(x) = \{i \in \{1, \cdots, t\} \mid x_i \neq 0\},$$

where $k = |S(x)|$ is the sparsity of x. If the support set is identified, the recovery of sparse signal is a routine task, which means the recovery of signals can be regarded as recovering the support set of x in the terms of the CS theorem. Generally speaking, for an $s \times t$ measurement matrix Φ with $s < t$, which can be used to compress x into a measurement vector $y \in R^s$ with s times:

$$y = \Phi x + \epsilon,$$

where ϵ is a noise term. To a large extent, the prosperity of the recovery process depends on the performance of the measurement matrix. In other word, if the matrix Φ is appropriate, $S(x)$ can be recovered by y, even though $s \ll t$.

As a mathematical tool, pooling design is noted as using in molecular biology, such as DNA library screening, nonunique probe selection, gene detection, etc. A pooling design is usually represented by a binary matrix, whose columns are indexed by items and rows are indexed by pools. The value of an entry at cell (i, j) equals to 1 or 0, if the i-th pool contains or does not contain the j-th item, respectively. Actually, we all known that it is inevitable for biological experiments to produce erroneous outcomes. Thereby, in order to make pooling design error tolerant, the concept of k^e-disjunct matrix (see [7]) is introduced. A binary matrix Φ is called k^e-disjunct if given any $k + 1$ columns of Φ with one appointed, there are

$e+1$ rows with a 1 in the appointed column and 0 in each of the other k columns. An k^0-disjunct matrix is actually called k-disjunct. D'yachkov et al. proposed the concept of fully k^e-disjunct matrices (see [8]). A k^e-disjunct matrix is fully k^e-disjunct if it is not c^a-disjunct whenever $c > k$ or $a > e$.

Definition 1.1.[9] A binary matrix Φ is $(k; e_1)$-disjunct if for any $k+1$ distinct columns $C_0, C_1, \ldots, C_k$,

$$| C_0 \setminus \bigcup_{i=1}^{k} C_i | \geq e_1,$$

and is $(k; e_2)$-inclusive if for any $k+1$ distinct columns $C_0, C_1, \ldots, C_k$,

$$| C_0 \bigcap (\bigcup_{i=1}^{k} C_i) | \leq e_2.$$

The disjunct and inclusive matrices act an crucial role in the recovery of support set $S(x)$ scheme.

Now, we introduce the link between pooling designs and support recovery [9]. Define the quantization operator $Q : R^s \to \{0,1\}^s$, which maps $v \in R^s$ to a binary vector $\bar{v}$ satisfying

$$\bar{v}_i = \begin{cases} 1, & \text{if } v_i > 0, \\ 0, & \text{if } v_i \leq 0. \end{cases}$$

For a given signal $x \in R^t$, let

$$S_+(x) = \{i \in S(x) \mid x_i > 0\}, \qquad S_-(x) = \{i \in S(x) \mid x_i < 0\}.$$

In order to complete support recovery, it is suffices to recover both $S_+(x)$ and $S_-(x)$. Suppose Φ is a disjunct and inclusive matrix, then we have

$$y = \Phi x + \epsilon, \quad \bar{y} = Q(y). \tag{1}$$

Our goal is to recover the position of positive entries from $\bar{y}$, which is the quantization result of the inaccurate measurement vector y. Consider the vector ϵ, its i-th entry leads to a destructive error if

$$Q(\Phi x)_i \neq Q(\Phi x + \epsilon)_i.$$

Therefore, those errors alter the result $\bar{y}$. The destructive parts need to be dealt with.

Lemma 1.2.[9] Let $x \in R^t$ be a sparse signal with $|S(x)| = k \ll t$. In the formulation (1), suppose Φ is an $s \times t$ $(k; e_1)$-disjunct and $(k; e_2)$-inclusive matrix and the noise vector ϵ causes l destructive errors with $l < \frac{e_1 - e_2}{2}$. Then $S_+(x)$ can be identified.

The algorithm can be used to recover $S(x)$ is as fellow:

Algorithm 1[9]

Input:

- A $(k; e_1)$-disjunct and $(k; e_2)$-inclusive $s \times t$ measurement matrix Φ
- A binary quantization result $\bar{y}$

Output:

- Support set $S(x)$

Procedure:

1: $S_+(x) \leftarrow \emptyset, S_-(x) \leftarrow \emptyset$
2: Compute the tolerable number of destructive errors l, where l is the largest integer smaller than $\frac{e_1 - e_2}{2}$
3: Compute $b = j - \bar{y}$, where j is an all one vector with length s
4: **for** each $1 \le i \le t$ **do**
5: Use Φ_i to denote the i-th column of Φ, compute $|\Phi_i^0| = \Phi_i^T b$
6: **if** $|\Phi_i^0| \le e_2 + l$ **then**
7: $S_+(x) \leftarrow S_+(x) \bigcup \{i\}$
8: **end if**
9: **end for**
10: Flip the entries of $\bar{y}$ by $0 \rightarrow 1, 1 \rightarrow 0$
11: Repeat steps 3-9 in which $S_+(x)$ substituted by $S_-(x)$
12: Return $S(x) = S_+(x) \bigcup S_-(x)$

The Lemma tells us that $S_+(x)$ can be identified when the CS matrix is regarded as a disjunct and inclusive matrix. It is suffice to exchange the roles of negative and positive entries by displacing x, y, ϵ with $-x, -y, -\epsilon$ respectively, the $S_-(x)$ also can be identified. Hence, the support set $S(x)$ can be recovered. In fact, the Lemma just means if $e_1 > e_2$, then the $S(x)$ can be recovered exactly. Based on the disjunct and inclusive properties of the measurement matrix, the above-mentioned recovery algorithm is provided in [9].

Recently, some deterministic constructions of sensing matrices have been presented. DeVore's polynomials over finite fields[10]; algebraic curves by Gao et al.[11]; Amini and Marvasti's bipolar matrix by BCH code[12] and its generalization[13]; additive combinatorics by Bourgain et al.[14]. Inspired by the above studying, we propose a new construction of pooling designs based on singular linear spaces over finite fields, which can be used to process signals effectively.

2. Singular linear spaces

In this section we shall introduce the concepts of subspaces of type (m, h)

in singular linear spaces, (see Wang et al. [15]) and provide several lemmas.

Let $\mathbb{F}_q$ be a finite field with q elements, where q is a prime power. For two non-negative integers n and l, $\mathbb{F}_q^{(n+l)}$ denotes the $(n+l)$-dimensional row vector space over $\mathbb{F}_q$. The set of all $(n+l)\times(n+l)$ nonsingular matrices over $\mathbb{F}_q$ of the form

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

where T_{11} and T_{22} are nonsingular $n \times n$ and $l \times l$ matrices, respectively, forms a group under matrix multiplication, called the singular general linear group of degree $n+l$ over $\mathbb{F}_q$ and denoted by $GL_{n+l,n}(\mathbb{F}_q)$. If $l=0$ (resp. $n=0$), $GL_{n,n}(\mathbb{F}_q) = GL_n(\mathbb{F}_q)$ (resp. $GL_{l,0}(\mathbb{F}_q) = GL_l(\mathbb{F}_q)$) is the general linear group of degree n (resp. l). (See Wan [16]) Let P be a m-dimensional subspace of $\mathbb{F}_q^{(n+l)}$, denote also by P an $m\times(n+l)$ matrix of rank m whose rows span the subspace P and call the matrix P a matrix representation of the subspace P. There is an action of $GL_{n+l,n}(\mathbb{F}_q)$ on $\mathbb{F}_q^{(n+l)}$ defined as follows

$$\mathbb{F}_q^{(n+l)} \times GL_{n+l,n}(\mathbb{F}_q) \to \mathbb{F}_q^{(n+l)}$$

$$((x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+l}),T) \mapsto (x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+l})T.$$

The above action induces an action on the set of subspaces of $\mathbb{F}_q^{(n+l)}$, i.e., a subspace P is carried by $T \in GL_{n+l,n}(\mathbb{F}_q)$ to the subspace PT. The vector space $\mathbb{F}_q^{(n+l)}$ together with the above group action, is called the $(n+l)$-dimensional singular linear space over $\mathbb{F}_q$. For $1 \le j \le n+l$, let e_j be the row vector in $\mathbb{F}_q^{(n+l)}$ whose j-th coordinate is 1 and all other coordinates are 0. Denote by E the l-dimensional subspace of $\mathbb{F}_q^{(n+l)}$ generated by $e_{n+1}, e_{n+2}, \ldots, e_{n+l}$. A m-dimensional subspace P of $\mathbb{F}_q^{(n+l)}$ is called a subspace of type (m,h) if $\dim(P\cap E) = h$. The collection of all the subspaces of types $(m,0)$ in $\mathbb{F}_q^{(n+l)}$, where $0 \le m \le n$, is the attenuated space. (see A.E. Brouwer et al. [17])

Lemma 2.1. Let V denote the $(n+l)$-dimensional row vector space over a finite field $\mathbb{F}_q$, and fix a subspace W of type $(n+l-d,h)$ contained in V. Let $\mathcal{M}_+(i_1,h_1;d,h;n+l,n)$ denote the set of all subspaces U of type (i_1,h_1) contained in V satisfying $U+W=V$, and let $N_+(i_1,h_1;d,h;n+l,n)$ denote the size of $\mathcal{M}_+(i_1,h_1;d,h;n+l,n)$. Then

$$N_+(i_1,h_1;d,h;n+l,n) = q^{d(n+l-i_1)} \begin{bmatrix} n+l-d-h \\ i_1-h_1-d \end{bmatrix}_q \begin{bmatrix} h \\ h_1 \end{bmatrix}_q. \qquad (2)$$

Proof. By the transitivity of $GL_{n+l,n}(\mathbb{F}_q)$ on the set of subspaces of the same type, we may choose the subspace W of type $(n+l-d, h)$ as the form

$$\begin{pmatrix} I^{(n+l-d-h)} & 0^{(n+l-d-h,d+h-l)} & 0 & 0 \\ 0 & 0 & I^{(h)} & 0^{(h,l-h)} \end{pmatrix}.$$

Let U has a matrix representation of the form

$$\begin{pmatrix} X^{(i_1-d-h_1,n+l-d-h)} & 0^{(i_1-d-h_1,d+h-l)} & 0 & 0 \\ Y^{(d,n+l-d-h)} & I^{(d+h-l)} & B^{(d,h)} & I^{(d,l-h)} \\ 0^{(h_1,n+l-d-h)} & 0 & A^{(h_1,h)} & 0^{(h_1,l-h)} \end{pmatrix},$$

where X is an $(i_1-d-h_1)\times(n+l-d-h)$ matrix of rank (i_1-d-h_1), Y is a $d\times(n+l-d-h)$ matrix, A is an $h_1\times h$ matrix of rank h_1, B is a $d\times h$ matrix. Then X is an (i_1-d-h_1)-subspace which contained in $I^{(n+l-d-h)}$. By Wan (2002b, Theorem 1.7 [16]), there are $\begin{bmatrix} n+l-d-h \\ i_1-h_1-d \end{bmatrix}_q$ choices for X. By the same token, A is an h_1-subspace which contained in $I^{(h)}$ and has $\begin{bmatrix} h \\ h_1 \end{bmatrix}_q$ choices. By the transitivity of $GL_{n+l,n}(\mathbb{F}_q)$, we may let $X = (I^{(i_1-d-h_1)} \quad 0^{(i_1-d-h_1,n+l-h-i_1+h_1)})$, $A = (I^{(h_1)} \quad 0^{(h_1,h-h_1)})$. Then U has the unique matrix representation of the form

$$\begin{pmatrix} I^{(i_1-d-h_1)} & 0^{(i_1-d-h_1,n+l-h-i_1+h_1)} & 0^{(i_1-d-h_1,h+d-l)} & 0^{(i_1-d-h_1,h_1)} & 0 & 0 \\ 0 & Y_1^{(d,n+l-h-i_1+h_1)} & 0^{(d,h+d-l)} & 0^{(d,h_1)} & B_1^{(d,h-h_1)} & 0^{(d,l-h)} \\ 0^{(h_1,i_1-d-h_1)} & 0 & 0^{(h_1,h+d-l)} & I^{(h_1)} & 0 & 0^{(h_1,l-h)} \end{pmatrix}$$

Hence $N_+(i_1,h_1;d,h;n+l,n) = q^{d(n+l-i_1)}\begin{bmatrix} n+l-d-h \\ i_1-h_1-d \end{bmatrix}_q \begin{bmatrix} h \\ h_1 \end{bmatrix}_q$. □

Lemma 2.2. Let V denote the $(n+l)$-dimensional row vector space over a finite field $\mathbb{F}_q$, and fix a subspace W of type $(n+l-d, h)$ contained in V. For a given subspace U_2 of type (i_2, h_2) contained in V satisfying $U_2+W=V$, let $N_+(i_1,h_1;i_2,h_2;d,h;n+l,n)$ denote the number of subspaces U_1 of type (i_1,h_1) contained in V satisfying $U_1+W=V$ and $U_1\subseteq U_2$. Then

$$N_+(i_1,h_1;i_2,h_2;d,h;n+l,n) = q^{d(i_2-i_1)}\begin{bmatrix} i_2-d-h_2 \\ i_1-h_1-d \end{bmatrix}_q \begin{bmatrix} h_2 \\ h_1 \end{bmatrix}_q. \qquad (3)$$

Proof. Since the subgroup $GL_{n+l,n}(\mathbb{F}_q)_W$ of $GL_{n+l,n}(\mathbb{F}_q)$ fixing W acts transitively on the set $\{U|U+W=V, \dim U = i_2\}$, the number $N_+(i_1,h_1;i_2,h_2;d,h;n+l,n)$ depends only on i_1 and i_2. By Lemma 2.1 and (2), we get $N_+(i_1,h_1;i_2,h_2;d,h;n+l,n) = q^{d(i_2-i_1)}\begin{bmatrix} i_2-d-h_2 \\ i_1-h_1-d \end{bmatrix}_q \begin{bmatrix} h_2 \\ h_1 \end{bmatrix}_q$. □

Lemma 2.3. Let V denote the $(n+l)$-dimensional row vector space over a finite field $\mathbb{F}_q$, and fix a subspace W of type $(n+l-d, h)$ contained in V. For a given subspace U_1 of type (i_1, h_1) contained in V satisfying $U_1 + W = V$, let $N_+'(i_1, h_1; i_2, h_2; d, h; n+l, n)$ denote the number of subspaces U_2 of type (i_2, h_2) contained in V satisfying $U_2 + W = V$ and $U_1 \subseteq U_2$. Then

$$N_+'(i_1, h_1; i_2, h_2; d, h; n+l, n) = \begin{bmatrix} n+l-h-i_1+h_1 \\ i_2-h_2-i_1+h_1 \end{bmatrix}_q \begin{bmatrix} h-h_1 \\ h_2-h_1 \end{bmatrix}_q. \quad (4)$$

Proof. Let

$M = \{(U_1, U_2) | U_1 \in \mathcal{M}_+(i_1, h_1; d, h; n+l, n), U_2 \in \mathcal{M}_+(i_2, h_2; d, h; n+l, n), U_1 \subseteq U_2\}$.

We compute the size of M in the following two ways.

For a fixed subspace U_1 of type (i_1, h_1), there are $N_+'(i_1, h_1; i_2, h_2; d, h; n+l, n)$ subspaces of type (i_2, h_2) containing U_1. By Lemma 2.1

$$|M| = N_+'(i_1, h_1; i_2, h_2; d, h; n+l, n) N_+(i_1, h_1; d, h; n+l, n). \quad (5)$$

For a fixed subspace U_2 of type (i_2, h_2), there are $N_+(i_1, h_1; i_2, h_2; d, h; n+l, n)$ subspaces of type (i_1, h_1) contained in U_2. By Lemma 2.1

$$|M| = N_+(i_1, h_1; i_2, h_2; d, h; n+l, n) N_+(i_2, h_2; d, h; n+l, n). \quad (6)$$

Combining (5), (6), (3) and (2), (4) holds. □

Lemma 2.4. Given integers $0 \le h_1 \le h \le l$ and $d \le i - h_1 \le n+l-h \le n+d$, the sequence $N_+(i, h_1; d, h; n+l, n)$ is unimodal and gets its peak at $i = \lfloor \frac{n+l-h}{2} \rfloor + h_1$.

Proof. By Lemma 2.1, if $i_1 < i_2$, then we have

$$\frac{N_+(i_1, h_1; d, h; n+l, n)}{N_+(i_2, h_1; d, h; n+l, n)}$$

$$= \frac{q^{d(n+l-i_1)} \begin{bmatrix} n+l-d-h \\ i_1-h_1-d \end{bmatrix}_q \begin{bmatrix} h \\ h_1 \end{bmatrix}_q}{q^{d(n+l-i_2)} \begin{bmatrix} n+l-d-h \\ i_2-h_1-d \end{bmatrix}_q \begin{bmatrix} h \\ h_1 \end{bmatrix}_q}$$

$$= q^{d(i_2-i_1)} \cdot \frac{\prod_{i=i_1-h_1-d+1}^{i_2-h_1-d} (q^i-1)}{\prod_{i=n+l-h-i_2+h_1+1}^{n+l-h-i_1+h_1} (q^i-1)}$$

$$= \frac{(q^{i_1-h_1+1}-q^d)(q^{i_1-h_1+2}-q^d)\dots(q^{i_2-h_1}-q^d)}{(q^{n+l-h-i_2+h_1+1}-1)(q^{n+l-h-i_2+h_1+2}-1)\dots(q^{n+l-h-i_1+h_1}-1)}$$

$$= \frac{q^{i_1-h_1+1}-q^d}{q^{n+l-h-i_1+h_1}-1} \cdot \frac{q^{i_1-h_1+2}-q^d}{q^{n+l-h-i_1+h_1-1}-1} \cdot \dots \cdot \frac{q^{i_2-h_1}-q^d}{q^{n+l-h-i_2+h_1+1}-1},$$

where $\frac{q^{i_1-h_1+1}-q^d}{q^{n+l-h-i_1+h_1}-1} < \frac{q^{i_1-h_1+2}-q^d}{q^{n+l-h-i_1+h_1-1}-1} < \dots < \frac{q^{i_2-h_1}-q^d}{q^{n+l-h-i_2+h_1+1}-1}$.

If $i_2 \le \lfloor \frac{n+l-h}{2} \rfloor + h_1$, then $i_2 - h_1 < n+l-h-i_2+h_1+1$, $\frac{q^{i_2-h_1}-q^d}{q^{n+l-h-i_2+h_1+1}-1} < 1$. Hence, when $h_1+d \le i_1 < i_2 \le \lfloor \frac{n+l-h}{2} \rfloor + h_1$, we have $\frac{N_+(i_1,h_1;d,h;n+l,n)}{N_+(i_2,h_1;d,h;n+l,n)} < 1$.

If $i_1 \ge \lfloor \frac{n+l-h}{2} \rfloor + h_1$, then $i_1 - h_1 + 1 > n+l-h-i_1+h_1$, $\frac{q^{i_1-h_1+1}-q^d}{q^{n+l-h-i_1+h_1}-1} > 1$. Hence, when $\lfloor \frac{n+l-h}{2} \rfloor + h_1 \le i_1 < i_2 \le n+l-h+h_1$, we have $\frac{N_+(i_1,h_1;d,h;n+l,n)}{N_+(i_2,h_1;d,h;n+l,n)} > 1$. □

Proposition 2.5.(Wan [16] Corollary1.9) Let $0 \le i_1 \le i_2 \le n$. Then the number $N'(i_1,i_2,n)$ of i_2-dimensional vector subspaces containing a given i_1-dimensional vector subspace $\mathbb{F}_q^{(n)}$ is equal to

$$\begin{bmatrix} n-i_1 \\ i_2-i_1 \end{bmatrix}_q .$$

3. The construction

In this section, we construct a family of inclusion matrices associated with subspaces of $\mathbb{F}_q^{(n+l)}$, then exhibit its disjunct property and the performance of signal recovery. Finally, we show our construction is superior to the DeVore's construction using polynomials over finite fields on the recovery of signals.

Definition 3.1. Given integers $0 \leq h_1 \leq h_2 \leq h \leq l$ and $d \leq i_1 - h_1 \leq i_2 - h_2 \leq n+l-h \leq n+d$. Let $M_+(i_1, h_1; i_2, h_2; d, h; n+l, n)$ be the binary matrix whose rows (resp. columns) are indexed by $\mathcal{M}_+(i_1, h_1; d, h; n+l, n)$ (resp. $\mathcal{M}_+(i_2, h_2; d, h; n+l, n)$). and with a 1 or 0 in the (i, j) position of the matrix, if the i-th subspace which belongs to $\mathcal{M}_+(i_1, h_1; d, h; n+l, n)$ is or is not contained in the j-th subspace which belongs to $\mathcal{M}_+(i_2, h_2; d, h; n+l, n)$, respectively.

By Lemma 2.1, 2.2 and 2.3, $M_+(i_1, h_1; i_2, h_2; d, h; n+l, n)$ is a $N_+(i_1, h_1; d, h; n+l, n) \times N_+(i_2, h_2; d, h; n+l, n)$ matrix, whose constant row (resp. column) weight is $N_+'(i_1, h_1; i_2, h_2; d, h; n+l, n)$ (resp. $N_+(i_1, h_1; i_2, h_2; d, h; n+l, n)$). Lemma 2.4 tells us how to choose i_2 so that the test to item is minimized.

Theorem 3.2. Given integers $0 \leq h_1 \leq h_2 - 2$, $h_2 \leq h \leq l$, $d \leq i_1 - h_1 \leq i_2 - h_2 - 2$, $i_2 - h_2 \leq n+l-h \leq n+d$ and let $t = N_+(i_1, h_1; i_2, h_2; d, h; n+l, n)$, $u = N_+(i_1, h_1; i_2 - 1, h_2; d, h; n+l, n)$, $v = N_+(i_1, h_1; i_2 - 1, h_2 - 1; d, h; n+l, n)$, $x = N_+(i_1, h_1; i_2 - 2, h_2; d, h; n+l, n)$, $y = N_+(i_1, h_1; i_2 - 2, h_2 - 1; d, h; n+l, n)$, $z = N_+(i_1, h_1; i_2 - 2, h_2 - 2; d, h; n+l, n)$ and $w = \max\{u - x, u - y, u - z, v - x, v - y, v - z\}$, if $1 \leq k \leq \lfloor \frac{t - \max\{u,v\} - 1}{w} \rfloor + 1$ then $M_+(i_1, h_1; i_2, h_2; d, h; n+l, n)$ is k^{e_1}-disjunct and k^{e_2}-inclusive, where $e_1 = t - \max\{u, v\} - (k-1)w - 1$ and $e_2 = \max\{u, v\} + (k-1)w + 1$. In particular, if $1 \leq k \leq \min\{\lfloor \frac{t - \max\{u,v\} - 1}{w} \rfloor + 1, q+1\}$, then $M_+(i_1, h_1; i_2, h_2; d, h; n+l, n)$ is fully k^{e_1}-disjunct.

Proof Let $P, P_1, P_2, \cdots, P_k$ be $k+1$ distinct columns of $M_+(i_1, h_1; i_2, h_2; d, h; n+l, n)$. In order to obtain the maximum numbers of subspaces of $\mathcal{M}_+(i_1, h_1; d, h; n+l, n)$, which contains in

$$P \cap \bigcup_{i=1}^{k} P_i = \bigcup_{i=1}^{k} (P \cap P_i),$$

we may assume that $\dim(P \cap P_i) = i_2 - 1$ and $\dim(P \cap P_i \cap P_j) = \dim((P \cap P_i) \cap (P \cap P_j)) = i_2 - 2$, for any two distinct i and j , where $1 \leq i, j \leq k$. Since $P \in \mathcal{M}_+(i_2, h_2; d, h; n+l, n)$, $P \cap P_j$ (resp. $P \cap P_i \cap P_j$) is a subspace of type $(i_2 - 1, h_2)$ or type $(i_2 - 1, h_2 - 1)$(resp. type $(i_2 - 2, h_2)$ or type $(i_2 - 2, h_2 - 1)$ or type $(i_2 - 2, h_2 - 2)$) of $\mathbb{F}_q^{(n+l)}$ by Lemma 2.1.

By Lemma 2.2, $x > 0$, $y > 0$ and $z > 0$. By Lemma 2.2, the number of subspaces of P not covered by $P_1, P_2, \cdots, P_k$ is at least

$$t - \max\{u,v\} - (k-1)[\max\{u,v\} - \min\{x,y,z\}]$$
$$= t - k\max\{u,v\} + (k-1) \times \min\{x,y,z\} = t - \max\{u,v\} - (k-1)w.$$

Therefore, we may take $e_1 = t - \max\{u,v\} - (k-1)w - 1$ under the assumption that k. Since $e_1 \geq 0$, we get

$$k \leq \lfloor \frac{t - \max\{u,v\} - 1}{w} \rfloor + 1.$$

At the same time, as the definition of k^{e_1}-disjunct matrix and k^{e_2}-inclusive matrix, we can get $e_2 = t - e_1 = \max\{u,v\} + (k-1)w + 1$ under the assumption that k. As $k \geq 1 > \frac{w - \max\{u,v\} - 1}{w}$, then $e_2 > 0$.

Now we show that the above lower bound $e_1 + 1$ can not be increased by an specific construction. For $P \cap P_1$, by Lemma 2.2, $N_+(i_1, h_1; i_2 - 2, h_2; d, h; n+l, n) \geq 1$, $N_+(i_1, h_1; i_2 - 2, h_2 - 1; d, h; n+l, n) \geq 1$ and $N_+(i_1, h_1; i_2 - 2, h_2 - 2; d, h; n+l, n) \geq 1$. Hence there exists an $(i_2 - 2)$-dimensional subspace contained in $P \cap P_1$, denoted by V, such that the number of subspace of type (i_1, h_1) contained in V is equal to $\min\{x,y,z\}$. By Proposition 2.5, the number of $(i_2 - 1)$-dimensional subspaces, which contain V and are contained in P, equals to $q + 1$, and each of these subspaces is a subspace of type $(i_2 - 1, h_2)$ or type $(i_2 - 1, h_2 - 1)$. For $1 \leq k \leq \min\{\lfloor \frac{t - \max\{u,v\} - 1}{w} \rfloor + 1, q + 1\}$, we choose k distinct $(i_2 - 1)$-dimensional subspaces between V and P, say $V_i, (1 \leq j \leq k)$. Since $N_+{}'(i_2 - 1, h_2; i_2, h_2; d, h; n+l, n) \geq 2$ and $N_+{}'(i_2 - 1, h_2 - 1; i_2, h_2; d, h; n+l, n) \geq 2$ by Lemma 2.3, for each V_i, we can choose a subspace of type (i_2, h_2) denoted by P_i, such that $P \cap P_i = V_i$. Hence, each pair of P_i and P_j overlaps at the same subspace V.

Now we have showed that $M_+(i_1, h_1; i_2, h_2; d, h; n+l, n)$ is k^{e_1}-disjunct but not k^{e_1+1}-disjunct. Meanwhile we assume that $M_+(i_1, h_1; i_2, h_2; d, h; n+l, n)$ is $(k+1)^{e_1'}$-disjunct. By the maximality of e_1, we infer that

$$e_1' \leq t - \max\{u,v\} - (k+1-1)w - 1 < t - \max\{u,v\} - (k-1)w - 1 = e_1.$$

Hence $M_+(i_1, h_1; i_2, h_2; d, h; n+l, n)$ is not $(k+1)^{e_1}$-disjunct. Consequently, $M_+(i_1, h_1; i_2, h_2;$
$d, h; n+l, n)$ is fully k^{e_1}-disjunct. This completes the proof. □

Theorem 3.3. Let $\Phi = M_+(i_1, h_1; i_2, h_2; d, h; n+l, n)$ and $x \in R^t$ is k-sparse. Then the support of x can be exactly recovery from $y = \Phi x + \epsilon$ when $k < \lfloor \frac{t - 2\max\{u,v\} - 2}{2w} \rfloor + 1$.

Proof By Theorem 3.2, Φ is a $(k;e_1)$-disjunct and $(k;e_2)$-inclusive matrix when

$$1 \leq k \leq \lfloor \frac{t-\max\{u,v\}-1}{w} \rfloor + 1,$$

where $e_1 = t - \max\{u,v\} - (k-1)w - 1, e_2 = \max\{u,v\} + (k-1)w + 1$. According to Lemma 1.2, we get that x can be exactly recovered from $y = \Phi x + \epsilon$ when $e_1 > e_2$, i.e.,

$$k < \lfloor \frac{t-2\max\{u,v\}-2}{2w} \rfloor + 1.$$

This completes the proof. □

Now, we compare a sensing matrix formed by singular linear spaces and a Gaussian random sensing matrix via numerical simulation. For a signal x, we use the algorithm in [9] to solve $S(x)$. If the recovery of support set $S(x)$ can be exactly recovered, we say the recovery of x is perfect. By Theorem 3.2, we get a 27×39 sensing matrix formed by singular linear spaces. Fig. 1 shows the perfect recovery percentage of this matrix and that of a 27×39 random Gaussian matrix. For each sparsity, 5000 input signals are used to compute the perfect recovery percentage. According to the simulation results, the matrix formed by singular linear spaces can be applied to recover signals effectively and outperforms the Gaussian matrix.

Next, Let's recall the DeVore's construction[10]. DeVore provided a kind of deterministic sensing matrix using polynomials over finite fields. For our comparison, we consider finite field of prime power order. Let $\mathbb{F}_{q'}$ be a finite field, where q' is a prime power. Given an integer r, where $0 < r < q'$, let $\mathbb{P}_r$ denotes the set $\{f(x)|\partial(f(x)) \leq r, x \in \mathbb{F}_{q'}\}$. Then there are $t' := {q'}^{r+1}$ such polynomials in it. Denote a null matrix by H with $q' \times q'$ large, and order the positions of H lexicographically as $(0,0), (0,1), \ldots, (q'-1, q'-2), (q'-1, q'-1)$. We classify the construction as three steps. First, insert one to a position of every row of H by the following way. Look $x \mapsto Y(x)$ as a mapping of $\mathbb{F}_{q'} \to \mathbb{F}_{q'}$, where $Y \in \mathbb{P}_r, x \in \mathbb{F}_{q'}$. then change the value of position $(x, Y(x))$ into 1. Every row exactly has a one. Second, Transform H into a column vector v_Y with $s' \times 1$ large, where $s' = {q'}^2$. Note that there are exactly q' ones in v_Y; one in the first q' entries, one in the next q' entries, and so on. Third, Recycle the above two steps for all the polynomials, which belongs to $\mathbb{P}_r$. Hence there are $t' := {q'}^{r+1}$ column vectors. At last, we obtain the matrix Φ'_0 with $s' \times t'$ large.

Lemma 3.4.[10] Suppose the matrix $\Phi' = \frac{1}{\sqrt{q'}}\Phi'_0$, then Φ' satisfies the RIP with $\delta = (k'-1)r/q'$ for any $k' < q'/r + 1$.

In order to compare the upper bound values on k and k', we will

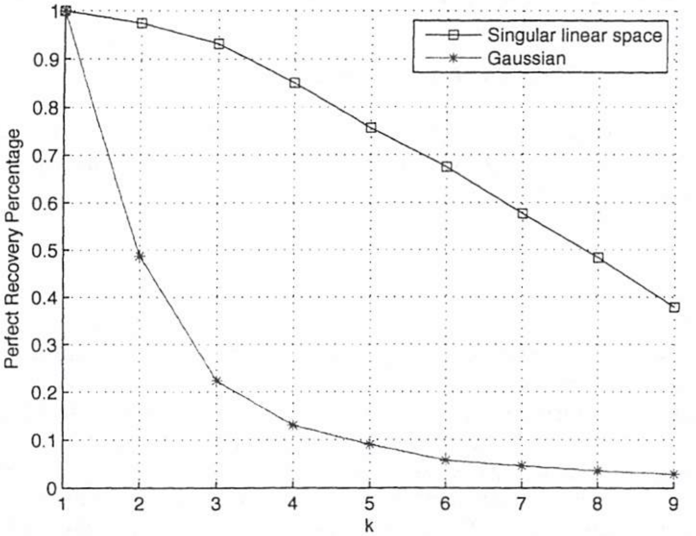

Fig. 1. (Perfect recovery perc entage of a matrix formed by singular linear spaces hspace*11.2mmand that of a random Gaussian matrix with the same size 27 × 39. For each k, 5000 input signals are used to compute the percentage.)

make the upper bound value on k specific. Let $h_1 + d \leq i_1 - h_1$, then $\max\{u, v\} = v, w = v - x$ and $k \leq \lfloor \frac{t-2v}{2(v-x)} \rfloor < \lfloor \frac{t-2v-2}{2(v-x)} \rfloor + 1$, where

$$\frac{t-2v}{2(v-x)}$$

$$= \frac{q^{d(i_2-i_1)}\begin{bmatrix} i_2-h_2-d \\ i_1-h_1-d \end{bmatrix}_q \begin{bmatrix} h_2 \\ h_1 \end{bmatrix}_q - 2q^{d(i_2-1-i_1)}\begin{bmatrix} i_2-h_2-d \\ i_1-h_1-d \end{bmatrix}_q \begin{bmatrix} h_2-1 \\ h_1 \end{bmatrix}_q}{2q^{d(i_2-1-i_1)}\begin{bmatrix} i_2-h_2-d \\ i_1-h_1-d \end{bmatrix}_q \begin{bmatrix} h_2-1 \\ h_1 \end{bmatrix}_q - 2q^{d(i_2-2-i_1)}\begin{bmatrix} i_2-2-h_2-d \\ i_1-h_1-d \end{bmatrix}_q \begin{bmatrix} h_2 \\ h_1 \end{bmatrix}_q}$$

$$= \frac{1 - \dfrac{2(q^{h_2-h_1}-1)}{q^d(q^{h_2}-1)}}{\dfrac{2(q^{h_2-h_1}-1)}{q^d(q^{h_2}-1)} - \dfrac{2(q^{i_2-h_2-i_1+h_1-1}-1)(q^{i_2-h_2-i_1+h_1}-1)}{q^{2d}(q^{i_2-h_2-d-1}-1)(q^{i_2-h_2-d}-1)}}.$$

Theorem 3.5. Suppose $h_1 = 0, i_1 = d \geq 2$, then the upper bound value on k' is smaller than that of k when $q = q' > 2$.

Proof As above-mentioned, let $h_1 = 0, i_1 = d \geq 2$, then $k \leq \lfloor \frac{t-2v}{2(v-x)} \rfloor = \lfloor \frac{q^{2d}-2q^d}{2(q^d-1)} \rfloor$. By Lemma 3.4, we have $k' \leq \lceil \frac{q'}{r} \rceil$. Given $q = q' > 2$. Since

$$\frac{q'}{r} = \frac{q}{r} \leq \frac{q}{2} \leq \frac{q^d - 2}{2} = \frac{q^d(q^d - 2)}{2q^d} = \frac{q^{2d} - 2q^d}{2q^d} < \frac{q^{2d} - 2q^d}{2(q^d - 1)}.$$

Hence, the upper bound value on k' is smaller than that of k. This completes the proof. □

We have proved that our construction is superior to the construction of DeVore under some conditions. By changing the numbers of parameters, we will obtain a type of different CS matrices.

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References

[1] R. Baraniuk. Compressive Sensing. IEEE Signal Processing Magazine, 2007, 24(4) : 118 – 121.

[2] Shi G, Liu D, Gao D, Liu Z, Lin J, Wang L. Advances in Theory and Application of Compressed Sensing. Acta Electronica Sinica, 2009, 37(5) : 1070 – 1081.

[3] E. Candès, T. Tao. Near-optimal signal recovery from random projections: Universal encoding strategies. IEEE Trans. Inf. Theory, 2006, 52(12) : 5406 – 5425.

[4] E. Candès, J. Romberg, T. Tao. Stable signal recovery from incomplete and inaccurate measurement. Communications on Pure and Applied Mathematices, 2006, 59(8) : 1207 – 1223.

[5] Y. Tsaig, D. Donoho. Extensions of compressed sensing. Signal Processing, 2006, 86(3) : 549 – 571.

[6] Wu H, Zhang X, Chen W. Measurement matrices in compressed sensing theory. Journal of military communication technology, 2012, 33(1) : 90 – 94.

[7] Macula AJ. A simple construction of $d-$disjunct matrices with certain constant weights. Discrete Math 1996, 162 : 311 – 312.

[8] D'yachkov AG, Macula AJ, Vilenkin PA. Nonadaptive and trivial two-stage group testing with error-correcting d^e- disjunct inclusion matrices. Springer Berlin Heidelberg, 2007, 16 : 71 – 83.

[9] Li S, Wei H, Ge, G. A combinatorial approach to support recovery. To be published.

[10] R. DeVore. Deterministic constructions of compressed sensing matrices. J. Complexity 2007, 23(4 – 6) : 918 – 925.

[11] Li S, Gao F, Ge G, Zhang S. Deterministic construction of compressed sensing matrices via algebraic curves. IEEE Trans. Inf. Theory 2012, 58(8) : 5035 – 5041.

[12] A. Amini, F. Marvasti. Deterministic construction of binary, bipolar and ternary compressed sensing matrices. IEEE Trans. Inf. Theory 2011, 57(4) : 2360 – 2370.

[13] A. Amini, V. Montazerhodjat, F. Marvasti. Matrices with small coherence using p-ary block codes. IEEE Trans. Signal Processing 2012, 60(1) : 172 – 181.

[14] J. Bourgain, S. Dilworth, K. Ford, S. Konyagin, D. Kutzarova. Explicit constructions of RIP matrices and related problems. Duke Math. J, 2011, 159(1) : 145 – 185.

[15] Wang K, Guo J, Li F. Singular linear space and its applications. Finite Fields Appl 2011, 17 : 395 – 406.

[16] Wan Z. Geometry of classical groups over finite fields, 2nd edn. Science, Beijing, 2002.

[17] A.E. Brouwer, A.M. Cohen, A. Neumaier. Distance-Regular Graphs. Springer-Verlag, Berlin, Heidelberg, 1989.