Spread Conditions for Some Hamiltonian Properties of a Graph

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Abstract

The spread of a graph G is defined as the difference between the largest and smallest eigenvalues of G. Using the lower bounds obtained by Liu and Liu in [4] on the spread of a graph, we in this note present spread conditions for some Hamiltonian properties of a graph.

Keywords : spread of a graph, Hamiltonian properties.

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We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. For a graph $G = (V, E)$, we use n and e to denote its order $|V|$ and size $|E|$, respectively. We define $\sigma_l(G)$ as $\min\{d(v_1)+d(v_2)+...+d(v_l) : \{v_1, v_2, ..., v_l\}$ is an independent set in $G\}$. For a subset V_1 of V, we define its average degree as $\sum_{v \in V_1} d(v)/|V_1|$. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path.

Let G be a graph G of order n. We use $\mu_1 \geq \mu_2 \geq ... \geq \mu_n$ to denote the eigenvalues of G. The spread, denoted $S(G)$, of G is defined as $\mu_1 - \mu_n$. Liu and Liu in [4] obtained the following results (Proposition 3.3 on Page 2730) on the spread of a graph.

Theorem 1. Suppose a graph G contains t ($t \geq 1$) independent vertices, say T, the average degree of which is d_0. Then

$$S(G) \geq 2\sqrt{\left(\frac{e - td_0}{n-t}\right)^2 + \frac{td_0^2}{n-t}} \geq 2d_0\sqrt{\frac{t}{n-t}}.$$

If equality holds between the first two expressions, then the vertex degrees are constant on T and also on $V \backslash T$ and each vertex in $V \backslash T$ is adjacent to the same number of vertices in T. If equality holds between the last two expressions, then G is bipartite with vertex parts T and $V \backslash T$.

In this note, we will use Theorem 1 to present spread conditions for Hamiltonicity and traceability of a graph. Namely, we will prove the following theorems.

Theorem 2. Let G be a k - connected ($k \geq 2$) graph with order n and size e. If

$$\frac{2\sigma_{k+1}}{k+1}\sqrt{\frac{k+1}{n-(k+1)}} \geq S(G),$$

then G is Hamiltonian or $K_{k,\,k+1}$.

Theorem 3. Let G be a k - connected ($k \geq 1$) graph with order n and size e. If

$$\frac{2\sigma_{k+2}}{k+2}\sqrt{\frac{k+2}{n-(k+2)}} \geq S(G),$$

then G is traceable or $K_{k,\,k+2}$.

Next, we will prove Theorem 2 and Theorem 3.

Proof of Theorem 2. Let G be a graph satisfying the conditions in Theorem 2. If G has a Hamiltonian cycle, then the proof is finished. Now we assume that G is not Hamiltonian. Choose a longest cycle C in G and give an orientation on C. Since G is not Hamiltonian, there exists a vertex $x_0 \in V(G) \backslash V(C)$. By Menger's theorem, we can find s ($s \geq k$) pairwise disjoint (except for x_0) paths P_1, P_2, ..., P_s between x_0 and $V(C)$. Let u_i be the end vertex of P_i on C, where $1 \leq i \leq s$. We use u_i^+ to denote the successor of u_i along the orientation of C, where $1 \leq i \leq s$. Then a standard proof in Hamiltonian graph theory yields that $S := \{x_0, u_1^+, u_2^+, ..., u_s^+\}$ is independent (otherwise G would have cycles which are longer than C). Relabel the vertices in S as w_1, w_2, ..., w_{s+1} satisfying $d(w_1) \leq d(w_2) \leq$

$\ldots \le d(w_{s+1})$. It can be easily proved that

$$\frac{\sum_{i=1}^{k+1} d(w_i)}{k+1} \le \frac{\sum_{i=1}^{k+2} d(w_i)}{k+2} \le \ldots \le \frac{\sum_{i=1}^{s+1} d(w_i)}{s+1} := d_{avg}.$$

By Theorem 1 and the assumptions in Theorem 2, we have

$$\frac{2\sigma_{k+1}}{k+1}\sqrt{\frac{k+1}{n-(k+1)}} \ge S(G) \ge 2\sqrt{\left(\frac{e-(s+1)d_{avg}}{n-(s+1)}\right)^2 + \frac{sd_{avg}^2}{n-(s+1)}}$$

$$\ge 2d_{avg}\sqrt{\frac{s+1}{n-(s+1)}} \ge \frac{2\sum_{i=1}^{k+1} d(w_i)}{k+1}\sqrt{\frac{k+1}{n-(k+1)}} \ge \frac{2\sigma_{k+1}}{k+1}\sqrt{\frac{k+1}{n-(k+1)}}.$$

By Theorem 1 again, we have that G is a bipartite graph with vertex parts S and $V \backslash S$ such that all the vertices in S have the same degree, say a, and all the vertices in $V \backslash S$ have the same degree, say b. we further have that S is an independent set of size $(k+1)$ in G such that the degree sum of all vertices in S is equal to σ_{k+1}.

Set $r := |V \backslash S| = n-(k+1)$. Since G is k - connected, $r \ge k$. If $r \ge (k+1)$, then in $V \backslash S$ we can choose $(k+1)$ independent vertices of the same degree b . Notice that S is an independent set of size $(k+1)$ in G such that the degree sum of all vertices in S is equal to σ_{k+1} and all the vertices in S have the same degree a, we have that $(k+1)b \ge (k+1)a$, namely, $b \ge a$. Since both rb and $(k+1)a$ count the number of edges between S and $V \backslash S$, we have $rb = (k+1)a$. Notice further that $r \ge (k+1)$ and $b \ge a$, we must have $a = b$ and $r = (k+1)$. Again since G is k - connected, $a = b \ge k$. Clearly, $a = b \le (k+1)$. Thus $a = b = (k+1)$ or $a = b = k$.

If $a = b = (k+1)$, then G is Hamiltonian, a contradiction. If $a = b = k$, recall that every 2 - connected m - regular graph on at most $3m$ vertices is Hamiltonian (see [3]), then G is Hamiltonian, a contradiction.

If $r = k$, then $k = r \ge a \ge k$. Thus $a = k$. Again notice that $rb = (k+1)a$, we have $b = (k+1)$. Hence G is $K_{k,k+1}$. QED

Proof of Theorem 3. Let G be a graph satisfying the conditions in Theorem 3. If G has a Hamiltonian path, then the proof is finished. Now we assume that G is not traceable. Choose a longest path P in G and give an orientation on P. Let y and z be the two end vertices of P. Since G is not traceable, there exists a vertex $x_0 \in V(G) \backslash V(P)$. By Menger's theorem, we can find s ($s \ge k$) pairwise disjoint (except for x_0) paths P_1, P_2, ..., P_s between x_0 and $V(P)$. Let u_i be the end vertex of P_i on P, where

$1 \le i \le s$. Since P is a longest path in G, $y \neq u_i$ and $z \neq u_i$, for each i with $1 \le i \le s$, otherwise G would have paths which are longer than P. We use u_i^+ to denote the successor of u_i along the orientation of C, where $1 \le i \le s$. Then a standard proof in Hamiltonian graph theory yields that $S := \{x_0, y, u_1^+, u_2^+, ..., u_s^+\}$ is independent (otherwise G would have paths which are longer than P). Relabel the vertices in S as w_1, w_2, ..., w_{s+2} satisfying $d(w_1) \le d(w_2) \le ... \le d(w_{s+2})$. It can be easily proved that

$$\frac{\sum_{i=1}^{k+2} d(w_i)}{k+2} \le \frac{\sum_{i=1}^{k+3} d(w_i)}{k+3} \le ... \le \frac{\sum_{i=1}^{s+2} d(w_i)}{s+2} := d_{avg}.$$

By Theorem 1 and the assumptions in Theorem 3, we have

$$\frac{2\sigma_{k+2}}{k+2}\sqrt{\frac{k+2}{n-(k+2)}} \ge S(G) \ge 2\sqrt{\left(\frac{e-(s+2)d_{avg}}{n-(s+2)}\right)^2 + \frac{sd_{avg}^2}{n-(s+2)}}$$

$$\ge 2d_{avg}\sqrt{\frac{s+2}{n-(s+2)}} \ge \frac{2\sum_{i=1}^{k+2} d(w_i)}{k+2}\sqrt{\frac{k+2}{n-(k+2)}} \ge \frac{2\sigma_{k+2}}{k+2}\sqrt{\frac{k+2}{n-(k+2)}}.$$

By Theorem 1 again, we have that G is a bipartite graph with vertex parts S and $V \backslash S$ such that all the vertices in S have the same degree, say a, and all the vertices in $V \backslash S$ have the same degree, say b. we further have that S is an independent set of size $(k+2)$ in G such that the degree sum of all vertices in S is equal to σ_{k+2}.

Set $r := |V \backslash S| = n - (k+2)$. Since G is k - connected, $r \ge k$. If $r \ge (k+2)$, then in $V \backslash S$ we can choose $(k+2)$ independent vertices of the same degree b . Notice that S is an independent set of size $(k+2)$ in G such that the degree sum of all vertices in S is equal to σ_{k+2} and all the vertices in S have the same degree a, we have that $(k+2)b \ge (k+2)a$, namely, $b \ge a$. Since both rb and $(k+2)a$ count the number of edges between S and $V \backslash S$, we have $rb = (k+2)a$. Notice further that $r \ge (k+2)$ and $b \ge a$, we must have $a = b$ and $r = (k+2)$. Again since G is k - connected, $a = b \ge k$. Clearly, $a = b \le (k+2)$. Thus $a = b = (k+2)$ or $a = b = (k+1)$ or $a = b = k$.

If $a = b = (k+2)$, then G is Hamiltonian, a contradiction. If $a = b = (k+1)$, recall that every connected m - regular graph on at most $3m+3$ vertices has a Hamiltonian path (see Theorem 2.5 in [2]), then G is traceable, a contradiction. If $a = b = k$, then again Theorem 2.5 in [2] implies that G is traceable, a contradiction.

If $r = (k+1)$, then we have $(k+1)b = (k+2)a$, $(k+1) \ge a \ge k$, and $(k+2) \ge b \ge k$. Thus $a = (k+1)$ and $b = (k+2)$. So G is traceable, a contradiction.

If $r = k$, then $k = r \geq a \geq k$. Thus $a = k$. Again notice that $rb = (k+2)a$, we have $b = (k+2)$. Hence G is $K_{k,\,k+2}$. QED

Obviously, Theorem 2 and Theorem 3 have the following Corollary 4 and Corollary 5, respectively.

Corollary 4. Let G be a k - connected ($k \geq 2$) graph with order n and size e. If

$$2\delta\sqrt{\frac{k+1}{n-(k+1)}} \geq S(G),$$

then G is Hamiltonian or $K_{k,\,k+1}$.

Corollary 5. Let G be a k - connected ($k \geq 1$) graph with order n and size e. If

$$2\delta\sqrt{\frac{k+2}{n-(k+2)}} \geq S(G),$$

then G is traceable or $K_{k,\,k+2}$.

References

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