On Radio Labeling Edge-Balanced Caterpillars

Katherine F. Benson

Abstract

A radio labeling of a simple connected graph G is a function $f : V(G) \to \mathbb{Z}^+$ such that for every two distinct vertices u and v of G

$$\text{distance}(u, v) + |f(u) - f(v)| \geq 1 + \text{diameter}(G).$$

The radio number of a graph G is the smallest integer M for which there exists a labeling f with $f(v) \leq M$ for all $v \in V(G)$. An edge-balanced caterpillar graph is a caterpillar graph that has an edge so that removing this edge results in two components with an equal number of vertices. In this paper, we determine the radio number of particular edge-balanced caterpillars as well as improve the lower bounds of the radio number of other edge-balanced caterpillars.

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1 Introduction

The problem that inspired radio labeling is what is called the channel assignment problem. This problem appears when radio frequencies are assigned to radio transmitters in a way that causes interference between them. The goal of solving this problem is to determine a method to assign radio frequencies to transmitters so that interference between transmitters is minimized. This problem was modeled with graph theoretic concepts by Hale [6]. This graph theoretic approach has vertices of a simple connected graph represent the radio transmitters and labels given to the vertices represent the radio frequencies assigned to each transmitter. There have been a few different approaches to solving this problem using graph theory. Many of these approaches have included some specific restriction on the possible labels for vertices given their distance from one another.

Distance-2 labeling and k-radio coloring methods were discussed by Chartrand and Zhang in [2]. Let G be a simple connected graph and let $d(u, v)$ denote the distance between vertices u and v of G. Let D be the diameter of G. Given k with $1 \leq k \leq D$, and $f : V(G) \to \mathbb{Z}^+$ a coloring of G, f is a k-radio coloring if

$$d(u,v) + |f(u) - f(v)| \geq k + 1 \qquad (1)$$

for all distinct vertices u, v in G. The goal is then to minimize the largest value used as a label when labeling G such that this condition is satisfied.

Distance-2 labeling is the specific k-radio coloring when k is 2. This type of labeling forces vertices that are distance 2 apart to have different labels and requires adjacent vertices to have labels whose absolute difference is at least 2. This type of labeling was incorporated into the channel assignment problem because it studied restrictions between vertices that were considered to be close or very close [5].

As Liu and Zhu discuss, however, there may be interference with transmitters farther than distance 2 apart [9]. This leads to considering k-radio colorings when $k \geq 2$. One specific type of these colorings is radio labeling which is k-radio coloring when $k = D$.

The radio numbers of some tree graphs have already been determined. In [9], Liu and Zhu found the radio number of paths. In [1], the radio number of spire graphs, paths with one leaf vertex off of the path, was determined. The radio numbers for m-ary trees were discussed in [7] and general bounds for the radio number of tress are found in [8]. In [10], Marinescu-Ghemeci determines the radio number of a very specific caterpillar graph. In this paper, we extend techniques used in [1] to improve the lower bound from [8] of the radio number of certain types of caterpillar graphs. For specific types of caterpillar graphs G, we develop an algorithm to produce an optimal labeling of G.

2 Background and Preliminary Work

Throughout this paper let G be a simple connected graph with n vertices. Let $V(G)$ denote the vertex set of G and $E(G)$ denote the edge set of G. For a component C of G, let $V(C)$ and $E(C)$ denote the vertex set and edge set of C respectively. For a given set S of vertices (or edges) of a graph, let the *order* of S, denoted $|S|$, be the number of vertices (or edges) in S. For two distinct vertices u and v in G, let the *distance* between u and v be denoted by $d(u,v)$. Let the diameter of G, the maximum distance found in G, be denoted by D.

A radio labeling of a graph G is a k-radio labeling[1] f for $k = D$. In this case, equation (1) is called the *radio condition*. The largest integer used as

[1]Some authors allow 0 as a label. In this paper, we do not allow 0 to be a label and adjust all relevant results we cite accordingly.

a label in f is called the *span* of f. The *radio number* of a graph G, denoted $rn(G)$, is the smallest possible span for a radio labeling of G. Equivalently, the *radio number* of G is the smallest integer M for which there exists a radio labeling f of G such that $f(v) \leq M$ for all $v \in V(G)$.

In this paper, we discuss bounds for radio numbers of certain types of tree graphs, specifically certain types of caterpillar graphs.

Definition. A caterpillar graph is a tree with $n = s+l$ vertices $v_1, \ldots, v_{s+l}$, consisting of a path $v_1, \ldots, v_s$, named the *spine* of G and terminal vertices $v_{s+1}, \ldots, v_{s+l}$ such that they are adjacent to some v_i for $2 \leq i \leq s-1$ which are called *leg vertices.* See an example in Figure 1.

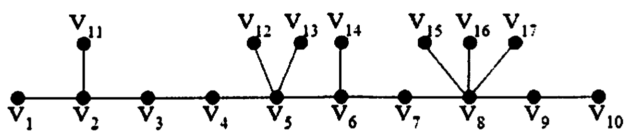

Figure 1: A Caterpillar with $s = 10$ vertices on the spine and $l = 7$ leg vertices.

We now develop some techniques that help in establishing a good lower bound for the radio number of caterpillar graphs. First we establish some terminology and notation we will use throughout the paper to help when relating the order vertices are labeled and a particular labeling function of a given graph G.

Definition. An *ordering* of the n vertices of a graph G is a bijection of the vertices of G to the set $\{x_1, \ldots, x_n\}$ where the subscript denotes the order the vertices are labeled.

Definition. Given an ordering $x_1, \ldots, x_n$ of the n vertices of a graph G, let the *associated* radio labeling be a function f with $f(x_1) = 1$ and defined inductively so that $f(x_i)$ is the smallest integer for which the radio condition is satisfied for all pairs x_i and x_j with $j < i$.

For the rest of this paper, unless otherwise indicated, for a graph G, we refer to $x_1, \ldots, x_n$ as the ordering of the vertices of G and call the associated radio labeling f.

Now consider the process of associating a radio labeling f to an ordering $x_1, \ldots, x_n$ of vertices in a graph G. Since a radio labeling f is a function from the vertices of G to the positive integers, we let $f(x_1) = 1$. As we label the rest of the vertices, at each step, we choose $f(x_{i+1})$ to be the smallest integer that satisfies the radio condition with all vertices $x_1, x_2, \ldots, x_i$.

When labeling x_{i+1}, a reasonable first consideration for $f(x_{i+1})$ is the positive integer k such that

$$k = D + 1 + f(x_i) - d(x_i, x_{i+1}).$$

Notice that if $f(x_{i+1}) = k$, then the radio condition between the successively labeled vertices x_i and x_{i+1} is an equality. However, this value for $f(x_{i+1})$ might not satisfy (1) with all of the previously labeled vertices $x_1, x_2, \ldots, x_{i-1}$. Let l be the smallest integer such that $k + l \geq D + 1 + f(x_j) - d(x_{i+1}, x_j)$ for all $1 \leq j \leq i - 1$.

For the radio condition to be satisfied for all pairs of vertices, we need to increase the value of $f(x_{i+1})$ so that $f(x_{i+1}) = k + l$. Since this is needed when labeling x_{i+1} after x_i, we will use the notation $J_f(x_i, x_{i+1})$ for l. Then when considering $f(x_{i+1})$ in terms of the successively labeled vertices x_i and x_{i+1}, we have the following:

$$f(x_{i+1}) = D + 1 + f(x_i) - d(x_i, x_{i+1}) + J_f(x_i, x_{i+1}).$$

If $J_f(x_i, x_{i+1}) > 0$, the radio condition is satisfied with a strict inequality for the pair of vertices x_i and x_{i+1}. This need to have a strict inequality for the radio condition between successively labeled vertices is what we will refer to as needing *jumps*. This is because we need to make an increase, or jump, in the value of $f(x_{i+1})$ beyond what is required when just considering the radio condition between the successively labeled vertices x_i and x_{i+1}. More formally, we have the following:

Definition. As in [7], for $x_1, \ldots, x_n$ an ordering of a simple connected graph G with associated radio labeling f, the *jump of f from x_i to x_{i+1}* is a non-negative integer $J_f(x_i, x_{i+1})$ such that

$$d(x_i, x_{i+1}) + f(x_{i+1}) - f(x_i) = D + 1 + J_f(x_i, x_{i+1}).$$

Definition. Given an ordering $x_1, \ldots, x_n$ of the vertices of a graph G and the associated radio labeling f, we say that f *requires jumps* if $\sum_{i=1}^{n-1} J_f(x_i, x_{i+1}) \geq 1$.

Many of the techniques and definitions in this paper rely on Propositions and Lemmas from [1]. We include them here for reference with some slight modifications to make the notation consistent with the rest of this paper.

Proposition 1. *(Proposition 3 of [1]) Let G be a simple connected graph with n vertices and let $x_1, \ldots, x_n$ be any ordering of the vertices of G with f the associated radio labeling. Then,*

$$f(x_n) = (n-1)(D+1) + f(x_1) - \sum_{i=1}^{n-1} d(x_i, x_{i+1}) + \sum_{i=1}^{n-1} J_f(x_i, x_{i+1}).$$

Proposition 2. *(Lemma 4 of [1]) Let G be a simple connected graph with n vertices. Then*

$$rn(G) \geq (n-1)(D+1) + 1 - \max \sum_{i=1}^{n-1} d(x_i, x_{i+1})$$

where the maximum is taken over all possible orderings $\{x_1, \ldots, x_n\}$ of the vertices of G.

From Proposition 2 we see that finding $\max \sum_{i=1}^{n-1} d(x_i, x_{i+1})$ for a graph G will give a lower bound for the radio number of G. As we will refer to this occurrence of maximizing $\sum_{i=1}^{n-1} d(x_i, x_{i+1})$, we have the following definition:

Definition. We call any ordering of the vertices of a graph G for which $\max \sum_{i=1}^{n-1} d(x_i, x_{i+1})$ is achieved, where the maximum is taken over all possible orderings of the vertices of G, a *distance maximizing ordering.*

Notice that Proposition 2 gives a preliminary lower bound for the radio number of a tree. This is the same lower bound as given by Liu in Theorem 3 of [8] but with different notation. In Liu's proof, she shows that $\sum_{i=0}^{m-2} d(u_{i+1}, u_i) \leq 2\omega(T) - 1$ where $\omega(T)$ is the weight of the tree T. The sum $\sum_{i=0}^{m-2} d(u_{i+1}, u_i)$ in [8] is equivalent to $\sum_{i=1}^{n-1} d(x_i, x_{i+1})$ in this paper. Therefore, according to Liu's proof, $\max \sum_{i=1}^{n-1} d(x_i, x_{i+1}) = 2\omega(G) - 1$ where the maximum is taken over all possible orderings of the vertices of G. Making this substitution, exchanging variables to match this paper's notation, and adjusting for that fact that Liu uses 0 as the first label in her labelings shows that the bound given in Theorem 3 of [8] is the same as the bound given in Proposition 2. In this paper, we improve this bound for particular types of caterpillar graphs.

The following lemma from [1] will be useful in techniques we develop to determine the value of $\sum_{i=1}^{n-1} d(x_i, x_{i+1})$ for particular graphs.

Lemma 1. *(Lemma 6 of [1]) Let G be a graph with vertices $v_1, \ldots, v_n$ and edges $e_1, \ldots, e_m$. Let $x_1, \ldots, x_n$ be an ordering of G. Let P_j be a fixed shortest path from x_j to x_{j+1}. Let $n_x(e_i)$ be the number of paths P_j that contain the edge e_i. Then the following hold:*

1. *Each edge can appear in any path P_j at most once.*
2. *Let $\{e^k_{i_1}, \ldots, e^k_{i_r}\}$ be the set of all the edges incident to x_k. Then $n_x(e^k_{i_1}) + \ldots + n_x(e^k_{i_r})$ is even unless $k = 1$ or $k = n$ in which case the sum is odd.*

3. *Suppose e_i is an edge so that removing it from the graph gives a graph with two components denoted A and B. Furthermore assume that if x_j and x_{j+1} are both contained in the same component, then so is P_j. Then $n_x(e_i) \leq 2min\{|V(A)|, |V(B)|\}$.*

4. *Let $\{e_{i_1}, ..., e_{i_r}\}$ be a set of edges so that no two of them are ever contained in the same P_j. Then $n_x(e_{i_1}) + ... + n_x(e_{i_r}) \leq n - 1$.*

Remark 1. *Let G be a graph with vertices $v_1, ..., v_n$ and edges $e_1, ..., e_m$. Let $N(e_i)$ be the maximal value of $n_x(e_i)$ for all possible orderings $x_1, \ldots x_n$ allowable under the conditions of Lemma 1. Then $\max \sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \sum_{j=1}^{m} N(e_j)$ where the maximum is taken over all possible orderings of the vertices of G.*

Remark 2. *Note that from (2) of Lemma 1, for a specific ordering $x_1, \ldots, x_n$ of the vertices of G, there needs to be at least one edge e_i in G such that $n_x(e_i)$ is odd.*

When G is a tree, paths are unique. Thus, for the rest of this paper, we will use the following notation.

Notation. Let G be a tree with vertices $v_1, ..., v_n$ and edges $e_1, ..., e_{n-1}$. Let $x_1, \ldots, x_n$ be an ordering of G. Let P_j be the unique path from x_j to x_{j+1}. Let $n_x(e_i)$ be the number of paths P_j that contain the edge e_i.

Since this paper focuses on radio labeling certain trees, the next three propositions follow from Lemma 1 when G is a tree.

Proposition 3. *Let G be a tree with edges $e_1, \ldots, e_{n-1}$ and an ordering $x_1, \ldots, x_n$ of its vertices. Let A_i and B_i denote the two components of G that result from the removal of edge e_i from G. Then*

$$N(e_i) = \begin{cases} n-1 & \text{if } \min\{|V(A_i)|, |V(B_i)|\} = \frac{n}{2} \\ 2\min\{|V(A_i)|, |V(B_i)|\} & \text{else.} \end{cases}$$

Proof. When G is a tree, removing one edge will result in a disconnected graph of two components and removing more than one edge will result in a disconnected graph with three or more components.

Thus, in a tree, removing just one edge, e_i will result in two disjoint components, A_i and B_i. Then for a given edge e_i of G, (3) of Lemma 1 gives that $n_x(e_i) \leq 2\min\{|V(A_i)|, |V(B_i)|\}$. Also, (4) of Lemma 1 shows that $N(e_i) \leq n - 1$ for all edges e_i. It follows that the maximum possible value for $n_x(e_i)$ for all possible orderings $x_1, \ldots, x_n$ and edges e_i is

$$N(e_i) = \begin{cases} n-1 & \text{if } \min\{|V(A_i)|, |V(B_i)|\} = \frac{n}{2} \\ 2\min\{|V(A_i)|, |V(B_i)|\} & \text{else} \end{cases}.$$

□

Proposition 4. *Let G be a tree with edges $e_1, \ldots, e_{n-1}$ and $x_1, \ldots, x_n$ an ordering on the vertices of G. If there is only one i such that $n_x(e_i)$ is odd, then x_1 and x_n are both incident to e_i.*

Proof. Let $\{e^1_{i_1}, \ldots, e^1_{i_r}\}$ be the set of edges incident to x_1 and $\{e^n_{j_1}, \ldots, e^n_{j_s}\}$ be the set of edges incident to x_n. Suppose by way of contradiction that x_1 and x_n are not adjacent. This means that $\{e^1_{i_1}, \ldots, e^1_{i_r}\} \cap \{e^n_{j_1}, \ldots, e^n_{j_s}\} = \emptyset$. Also, by (2) of Lemma 1, $\sum_{k=1}^{r} n_x(e^1_{i_k})$ and $\sum_{k=1}^{s} n_x(e^n_{j_k})$ must both be odd which means that both $\{e^1_{i_1}, \ldots, e^1_{i_r}\}$ and $\{e^n_{j_1}, \ldots, e^n_{j_s}\}$ have at least one edge such that $n_x(e)$ is odd. Since these two sets have no common members, this means there are at least two edges e in G such that $n_x(e)$ is odd, a contradiction to our assumption. Therefore, x_1 and x_n are both incident to the edge e_i such that $N(e_i)$ is odd. □

In this paper, we are considering particular types of trees. The following proposition provides a way to describe $\sum_{i=1}^{n-1} d(x_i, x_{i+1})$ for a tree G and ordering $x_1, \ldots, x_n$ in terms of the $n_x(e_i)$ values for the edges e_i of G.

Proposition 5. *Let G be a tree with ordering $x_1, \ldots, x_n$ and associated radio labeling f. Let $e_1, e_2, \ldots, e_{n-1}$ be the edges of G. Then* $\sum_{i=1}^{n-1} d(x_i, x_{i+1}) = \sum_{i=1}^{n-1} n_x(e_i)$.

Proof. Consider the path P_j between x_j and x_{j+1}. Suppose this path is of length k. Since the length of this path is the shortest length of a path between x_j and x_{j+1}, it follows that $d(x_j, x_{j+1}) = k$. Thus, $d(x_j, x_{j+1})$ contributes k to the total $\sum_{i=1}^{n-1} d(x_j, x_{j+1})$.

Also, since there are k edges in P_j, this path contributes 1 to the $n_x(e_i)$ value for each of the k edges e_i in the path. Therefore, P_j contributes k to the total sum $\sum_{i=1}^{n-1} n_x(e_i)$.

Since the above arguments are true for each j, $1 \leq j \leq n-1$, it follows that
$\sum_{i=1}^{n-1} d(x_j, x_{j+1}) = \sum_{i=1}^{n-1} n_x(e_i)$. □

Note that for a tree graph G, Proposition 5 implies that to find a distance maximizing ordering, we need to maximize $\sum_{e \in E(G)} n_x(e)$.

The following definition for trees in general will help us specify the type of caterpillar we will consider for this paper.

Definition. For a tree G on n vertices with edges $e_1, \ldots, e_{n-1}$, a *center edge*, e_c, is an edge with largest $N(e_i)$ value.

Note that the removal of a center edge results in a disconnected graph with two components which we will denote as A and B.

3 Caterpillar Preliminaries

There are four main categories of caterpillar graphs in terms of the center edge definition. There could be one center edge e_c where $N(e_c)$ is odd, there could be one center edge e_c where $N(e_c)$ is even, there could be two center edges, or there could be more than two center edges. Notice that the only way for a caterpillar graph to have more than two center edges is if the caterpillar is a star graph, whose radio number has been determined in [4].

In this paper, we focus on a particular type of caterpillar graph that has one center edge.

Definition. A caterpillar is *edge-balanced* if there is an edge so that removing this edge results in exactly two components with an equal number of vertices.

To better understand the structure of edge-balanced caterpillars, we have the following result.

Proposition 6. *Let G be a tree with n vertices and one center edge e_c. The value of $N(e_c)$ is odd if and only if n is even and $|V(A)| = |V(B)| = \frac{n}{2}$.*

Proof. First suppose $N(e_c)$ is odd. Let A and B be the components of G after the removal of e_c. Suppose by way of contradiction that $|V(A)| \neq |V(B)|$. Without loss of generality, suppose $|V(A)| > |V(B)|$. Notice that $|V(B)| < \frac{n}{2}$. From Proposition 3, $N(e_c) = 2|V(B)|$ which is even, contradicting the assumption that $N(e_c)$ is odd. Therefore, $|V(A)| = \frac{n}{2} = |V(B)|$.

Now suppose e_c is the only center edge and $|V(A)| = \frac{n}{2} = |V(B)|$. Since $|V(A)| = \frac{n}{2} = |V(B)|$, $\min\{|V(A)|, |V(B)|\} = \frac{n}{2}$. Thus by Proposition 3, $N(e_c) = n - 1$ which is odd. □

Remark 3. *In terms of the center edge definition, an edge-balanced caterpillar is a caterpillar with an even number of vertices and one center edge where $N(e_c)$ is odd.*

Notation. Let G be an edge-balanced caterpillar with n vertices. Name the vertices of G as follows: The vertices of the spine will be denoted $u_1, ..., u_s$ (note that $D = s - 1$). If there are t leg vertices adjacent to u_r, we will denote them $l_{r-1}^1, ..., l_{r-1}^t$ if they are to the left of the center edge and $l_{r+1}^1, ..., l_{r+1}^t$ if they are to the right. See Figure 2.

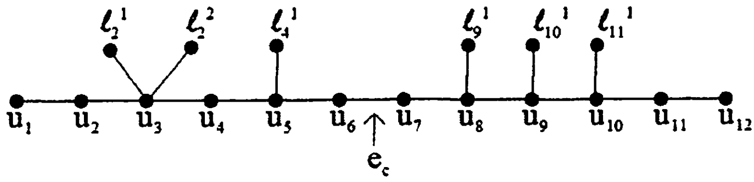

Figure 2: An edge-balanced caterpillar with nine vertices on each side of the center edge e_c.

Note that the distance between any two vertices on opposite sides of the center edge is given by the absolute difference of their subscripts.

For an edge-balanced caterpillar G, let u_{c_a} and u_{c_b} be the vertices on the spine of G incident to e_c. This means $1 \le c_a < c_b \le s$ with $c_a + 1 = c_b$. Notice that this means we refer to A as the component to the left of the center edge and B as the component to the right of the center edge.

Now we use ideas from Section 2 along with the structure of edge-balanced caterpillars to determine when $\sum_{i=1}^{n-1} n_x(e_i)$ is maximized for a specific ordering $x_1, \ldots, x_n$.

Proposition 7. *Let G be an edge-balanced caterpillar with n vertices. Let $x_1, \ldots, x_n$ be an ordering of the vertices of G. Let $e_1, e_2, \ldots, e_{n-1}$ be the edges of G. Then the sum $\sum_{i=1}^{n-1} n_x(e_i)$ is maximized when $n_x(e_i) = N(e_i)$ for all $e_i \in E(G)$. Furthermore, when this maximized sum occurs, there is only one edge in $E(G)$ such that $n_x(e)$ is odd and this edge is e_c.*

Proof. By Proposition 3, the only time $N(e)$ is odd for some edge e in a tree is when $N(e) = n - 1$. By Propositions 6 and 3 and since G is an edge-balanced caterpillar, $N(e_c) = n - 1$. Note that $N(e)$ for all other edges of G is even. Remark 2 indicates that for a particular ordering $x_1, \ldots, x_n$, there must be at least one edge e with $n_x(e)$ odd. Since $N(e_c)$ is odd, $\sum_{i=1}^{n-1} n_x(e_i)$ is maximized when $n_x(e_i) = N(e_i)$ for all $e_i \in E(G)$ for a particular ordering $x_1, \ldots, x_n$.

Thus, by the above argument, it is seen that when $\sum_{i=1}^{n-1} n_x(e_i)$ is maximized, there is only one edge with an odd $n_x(e)$ value and this edge is e_c. □

Corollary 1. *Let G be an edge-balanced caterpillar with edges $e_1, \ldots, e_{n-1}$. Let $x_1, \ldots, x_n$ be a distance maximizing ordering of the vertices of G. Then the vertices x_1 and x_n are adjacent and $\{x_1, x_n\} = \{u_{c_a}, u_{c_b}\}$.*

Proof. By Proposition 5, $\sum_{i=1}^{n-1} d(x_i, x_{i+1}) = \sum_{i=1}^{n-1} n_x(e_i)$. Since $x_1, \ldots, x_n$ is a distance maximizing ordering, this means $\sum_{i=1}^{n-1} n_x(e_i)$ is maximized. By Proposition 7, there is only one edge such that $n_x(e_i)$ is odd and that edge is e_c. Proposition 4 shows that x_1 and x_n are adjacent such that both are incident to e_c. It follows that $\{x_1, x_n\} = \{u_{c_a}, u_{c_b}\}$. □

Corollary 2. *Let $x_1, \ldots, x_n$ be a distance maximizing ordering of the vertices of an edge-balanced caterpillar G. Vertices that are successive in the ordering alternate between component A and component B, i.e. for $1 \le i \le n-1$, if x_i is in component A, then x_{i+1} is in component B and if x_i is in component B, then x_{i+1} is in component A.*

Proof. By Proposition 5, $\sum_{i=1}^{n-1} d(x_i, x_{i+1}) = \sum_{i=1}^{n-1} n_x(e_i)$. Proposition 7 shows that $\sum_{i=1}^{n-1} n_x(e_i)$ is maximized when $n_x(e_i) = N(e_i)$ for all i. From Proposition 3, $n_x(e_c) = n - 1$. This means there are $n-1$ paths P_j from x_j to x_{j+1} that include the edge e_c. Since there are only $n-1$ paths P_j from x_j to x_{j+1}, this means every path from x_j to x_{j+1} includes the edge e_c. Since e_c divides G into components A and B, this shows that if x_j is in A, then x_{j+1} is in B and vice versa. □

4 Algorithm for Edge-Balanced Caterpillars

In this section, we propose an algorithm for ordering the vertices of an edge-balanced caterpillar to provide an optimal radio labeling of that caterpillar.

Consider Table 1. We will construct this type of table to help us determine an ordering for a radio labeling of an edge-balanced caterpillar. For a particular edge-balanced caterpillar G, a table can be constructed in the same manner as Table 1 where the numbers 1 through n are placed into the cells as shown in Table 1. The last number placed in the table is $n-2$. Notice that $n-2$ will be in the first column or the fourth column. We will use this table to divide the vertices of G into four groups to help in determining a distance maximizing labeling. We will consider this process for two isomorphic drawings of edge-balanced caterpillars.

As described in [3], two graphs that only differ in how they are drawn and vertices named are called isomorphic graphs. Given an edge-balanced caterpillar G, let H be defined by the isomorphism $\phi : V(G) \to V(H)$ such that

$$\begin{cases} \phi(u_i) = \tilde{u}_{s-i+1} & \text{if } u_i \text{ is on the spine of } G \\ \phi(l_i^k) = \tilde{l}_{s-i+1}^k & \text{if } l_i^k \text{ is a leg of } G \end{cases}$$

where vertices denoted with a tilde are vertices of H.

Note that informally ϕ flips the graph G so that the leftmost spine vertex is now the rightmost spine vertex and vice versa.

Figure 3 shows the edge-balanced caterpillar G from Figure 2 as well as the graph H given by the above isomorphism ϕ.

<table>
<tr><th colspan="2">Group 1</th><th colspan="2">Group 2</th></tr>
<tr><th>Column 1</th><th>Column 2</th><th>Column 3</th><th>Column 4</th></tr>
<tr><td>2</td><td>1</td><td>$n-1$</td><td>n</td></tr>
<tr><td>6</td><td rowspan="2">7</td><td>3</td><td rowspan="2">4</td></tr>
<tr><td>8</td><td>5</td></tr>
<tr><td>12</td><td rowspan="2">13</td><td>9</td><td rowspan="2">10</td></tr>
<tr><td>14</td><td>11</td></tr>
<tr><td>18</td><td rowspan="2">19</td><td>15</td><td rowspan="2">16</td></tr>
<tr><td>20</td><td>17</td></tr>
<tr><td>⋮</td><td>⋮</td><td>⋮</td><td>⋮</td></tr>
<tr><td>$j+3$</td><td rowspan="2">$j+4$</td><td>j</td><td rowspan="2">$j+1$</td></tr>
<tr><td>$j+5$</td><td>$j+2$</td></tr>
<tr><td>⋮</td><td>⋮</td><td>⋮</td><td>⋮</td></tr>
</table>

Table 1: Grid for Edge-Balanced Caterpillars.

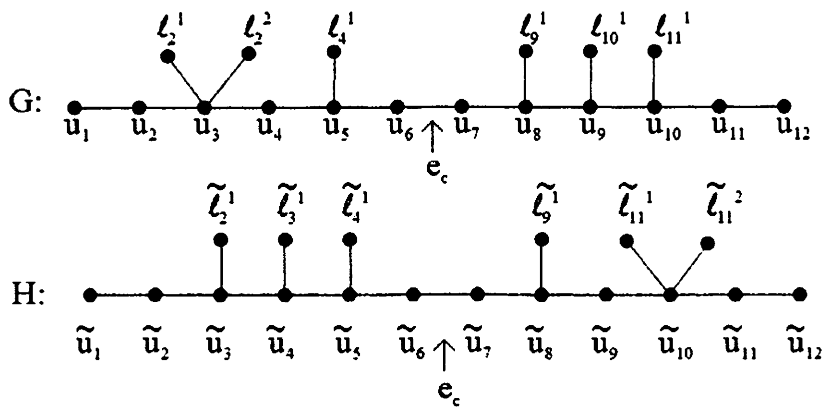

Figure 3: Two isomorphic drawings of an edge-balanced caterpillar.

We will use two copies of Table 1 to determine an ordering of the vertices of G and an ordering of the vertices of H using the algorithm below.

Algorithm 1. Consider an edge-balanced caterpillar G with n vertices. Construct a table like Table 1 with n numbered cells.

Place the names of the vertices of G in the table as follows:

- In Column 1, consecutively insert vertices to the left of the center edge, starting with u_1 keeping the subscripts in non-decreasing order where spine vertices are inserted before leg vertices with the same subscripts.

- In Column 2, consecutively insert vertices from the right side of the center edge starting with u_{c_b} keeping the subscripts in non-decreasing order and inserting leg vertices before spine vertices with the same subscript.
- In Column 3, consecutively insert vertices to the right of the center edge starting with u_s keeping the subscripts in non-increasing order where spine vertices are inserted before leg vertices with the same subscript.
- In Column 4, Consecutively insert vertices to the left of the center edge, starting with u_{c_a} keeping the subscripts in non-increasing order and inserting leg vertices before spine vertices with the same subscript.

Following Algorithm 1 for both drawings G and H of an edge-balanced caterpillar provides two filled in tables. For each table, when the table has been completely filled in, each vertex of the caterpillar is contained in exactly one numbered cell of the table.

Applying the process of Algorithm 1 to the caterpillars in Figure 3 gives Tables 2 and 3.

Group 1				Group 2			
Column 1		Column 2		Column 3		Column 4	
u_1	2	u_7	1	u_{12}	17	u_6	18
u_2	6	u_8	7	u_{11}	3	u_5	4
l_2^1	8			l_{11}^1	5		
l_2^2	12	l_9^1	13	u_{10}	9	l_4^1	10
u_3	14			l_{10}^1	11		
				u_9	15	u_4	16

Table 2: Table for the G drawing of the edge-balanced caterpillar of Figure 2 given by Algorithm 1.

Note that all the vertices to the right of the center edge are in Columns 2 and 3 while vertices to the left of the center edge are in Columns 1 and 4. Since the center edge divides G (or H) into two components with $\frac{n}{2}$ vertices each, this means that the total number of vertices in the middle two columns is $\frac{n}{2}$ and the total number of vertices in the outside columns is $\frac{n}{2}$.

Group 1				Group 2			
Column 1		Column 2		Column 3		Column 4	
$\tilde{u}_1$	2	$\tilde{u}_7$	1	$\tilde{u}_{12}$	17	$\tilde{u}_6$	18
$\tilde{u}_2$	6	$\tilde{u}_8$	7	$\tilde{u}_{11}$	3	$\tilde{u}_5$	4
$\tilde{l}_2^1$	8			$\tilde{l}_{11}^2$	5		
$\tilde{u}_3$	12	$\tilde{l}_9^1$	13	$\tilde{l}_{11}^1$	9	$\tilde{l}_4^1$	10
$\tilde{l}_3^1$	14			$\tilde{u}_{10}$	11		
				$\tilde{u}_9$	15	$\tilde{u}_4$	16

Table 3: Table for the H drawing of the edge-balanced caterpillar of Figure 2 given by Algorithm 1.

The numbers in the cells of the table with the names of the vertices are the subscripts i for the ordering of the vertices of G (or H) given by Algorithm 1. These orderings and grouping of the vertices is seen in Figure 4.

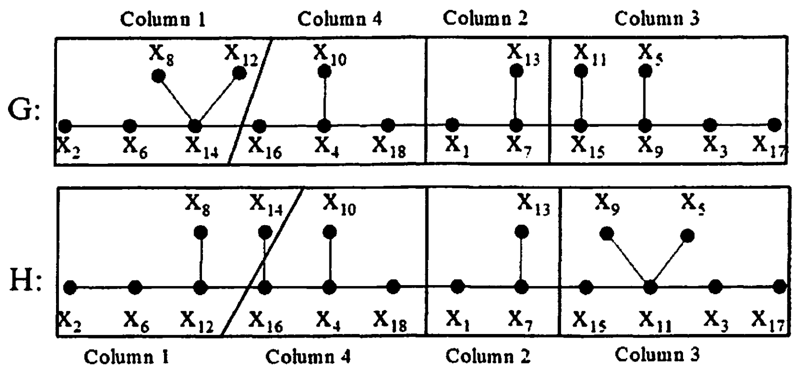

Figure 4: The ordering given by Algorithm 1 for the graphs in Figure 3. The vertices are grouped together based on the columns of Tables 2 and 3 in which the vertices are placed.

The idea of this algorithm is to alternate labeling a vertex in component A with labeling a vertex in component B. Furthermore, for a vertex x_i in the group of Column 1 vertices, the vertices x_{i-1} and x_{i+1} alternate between being in Column 2 or 3 of the table. [Analogously, for x_i in Column 3, x_{i-1} and x_{i+1} are in Columns 1 or 4.] This is done to ensure that one of $d(x_i, x_{i-1})$ or $d(x_i, x_{i+1})$ is relatively small for the given graph. This helps reduce the number of jumps needed in the associated labeling.

We want to determine an ordering of the vertices of G whose associated radio labeling's span is the radio number of G. We introduce the following definitions and notation to help us determine when Algorithm 1 gives such an ordering.

Notation. Let $x_1, \ldots, x_n$ be an ordering of the vertices of G. For a fixed i let $\alpha_{x_i}, \beta_{x_i}$ be the vertices x_{i-1} and x_{i+1} with the names chosen so that $d(x_i, \alpha_{x_i}) \leq d(x_i, \beta_{x_i})$. Note: for $i = 1$, consider x_2 as α_{x_i} and for $i = n$, consider x_{n-1} as α_{x_i}.

Definition. In a caterpillar G with an ordering $x_1, \ldots, x_n$ of its vertices, for a given i, let $t_{\alpha_{x_i}} = \begin{cases} 1 \text{ if } \alpha_{x_i} \text{ is a leg} \\ 0 \text{ otherwise.} \end{cases}$

Let G be an edge-balanced caterpillar and H be the isomorphic drawing of G given by ϕ. Let $y_1, \ldots, y_n$ be the ordering of the vertices of G given by Algorithm 1 and let $\tilde{y}_1, \ldots, \tilde{y}_n$ be the ordering of the vertices of H given by Algorithm 1.

Definition. An edge-balanced caterpillar is a *jumpless caterpillar* if the following conditions hold for the ordering given by Algorithm 1 of at least one of the graph representations G or H:

1. The distance between any pair of vertices that are in horizontally adjacent cells in Group 1 (respectively Group 2) is at most $\frac{D+1}{2} + t$ where t is 1 if the vertex in Column 2 (respectively Column 4) is a leg vertex and 0 otherwise.

2. $d(y_{n-2}, y_{n-3}) \leq \frac{D+1}{2} + t_{\alpha_{y_{n-2}}}$ (or $d(\tilde{y}_{n-2}, \tilde{y}_{n-3}) \leq \frac{D+1}{2} + t_{\alpha_{\tilde{y}_{n-2}}}$)

Remark 4. *If an edge-balanced caterpillar G is a jumpless caterpillar, we will represent it so that the corresponding ordering from Algorithm 1 satisfies the conditions in the definition of a jumpless caterpillar. Note that we may need to use the drawing H for this.*

For ease of notation, for the rest of this paper, we let $y_1, y_2, \ldots, y_n$ represent the ordering of the vertices of an edge-balanced caterpillar as given by Algorithm 1 and refer to this as the ordering given by Algorithm 1 even if using the H drawing of the graph.

Notice that the orderings given in Tables 2 and 3 satisfy the conditions of a jumpless caterpillar. Thus, the graph G of Figure 2 is a jumpless caterpillar.

Proposition 8. *Let G be an edge-balanced caterpillar with $y_1, \ldots, y_n$ the ordering of vertices given by Algorithm 1. Then this ordering is a distance maximizing ordering.*

Proof. First note that the structure of an edge-balanced caterpillar G means that e_c divides G into two components, each with $\frac{n}{2}$ vertices. Thus, $N(e_c) = n-1$.

Under Algorithm 1, y_1 and y_n are adjacent and both are incident to e_c. It can be checked that the pattern of Algorithm 1, which alternates labeling a vertex in A and then a vertex in B, causes $n_y(e) = N(e)$ for all edges in G.

Thus, by Proposition 7, $\sum_{i=1}^{n-1} n_y(e_i)$ is maximized and therefore, by Proposition 5, $\sum_{i=1}^{n-1} d(y_i, y_{i+1})$ is maximized. Thus, $y_1, \ldots, y_n$ is a distance maximizing ordering of G. □

The following lemma tells how the location of y_i and α_{y_i} in Table 1 are related which will be incorporated in the proofs of upcoming theorems.

Lemma 2. *Let G be an edge-balanced caterpillar. Let y_{i-1}, y_i, y_{i+1} be a triple of vertices under the ordering given by Algorithm 1, with $\{y_{i-1}, y_{i+1}\} = \{\alpha_{y_i}, \beta_{y_i}\}$ such that $d(y_i, \alpha_{y_i}) \leq d(y_i, \beta_{y_i})$. When $y_i \notin \{y_1, y_{n-2}, y_n\}$, the following statements are true:*

- *If y_i is entered in Column 1 of Table 1, then α_{y_i} is entered in Column 2 of Table 1.*
- *If y_i is entered in Column 2 of Table 1, then α_{y_i} is entered in Column 1 of Table 1. In particular, $\alpha_{y_i} = y_{i+1}$.*
- *If y_i is entered in Column 3 of Table 1, then α_{y_i} is entered in Column 4 of Table 1.*
- *If y_i is entered in Column 4 of Table 1, then α_{y_i} is entered in Column 3 of Table 1. In particular, $\alpha_{y_i} = y_{i+1}$.*

Also, $\alpha_{y_1} = y_2$, $\alpha_{y_{n-2}} = y_{n-3}$, and $\alpha_{y_n} = y_{n-1}$.

In particular, α_{y_i} is always in a cell that is horizontally adjacent to the cell for y_i where both α_{y_i} and y_i are in Group 1 or both are in Group 2 of Table 1.

Proof. First, we consider the case when $y_i \notin \{y_1, y_{n-2}, y_n\}$.

Case I: Suppose y_i is in Column 1 of Table 1.

Then, by the structure of the table, y_{i-1}, y_{i+1} are in Columns 2 and 3 of Table 1. Let $\{u, v\} = \{\alpha_{y_i}, \beta_{y_i}\}$ with u in Column 2 and v in Column 3. Under the process of Algorithm 1, $d(y_i, u) \leq d(y_i, v)$. When the inequality is strict, α_{y_i} is the vertex in Column 2.

If $d(y_i, u) = d(y_i, v)$, either both u and v are leg vertices or u is a leg vertex and v is on the spine of G. Note that either way, a leg vertex is in Column 2. By convention, let α_{y_i} be that leg vertex in Column 2.

Case II: Suppose y_i is in Column 2 of Table 1.

Then, by the structure of the table, both y_{i-1} and y_{i+1} are in Column 1. Therefore, α_{y_i} is in Column 1.

In particular, by Algorithm 1, $d(y_{i-1}, y_i) \geq d(y_i, y_{i+1})$ when y_{i-1}, y_{i+1} are in Column 1 and y_i is in Column 2 of Table 1. The distances are equal when both y_{i-1} and y_{i+1} are leg vertices or y_{i-1} is on the spine of G and y_{i+1} is a leg vertex. Thus, by convention, when the distances are equal, let α_{y_i} be the leg vertex entered into the $i+1$ cell of Table 1.

Case III: Suppose y_i is in Column 3 of Table 1.

The proof is analogous to the proof of Case I.

Case IV: Suppose y_i is in Column 4 of Table 1.

The proof is analogous to the proof of Case II.

Now we consider the case when y_i is in $\{y_1, y_{n-2}, y_n\}$.

When $y_i = y_1$, then it is not part of a triple of vertices y_{i-1}, y_i, y_{i+1}. In this case, as before, consider y_2 as α_{y_1}. Note that α_{y_1} is in Column 1 of Table 1. Analogously, when $y_i = y_n$ then it is not part of a triple of vertices y_{i-1}, y_i, y_{i+1}. In this case, consider y_{n-1} as α_{y_n}. Note that α_{y_n} is in Column 3 of Table 1.

When $y_i = y_{n-2}$, y_i is in Column 1 or Column 4 of Table 1. If y_{n-2} is in Column 4, then both y_{n-3} and y_{n-1} are in Column 3. If y_{n-2} is in Column 1, then either both y_{n-3} and y_{n-1} are in Column 3 or y_{n-3} is in Column 2 and y_{n-1} is in Column 3. In each case, by the process of Algorithm 1, $d(u_{c_b}, y_{n-3}) \leq d(u_{c_b}, y_{n-1})$. In the case where the distances are equal, y_{n-3} is a leg vertex and y_{n-1} is on the spine. In that case, we choose $\alpha_{y_{n-2}} = y_{n-3}$, the leg vertex. Therefore, $\alpha_{y_{n-2}}$ is y_{n-3} in all cases.

In all of the above cases, it can be checked that y_i and α_{y_i} are in horizontally adjacent cells in Group 1 of Table 1 or in horizontally adjacent cells in Group 2 of Table 1. □

We will use the following definition in the proof of the next theorem which defines a labeling that uses the ordering of the vertices given by Algorithm 1. The theorem also shows that this labeling is the radio labeling associated with the ordering given by Algorithm 1 when G is jumpless.

Definition. For a caterpillar G, let $m_i := d(x_i, \alpha_{x_i}) - (\frac{D+1}{2} + t_{\alpha_{x_i}})$, if the quantity is positive and zero otherwise.

Theorem 1. *Let G be an edge-balanced caterpillar with ordering $y_1, y_2, \ldots, y_n$ of vertices as given by Algorithm 1. Define a labeling g such that $g(y_1) = 1$ and $g(y_{i+1}) = D + 1 - d(y_i, y_{i+1}) + g(y_i)$ for all i, $1 \leq i \leq n-1$. If G is a jumpless caterpillar, then g is a radio labeling of G and is therefore the associated radio labeling to the ordering given by Algorithm 1.*

Proof. We begin by showing that $m_i = 0$ for all i.

We start by considering when $y_i \notin \{y_1, y_{n-2}, y_n\}$. Then we have the following cases:

Case I: Consider a vertex in Column 1 or Column 3 of Table 1 as y_i in a triple of vertices y_{i-1}, y_i, y_{i+1}. By Lemma 2, α_{y_i} and y_i are in horizontally adjacent cells in Group 1 or in Group 2 of Table 1. Since G is a jumpless caterpillar, this means $d(y_i, \alpha_{y_i}) \leq \frac{D+1}{2} + t_{\alpha_{y_i}}$. Thus, $m_i = 0$ for all y_i when y_i is in Columns 1 or 3 of Table 1.

Case II: Consider a vertex in Column 2 or Column 4 of Table 1 as y_i in a triple of vertices y_{i-1}, y_i, y_{i+1}. Then we consider the following two cases:

Subcase A: Suppose y_i is on the spine of G. Notice that both y_{i-1} and y_{i+1} are in cells that are horizontally adjacent to the cell for y_i such that all three vertices are in Group 1 or all three are in Group 2 of Table 1. Due to this and the fact that G is a jumpless caterpillar, $d(y_i, y_{i-1}) \leq \frac{D+1}{2} + 0$ and $d(y_i, y_{i+1}) \leq \frac{D+1}{2} + 0$. Note that this means $d(y_i, \alpha_{y_i}) \leq \frac{D+1}{2}$.

1. Suppose α_{y_i} is on the spine of G. Then $m_i = d(y_i, \alpha_{y_i}) - (\frac{D+1}{2} + 0) \leq \frac{D+1}{2} - (\frac{D+1}{2} + 0) = 0$ so by definition of m_i, it follows that $m_i = 0$.

2. Suppose α_{y_i} is a leg vertex. Then $m_i = d(y_i, \alpha_{y_i}) - (\frac{D+1}{2} + 1) \leq \frac{D+1}{2} - (\frac{D+1}{2} + 1) = -1$ so by definition of m_i, $m_i = 0$.

Therefore, when y_i is on the spine of G, it follows that $m_i = 0$.

Subcase B: Suppose y_i is a leg vertex of G. Notice that both y_{i-1} and y_{i+1} are in cells that are horizontally adjacent to the cell for y_i such that all three vertices are in Group 1 or all three vertices are in Group 2 of Table 1. Due to this and the definition of G being a jumpless caterpillar, $d(y_i, y_{i-1}) \leq \frac{D+1}{2} + 1$ and $d(y_i, y_{i+1}) \leq \frac{D+1}{2} + 1$. This means $d(y_i, \alpha_{y_i}) \leq \frac{D+1}{2} + 1$.

1. Suppose both y_{i-1} and y_{i+1} are on the spine of G. Then, $t_{\alpha_{y_i}} = 0$. Thus, $m_i = d(y_i, \alpha_{y_i}) - \frac{D+1}{2}$. Suppose by way of contradiction that $m_i \neq 0$. The only time this would occur is if $d(y_i, \alpha_{y_i}) = \frac{D+1}{2} + 1$. If this were the case, notice that both $d(y_i, y_{i-1}) = \frac{D+1}{2} + 1$ and $d(y_i, y_{i+1}) = \frac{D+1}{2} + 1$ because if one were smaller, then $d(y_i, \alpha_{y_i})$ would be smaller. However, by Algorithm 1, y_{i-1} and y_{i+1} are in the same component of G. This contradicts the fact that in a caterpillar, there is a unique vertex on the spine in component A (or component B) that is distance $\frac{D+1}{2} + 1$ from y_i.

Therefore, if both y_{i-1} and y_{i+1} are on the spine, $m_i = 0$.

2. Suppose at least one of y_{i-1} or y_{i+1} is a leg vertex. Notice that if $t_{\alpha_{y_i}} = 1$, then $m_i \leq \frac{D+1}{2} + 1 - (\frac{D+1}{2} + 1)$ shows that $m_i = 0$. Also, if $d(y_i, \alpha_{y_i}) < \frac{D+1}{2} + 1$, $m_i \leq \frac{D+1}{2} - (\frac{D+1}{2} + t_{\alpha_{y_i}})$ which implies that $m_i = 0$.

Suppose by contradiction that $t_{\alpha_{y_i}} = 0$ and $d(y_i, \alpha_{y_i}) = \frac{D+1}{2} + 1$. This would mean that α_{y_i} is on the spine and $d(y_{i-1}, y_i) = \frac{D+1}{2} = d(y_i, y_{i+1})$ because otherwise $d(y_i, \alpha_{y_i}) < \frac{D+1}{2} + 1$. This contradicts the convention found in Lemma 2 that says if $d(y_{i-1}, y_i) = d(y_i, y_{i+1})$ and one of y_{i-1}, y_{i+1} is a leg, then let α_{y_i} be the leg vertex.

Therefore in all instances where at least one of y_{i-1}, y_{i+1} is a leg, $m_i = 0$.

We now consider when $y_i \in \{y_1, y_{n-2}, y_n\}$.

When $y_i = y_1$, it is not part of a triple of vertices y_{i-1}, y_i, y_{i+1}, As before, we consider y_2 as α_{y_1}. By Algorithm 1, y_2 is on the spine of G and y_1 and y_2 are in horizontally adjacent cells in Group 1 of Table 1. Since G is jumpless, this means $d(y_1, \alpha_{y_1}) = d(y_1, y_2) \leq \frac{D+1}{2} + 0$. Thus, $m_1 = d(y_1, \alpha_{y_1}) - (\frac{D+1}{2} + 0) \leq \frac{D+1}{2} - \frac{D+1}{2} = 0$.

Using an analogous argument, we see that $m_n = 0$.

Now consider when y_{n-2} in the triple of vertices $y_{n-3}, y_{n-2}, y_{n-1}$. From Lemma 2, we know that $\alpha_{y_{n-2}} = y_{n-3}$. Thus, from condition (2) of the definition of G being a jumpless caterpillar, $m_{n-2} = d(y_{n-2}, \alpha_{y_{n-2}}) - (\frac{D+1}{2} + t_{\alpha_{y_{n-2}}}) \leq \frac{D+1}{2} + t_{\alpha_{y_{n-2}}} - (\frac{D+1}{2} + t_{\alpha_{y_{n-2}}}) = 0$.

Therefore, when G is a jumpless caterpillar, $m_i = 0$ for all i. Notice that this means $d(y_i, \alpha_{y_i}) \leq \frac{D+1}{2} + t_{\alpha_{y_i}}$ for $1 \leq i \leq n$.

Now, consider the labeling g such that $g(y_1) = 1$ and $g(y_{i+1}) = D + 1 - d(y_i, y_{i+1}) + g(y_i)$ for $1 \leq i \leq n - 1$. We claim g is a radio labeling of G.

By the definition of g, the radio condition is satisfied for any pair of vertices y_i, y_{i+1}.

We will next verify the radio condition for pairs of vertices y_{i-1}, y_{i+1}. Notice that

$$d(\alpha_{y_i}, \beta_{y_i}) = d(y_i, \beta_{y_i}) - d(y_i, \alpha_{y_i}) + s_{\alpha_{y_i}}$$

where $s_{\alpha_{y_i}} = 0$ if α_{y_i} is on the spine of G and $s_{\alpha_{y_i}} = 2$ if α_{y_i} is a leg vertex. From the definition of g it follows that,

$$d(y_i, \alpha_{y_i}) + |g(y_i) - g(\alpha_{y_i})| = D + 1 \text{ and}$$
$$d(y_i, \beta_{y_i}) + |g(y_i) - g(\beta_{y_i})| = D + 1.$$

Consider the case when $g(\alpha_{y_i}) < g(y_i) < g(\beta_{y_i})$. (The other case is proven similarly.) We start with the left hand side of (1) for the vertices α_{y_i} and β_{y_i} and make a series of substitutions as follows:

$$d(\alpha_{y_i}, \beta_{y_i}) + g(\beta_{y_i}) - g(\alpha_{y_i})$$

$$= d(y_i, \beta_{y_i}) - d(y_i, \alpha_{y_i}) + s_{\alpha_{y_i}} + g(\beta_{y_i}) - g(y_i) + g(y_i) - g(\alpha_{y_i})$$
$$= d(y_i, \beta_{y_i}) - d(y_i, \alpha_{y_i}) + s_{\alpha_{y_i}} + D + 1 - d(\beta_{y_i}, y_i) + D + 1 - d(\alpha_{y_i}, y_i)$$
$$= 2D + 2 - 2d(y_i, \alpha_{y_i}) + s_{\alpha_{y_i}}$$

$$\geq 2D + 2 - 2\left(\frac{D+1}{2} + t_{\alpha_{y_i}}\right) + s_{\alpha_{y_i}}$$
$$= 2D + 2 - D - 1 - 2t_{\alpha_{y_i}} + s_{\alpha_{y_i}}$$
$$= D + 1.$$

Therefore the radio condition is satisfied for vertices y_{i-1} and y_{i+1}.

To show the radio condition is satisfied for pairs of vertices y_i and y_j where $j = i + k$ for $k \geq 3$, we first notice the following. By the definition of g, $g(y_{i+1}) - g(y_i) = D + 1 - d(y_i, y_{i+1})$. Also, from the fact that G is a jumpless caterpillar, for all pairs of vertices y_i and y_{i+1} that are in horizontally adjacent cells with both vertices in Group 1 or both vertices in Group 2 of Table 1, $d(y_i, y_{i+1}) \leq \frac{D+1}{2} + 1$. Thus, for y_i and y_{i+1} in horizontally adjacent cells both in Group 1 or both in Group 2 of Table 1, we have that

$$\begin{aligned} g(y_{i+1}) - g(y_i) &= D + 1 - d(y_i, y_{i+1}) \\ &\geq D + 1 - \left(\frac{D+1}{2} + 1\right) \\ &= D - \frac{D+1}{2} \\ &= \frac{D-1}{2} \end{aligned} \tag{2}$$

Now consider the pair of vertices y_i and y_j where $j = i + k$ for some positive integer $k \geq 3$. Then

$$g(y_j) - g(y_i) \geq g(y_{i+3}) - g(y_i)$$
$$= g(y_{i+3}) - g(y_{i+2}) + g(y_{i+2}) - g(y_{i+1}) + g(y_{i+1}) - g(y_i). \tag{3}$$

From Algorithm 1, two of the label differences for a pair of successively labeled vertices in (3) correspond to vertices that are in horizontally adjacent cells of Table 1 in Group 1 or horizontally adjacent cells in Group 2. For those two pairs, we get a bound from (2). The other label difference is at least 1 because all labels are unique. Thus, (2) and (3) give

$$g(y_j) - g(y_i) \geq \tfrac{D-1}{2} + \tfrac{D-1}{2} + 1 = D.$$

Also, since $d(y_i, y_j) \geq 1$, it follows that $g(y_j) - g(y_i) + d(y_i, y_j) \geq D + 1$. Thus, the radio condition is satisfied for y_i and y_j whenever $|i - j| \geq 3$.

Therefore, g is a radio labeling of G. □

Corollary 3. *Let G be an edge-balanced caterpillar. If G is a jumpless caterpillar, then $rn(G) = g(y_n)$.*

Proof. From Proposition 8, the ordering $y_1, \ldots, y_n$ given by Algorithm 1 is a distance maximizing ordering of G. From Theorem 1, we know that since G is a jumpless caterpillar, g is a radio labeling. By how the labeling g in Theorem 1 was defined, $g(y_{i+1}) - g(y_i) = D + 1 - d(y_i, y_{i+1})$ for $1 \leq i \leq n-1$. Summing these $n-1$ equations and solving for $g(y_n)$ gives $g(y_n) = (n-1)(D+1) + 1 - \max \sum_{i=1}^{n-1} d(y_i, y_{i+1})$. From Proposition 2, it follows that $rn(G) = g(y_n)$. □

A technique used in the proof of Theorem 1 is useful when considering characteristics of a distance maximizing ordering of an edge-balanced caterpillar that does not require jumps. We include this in the next proposition.

Proposition 9. *Let G be an edge-balanced caterpillar. Let $x_1, \ldots, x_n$ be a distance maximizing ordering of the vertices of G such that the associated radio labeling f does not require jumps. Then for every i, $d(x_i, \alpha_{x_i}) \leq \frac{D+1}{2} + t_{\alpha_{x_i}}$.*

Proof. First we consider when $2 \leq i \leq n-1$. Since f does not require jumps, $\sum_{i=1}^{n-1} J_f(x_i, x_{i+1}) = 0$ which means that $J_f(x_i, x_{i+1}) = 0$ for $1 \leq i \leq n-1$. From this we have,

$$|f(x_i) - f(\alpha_{x_i})| = D + 1 - d(x_i, \alpha_{x_i}) \text{ and}$$
$$|f(x_i) - f(\beta_{x_i})| = D + 1 - d(x_i, \beta_{x_i}).$$

Notice that $d(\alpha_{x_i}, \beta_{x_i}) = d(x_i, \beta_{x_i}) - d(x_i, \alpha_{x_i}) + s_{\alpha_{x_i}}$ where $s_{\alpha_{x_i}} = 0$ if α_{x_i} is on the spine of G and $s_{\alpha_{x_i}} = 2$ if α_{x_i} is a leg vertex.

Consider the radio condition for x_{i-1} and x_{i+1}:

$$\begin{aligned}
f(x_{i+1}) - f(x_i) + f(x_i) - f(x_{i-1}) &\geq D + 1 - d(x_{i-1}, x_{i+1}) \\
\Rightarrow 2D + 2 - d(x_i, \alpha_{x_i}) - d(x_i, \beta_{x_i}) &\geq D + 1 - d(\alpha_{x_i}, \beta_{x_i}) \\
\Rightarrow D + 1 &\geq d(x_i, \alpha_{x_i}) + d(x_i, \beta_{x_i}) - d(\alpha_{x_i}, \beta_{x_i}) \\
\Rightarrow D + 1 &\geq d(x_i, \alpha_{x_i}) + d(x_i, \beta_{x_i}) \\
&\quad -[d(x_i, \beta_{x_i}) - d(x_i, \alpha_{x_i}) + s_{\alpha_{x_i}}] \\
\Rightarrow D + 1 &\geq 2d(x_i, \alpha_{x_i}) - s_{\alpha_{x_i}} \\
\Rightarrow \frac{D + 1 + s_{\alpha_{x_i}}}{2} &\geq d(x_i, \alpha_{x_i}) \\
\Rightarrow \frac{D+1}{2} + t_{\alpha_{x_i}} &\geq d(x_i, \alpha_{x_i}). \qquad (4)
\end{aligned}$$

Now consider when $i = 1$. By convention, $\alpha_{x_1} = x_2$. Suppose by contradiction that $d(x_1, x_2) = d(x_1, \alpha_{x_1}) > \frac{D+1}{2} + t_{\alpha_{x_1}} \geq \frac{D+1}{2}$.

Since $x_1, \ldots, x_n$ is a distance maximizing ordering, by Corollary 1, $x_1 \in \{u_{c_a}, u_{c_b}\}$. Suppose $x_1 = u_{c_a}$ (the proof is analogous when $x_1 = u_{c_b}$). It follows from Corollary 2 that $x_2 \in B$ and $x_3 \in A$. By the structure of an edge-balanced caterpillar, for $u \neq u_{c_a} \in A$ and $v \neq u_{c_b} \in B$, $d(u, v) > d(u_{c_a}, v)$. It follows that $d(x_2, x_1) = d(x_2, u_{c_a}) < d(x_2, x_3)$ which means that $x_1 = \alpha_{x_2}$. Also, by the structure of G, $x_1 = u_{c_a}$ is on the spine of G so $t_{\alpha_{x_2}} = 0$. Therefore, by (4), $d(x_2, x_1) \leq \frac{D+1}{2}$, which is a contraction to our assumption. Therefore, $d(x_1, \alpha_{x_1}) \leq \frac{D+1}{2} + t_{\alpha_{x_i}}$.

A similar argument can be used to show that $d(x_n, \alpha_{x_n}) \leq \frac{D+1}{2} + t_{\alpha_{x_n}}$. □

For an edge-balanced caterpillar G with $x_1, \ldots, x_n$ an ordering of its vertices, the occurrence of a vertex x_i being considered as α_{x_j} in relation to a vertex x_j is important in the arguments of the next theorem. Consider x_i for $3 \leq i \leq n-2$. This vertex is labeled after x_{i-1} and before x_{i+1}. Then x_i is in two triples of successively labeled vertices such that x_i is not the middle vertex of the triple, namely, the triples $\{x_{i-2}, x_{i-1}, x_i\}$ and $\{x_i, x_{i+1}, x_{i+2}\}$. Therefore, it is possible that x_i is $\alpha_{x_{i-1}}$ and/or $\alpha_{x_{i+1}}$. By definition, x_2 is α_{x_1}, but x_2 could also be α_{x_3} since it is part of the triple $\{x_2, x_3, x_4\}$. Similarly, $x_{n-1} = \alpha_{x_n}$ and could also be $\alpha_{x_{n-2}}$ since it is part of the triple $\{x_{n-3}, x_{n-2}, x_{n-1}\}$. This leads to the following definition.

Definition. In a caterpillar G with an ordering of vertices $x_1, \ldots, x_n$, for $1 < i < n$, if x_i is considered as $\alpha_{x_{i-1}}$ or $\alpha_{x_{i+1}}$, then x_i is called an *alpha vertex.*

For the specific cases of x_1 and x_n, we define alpha vertices as follows:

- x_1 is an alpha vertex if $x_1 = \alpha_{x_2}$.
- x_n is an alpha vertex if $x_n = \alpha_{x_{n-1}}$.

Notice that a vertex x_i, $3 \leq i \leq n-2$, could be an alpha vertex for zero, one, or two vertices; a vertex x_i for $i = 2, n-1$ could be an alpha vertex for one or two vertices; and x_1 and x_n can be an alpha vertex for zero or one vertex.

We will use the above definition to make arguments based on how many times certain vertices are considered to be alpha vertices under a given ordering of vertices of a caterpillar G in the proof of the next theorem. This theorem improves the lower bound of the radio number of edge-balanced caterpillars that are not jumpless. We prove this by first showing that the bound from Proposition 2 would be increased if an ordering of the vertices is not distance maximizing. Then we show that the radio labeling associated to the distance maximizing ordering of Algorithm 1 requires jumps when G is not jumpless. Finally, we show that no other distance maximizing ordering has associated radio labeling that does not require jumps. We do this by supposing one exists and then reach a contradiction based on the structure of G being jumpless and comparison to the ordering $y_1, \ldots y_n$

from Algorithm 1. This requires us to consider various cases based on the structure of G.

Since the arguments in the proof of this theorem can be complicated, we include an example of a graph from one of the situations following the proof.

Theorem 2. *Let G be an edge-balanced caterpillar with n vertices. If G is not a jumpless caterpillar, then*

$$rn(G) \geq (n-1)(D+1) + 1 - \max(\textstyle\sum_{i=1}^{n-1} d(x_i, x_{i+1})) + 1$$

where the maximum is taken over all possible orderings of the vertices of G.

Proof. First we consider an ordering $x_1, \ldots, x_n$ of the vertices of G that is not a distance maximizing ordering. It follows that $\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \max\left(\sum_{i=1}^{n-1} d(x_i, x_{i+1})\right) - 1$ where the maximum is taken over all possible orderings of the vertices of G. Thus, the bound follows from Proposition 1.

Next, we consider the ordering $y_1 \ldots, y_n$ of the vertices of G given by Algorithm 1. By Proposition 8, this is a distance maximizing ordering. We will show that the associated radio labeling to this ordering as well as to any other distance maximizing ordering of the vertices of G requires jumps.

By the hypothesis, G is not a jumpless caterpillar. Then for the ordering $y_1, \ldots, y_n$ of the vertices of G given by Algorithm 1, either

(i) there exists a pair of vertices in horizontally adjacent cells of Group 1 (or Group 2) of the table given by Algorithm 1 such that their distance is greater than $\frac{D+1}{2} + t$ where t is 1 if the vertex in Column 2 (or Column 4) is a leg vertex and 0 otherwise, or

(ii) $d(y_{n-2}, y_{n-3}) > \frac{D+1}{2} + t_{\alpha_{y_{n-2}}}$.

Let h be the associated radio labeling to the ordering $y_1, \ldots, y_n$.

Case I: Suppose condition (i) is satisfied.

Consider the vertex of this pair that is in Column 1 (or Column 3) as y_i for some $i \neq n-2$. By Lemma 2, α_{y_i} is in Column 2 (or Column 4) and thus it follows that $d(y_i, \alpha_{y_i}) > \frac{D+1}{2} + t_{\alpha_{y_i}}$. The ordering $y_1, \ldots, y_n$ is a distance maximizing ordering of the vertices of G and thus by the contrapositive of Proposition 9, the associated radio labeling requires jumps. Thus, $h(y_n) \geq (n-1)(D+1) + h(y_1) - (\sum_{i=1}^{n-1} d(y_i, y_{i+1})) + 1$.

Suppose by contradiction that there exists another distance maximizing ordering $x_1, \ldots, x_n$ of the vertices of G with associated radio labeling f such that f does not require jumps. From Proposition 9, this means that for all j, $d(x_j, \alpha_{x_j}) \leq \frac{D+1}{2} + t_{\alpha_{x_j}}$.

Suppose x_j is the same vertex as y_i. From the above assumptions, $d(y_i, \alpha_{y_i}) > \frac{D+1}{2} + t_{\alpha_{y_i}}$ and $d(x_j, \alpha_{x_j}) \leq \frac{D+1}{2} + t_{\alpha_{x_j}}$ where $\alpha_{x_j} \neq \alpha_{y_i}$. This means that $d(x_j, \alpha_{x_j}) \leq d(y_i, \alpha_{y_i})$.

Claim: If the pair of vertices is $\{y_1, y_2\}$ or $\{y_{n-1}, y_n\}$, by the structure of an edge-balanced caterpillar, no such α_{x_j} exists.

Proof of Claim: Consider when $i = 2$. Then $y_2 = u_1$ is in Column 1 and $y_1 = \alpha_{y_2} = u_{c_b}$ is in Column 2. Also, $d(y_2, \alpha_{y_2}) > \frac{D+1}{2}$. Notice that x_j, which is the same vertex as y_2, is in component A. Since $x_1, \ldots, x_n$ is a distance maximizing ordering of the vertices of G, by Corollary 2, x_{j-1} and x_{j+1} are in component B. However, by the structure of an edge-balanced caterpillar, every vertex $w \neq u_{c_b}$ in component B is such that $d(u_1, w) > d(u_1, u_{c_b})$ which means $d(x_j, w) > d(x_j, u_{c_b}) = d(y_2, \alpha_{y_2})$. Therefore, it is not possible to have $d(x_j, \alpha_{x_j}) \leq d(y_2, \alpha_{y_2})$. Thus $d(x_j, \alpha_{x_j}) > \frac{D+1}{2} + t_{\alpha_{x_j}}$ and f requires jumps. A similar argument shows that no such α_{x_j} exists when $i = n - 1$ and thus the claim has been proven.

By the above claim, if the pair of vertices satisfying condition (i) is $\{y_1, y_2\}$ or $\{y_{n-1}, y_n\}$, we have already reached a contradiction to the assumption that f does not require jumps.

Now we consider when $i \neq 2, n-1$ and look at the following cases to reach a contradiction to the assumption that f does not require jumps.

Subcase A: $d(x_j, \alpha_{x_j}) < d(y_i, \alpha_{y_i})$.

Since the arguments for y_i in Column 1 or y_i in Column 3 of Table 1 are analogous, we give the argument only once. We suppose y_i is in Column 1 for this proof.

Let $\mathscr{A}$ be the set of all vertices that are entered into cells above the cell for α_{y_i} in Column 2 of Table 1. Let $\mathscr{B}$ be the set of all vertices that are entered into cells above the cell for y_i in Column 1 of Table 1.

Claim: α_{x_j} is in $\mathscr{A}$.

Proof of Claim: Under Algorithm 1, vertices are entered into Column 2 of Table 1 so that the subscripts of the vertices are in non-decreasing order and leg vertices are entered before spine vertices with the same subscript. A vertex v is entered in the table above α_{y_i} means $d(u_{c_b}, v) \leq d(u_{c_b}, \alpha_{y_i})$. Since $d(x_j, \alpha_{x_j}) < d(y_i, \alpha_{y_i})$, it follows that $d(\alpha_{x_j}, u_{c_b}) < d(\alpha_{y_i}, u_{c_b})$. Thus, $\alpha_{x_j} \in \mathscr{A}$ and we have proven the claim.

We consider two possible situations depending on where y_i is located in Table 1. Consider arbitrary entries into Columns 1 and 2 of Table 1: cells $m, m+1, m+2$ where m and $m+2$ denote cells in Column 1 whose entries have their associated alpha vertex in the $m+1$ cell of Column 2.

1. y_i is in the m entry of Table 1 (meaning that $m = i$ in this case).

By the structure of Table 1, we see that $|\mathscr{B}| = 2|\mathscr{A}| - 1$.

Now consider the elements in $\mathscr{A}$. In a distance maximizing ordering of the vertices of G, every element in $\mathscr{A}$ except for u_{c_b} could be an alpha vertex for two vertices in component A. The vertex u_{c_b} can be an alpha vertex for only one vertex in component A. Thus, in general, the possible number of uses of vertices in $\mathscr{A}$ as alpha vertices under a distance maximizing ordering is $2|\mathscr{A}| - 1$.

For the distance maximizing ordering $x_1, \ldots, x_n$, vertex α_{x_j} has already been used as an alpha vertex for one vertex of component A. Therefore, there are $2|\mathscr{A}| - 2$ remaining possible number of uses of vertices in $\mathscr{A}$ as alpha vertices under the ordering $x_1, \ldots, x_n$. Since $|\mathscr{B}| = 2|\mathscr{A}| - 1 > 2|\mathscr{A}| - 2$, we conclude that there exists at least one vertex x_k in $\mathscr{B}$ such that α_{x_k} is not in $\mathscr{A}$ but is in component B.

By nature of how the sets $\mathscr{A}$ and $\mathscr{B}$ were formed,

$$d(u_{c_b}, \alpha_{y_i}) \leq d(u_{c_b}, \alpha_{x_k}) \text{ and}$$

$$d(y_i, u_{c_a}) \leq d(x_k, u_{c_a}). \tag{5}$$

Since $d(x_k, \alpha_{x_k}) = d(x_k, u_{c_a}) + d(u_{c_a}, u_{c_b}) + d(u_{c_b}, \alpha_{x_k})$ and $d(y_i, \alpha_{y_i}) = d(y_i, u_{c_a}) + d(u_{c_a}, u_{c_b}) + d(u_{c_b}, \alpha_{y_i})$, whenever at least one of the inequalities of (5) is strict, $d(x_k, \alpha_{x_k}) > d(y_i, \alpha_{y_i})$. By hypothesis, it follows that $d(x_k, \alpha_{x_k}) > d(y_i, \alpha_{y_i}) \geq \frac{D+1}{2} + 1$ which implies that $d(x_k, \alpha_{x_k}) > \frac{D+1}{2} + 1$. By contrapositive of Proposition 9, this means the associated radio labeling f for the ordering $x_1, \ldots, x_n$ requires jumps, contradicting the assumption.

To consider when the inequalities of (5) are both equalities, we notice that $\alpha_{x_k} \notin \mathscr{A}$ means that α_{x_k} is entered in Column 2 below α_{y_i}, is α_{y_i}, or is entered into Column 3 of Table 1. Also, recall that leg vertices are entered into Column 2 before spine vertices with the same subscript.

Note that if $\alpha_{x_k} = \alpha_{y_i}$, then since $d(x_k, \alpha_{x_k}) = d(y_i, \alpha_{y_i}) > \frac{D+1}{2} + t_{\alpha_{y_i}} = \frac{D+1}{2} + t_{\alpha_{x_k}}$, by the contrapositive of Proposition 9, f requires jumps, which is a contradiction to the assumption.

Now suppose $\alpha_{x_k} \neq \alpha_{y_i}$. Since $d(u_{c_b}, \alpha_{y_i}) = d(u_{c_b}, \alpha_{x_k})$, the vertices α_{y_i} and α_{x_k} have the same subscript in the original edge-balanced caterpillar notation. Therefore, either both α_{x_k} and α_{y_i} are leg vertices or α_{x_k} is a vertex on the spine of G while α_{y_i} is a leg vertex.

Since α_{y_i} is a leg vertex, $t_{\alpha_{y_i}} = 1$ and thus $d(y_i, \alpha_{y_i}) > \frac{D+1}{2} + 1$. Since $d(y_i, \alpha_{y_i}) = d(x_k, \alpha_{x_k})$, it follows that $d(x_k, \alpha_{x_k}) > \frac{D+1}{2} + 1 \geq \frac{D+1}{2} + t_{\alpha_{x_k}}$. Therefore, by the contrapositive of Proposition 9, the associated radio labeling f for the ordering $x_1, \ldots, x_n$ requires jumps, which is a contradiction to the assumption.

2. y_i is in the $m + 2$ entry of the table (meaning $i = m + 2$ in this case).

By the structure of Table 1, we see that $|\mathscr{B}| = 2|\mathscr{A}|$. Notice that $\mathscr{B}$ has the vertex entered in cell m which is why the set $\mathscr{B}$ in this case has one more element than the set $\mathscr{B}$ of Case I: Subcase A:1.

By the same arguments as in Case I: Subcase A:1, $\alpha_{x_j} \in \mathscr{A}$. Since the set $\mathscr{A}$ is the same as in that case, we use the same argument to see that the number of possible uses of vertices in $\mathscr{A}$ as alpha vertices that have not been used yet under the ordering $x_1, \ldots, x_n$ is $2|\mathscr{A}| - 2$. Since $|\mathscr{B}| = 2|\mathscr{A}| > 2|\mathscr{A}| - 2$, we conclude that there exists at least one vertex x_k in $\mathscr{B}$ such that α_{x_k} is not in $\mathscr{A}$ but is in component B.

The same arguments as in Case I: Subcase A:1 show that f requires a jump which is a contradiction to the assumption.

Subcase B: $d(x_j, \alpha_{x_j}) = d(y_i, \alpha_{y_i})$.

Note that the only way this can happen is if α_{y_i} is a vertex on the spine of G, α_{x_j} is a leg vertex, and $d(x_j, \alpha_{x_j}) = \frac{D+1}{2} + 1 = d(y_i, \alpha_{y_i})$. Also, this means that α_{y_i} and α_{x_j} have the same subscript in the original edge-balanced caterpillar notation.

As before, since the arguments for y_i in Column 1 or y_i in Column 3 of Table 1 are analogous, we give the argument only once. We suppose y_i is in Column 1 for this proof.

From Lemma 2, we know α_{y_i} is entered into Column 2 of Table 1. Let $\mathscr{A}$ be the set of all vertices that are entered into cells above the cell for α_{y_i} in Column 2 of Table 1 by Algorithm 1. Let $\mathscr{B}$ be the set of all vertices that are entered into cells above the cell for y_i in Column 1 of Table 1 by Algorithm 1.

Algorithm 1 inserts leg vertices into Column 2 before spine vertices with the same subscript in the edge-balanced caterpillar notation. Thus, since α_{x_j} is a leg vertex and α_{y_i} is on the spine of G, it follows that $\alpha_{x_j} \in \mathscr{A}$. The proof now follows the proof of Case I: Subcase A.

In all of the above cases, we have shown that when G is not a jumpless caterpillar such that condition (i) above is satisfied, the labeling associated with an arbitrary distance maximizing ordering requires jumps which is a contradiction to our assumption. Therefore, from Propositions 1 and 2 and the definition of a labeling requiring jumps, we have that

$$rn(G) \geq (n-1)(D+1) + 1 - \max\left(\sum_{i=1}^{n-1} d(x_i, x_{i+1})\right) + 1.$$

where the maximum is taken over all possible orderings of the vertices of G.

Case II: Suppose condition (ii) is satisfied.

By Lemma 2 $\alpha_{y_{n-2}} = y_{n-3}$. Condition (ii) shows that $d(y_{n-2}, \alpha_{y_{n-2}}) > \frac{D+1}{2} + t_{\alpha_{y_{n-2}}}$. By Proposition 8, $y_1, \ldots, y_n$ is a distance maximizing ordering of the vertices of G and thus by the contrapositive of Proposition 9, the associated radio labeling requires jumps. Thus, $h(y_n) \geq (n-1)(D+1) + h(y_1) - (\sum_{i=1}^{n-1} d(y_i, y_{i+1})) + 1$.

Let $x_1, \ldots, x_n$ be an arbitrary distance maximizing ordering of the vertices of G. Suppose by contradiction that the associated radio labeling f does not require jumps. From Proposition 9, this means that for all j, $d(x_j, \alpha_{x_j}) \leq \frac{D+1}{2} + t_{\alpha_{x_j}}$.

Suppose x_j is the same vertex as y_{n-2}. From the above assumptions, $d(y_{n-2}, \alpha_{y_{n-2}}) > \frac{D+1}{2} + t_{\alpha_{y_{n-2}}}$ and $d(x_j, \alpha_{x_j}) \leq \frac{D+1}{2} + t_{\alpha_{x_j}}$. This means $d(x_j, \alpha_{x_j}) \leq d(y_{n-2}, \alpha_{y_{n-2}})$.

Now we consider the following cases to find a contradiction to the assumption that f does not require jumps.

Subcase A: $d(x_j, \alpha_{x_j}) < d(y_{n-2}, \alpha_{y_{n-2}})$.

1. y_{n-2} is in Column 1 of Table 1.

Notice that $y_{n-3} = \alpha_{y_{n-2}}$ could be in Column 2 or Column 3 of Table 1.

a) y_{n-3} is in Column 2 of Table 1.

This means that for cells $m, m+1, m+2$ where m and $m+2$ are in Column 1 and $m+1$ is in Column 2 of Table 1, y_{n-2} is in the $m+2$ entry. Therefore, the proof of the case is the same argument as Case I: Subcase A:2 with y_{n-2} as y_i.

b) y_{n-3} is in Column 3 of Table 1.

Let $\mathscr{A}$ be the set of all vertices entered into cells in Column 2 of Table 1 by Algorithm 1. Let $\mathscr{B}$ be the set of all vertices entered into cells above the cell for y_{n-2} in Column 1 of Table 1. Note that $|\mathscr{B}| = 2|\mathscr{A}| - 1$.

Claim: α_{x_j} is in $\mathscr{A}$.

Proof of Claim: Since $d(x_j, \alpha_{x_j}) < d(y_{n-2}, \alpha_{y_{n-2}})$, it follows that $d(u_{c_b}, \alpha_{x_j}) < d(u_{c_b}, \alpha_{y_{n-2}})$. In Algorithm 1, vertices are entered into Column 3 in non-increasing order which implies that all vertices v in Column 3 are such that $d(u_{c_b}, v) \geq d(u_{c_b}, \alpha_{y_{n-2}})$. Since $\alpha_{y_{n-2}}$ is the last vertex entered into Column 3 and $d(u_{c_b}, \alpha_{x_j}) < d(u_{c_b}, \alpha_{y_{n-2}})$, it follows that α_{x_j} is in Column 2 of Table 1. Therefore, α_{x_j} is in $\mathscr{A}$ and the claim has been proven.

In a distance maximizing ordering of G, every element in $\mathscr{A}$ except for u_{c_b} could be an alpha vertex for two vertices in component A. The vertex u_{c_b} can be an alpha vertex for only one vertex in component A. Thus, in general, the possible number of uses of vertices in $\mathscr{A}$ as alpha vertices under a distance maximizing ordering is $2|\mathscr{A}| - 1$.

In the distance maximizing ordering $x_1, \ldots, x_n$, the vertex α_{x_j} has already been used as an alpha vertex for one vertex in component A. Therefore, the remaining possible number of uses of vertices in $\mathscr{A}$ as alpha vertices under the ordering $x_1, \ldots, x_n$ is $2|\mathscr{A}| - 2$. Since $|\mathscr{B}| = 2|\mathscr{A}| - 1 > 2|\mathscr{A}| - 2$, we conclude that there exists at least one vertex x_k in $\mathscr{B}$ such that α_{x_k} is not in $\mathscr{A}$ but is in component B. By nature of how the sets $\mathscr{A}$ and $\mathscr{B}$ were formed,

$$d(u_{c_b}, \alpha_{y_{n-2}}) \leq d(u_{c_b}, \alpha_{x_k}) \text{ and}$$

$$d(y_{n-2}, u_{c_a}) \leq d(x_k, u_{c_a}). \tag{6}$$

Since $d(y_{n-2}, \alpha_{y_{n-2}}) = d(y_{n-2}, u_{c_a}) + d(u_{c_a}, u_{c_b}) + d(u_{c_b}, \alpha_{y_{n-2}})$ and $d(x_k, \alpha_{x_k}) = d(x_k, u_{c_a}) + d(u_{c_a}, u_{c_b}) + d(u_{c_b}, \alpha_{x_k})$, whenever one of the above inequalities is strict, $d(y_{n-2}, \alpha_{y_{n-2}}) < d(x_k, \alpha_{x_k})$. Thus, if one of the inequalities of (6) is strict, we have that $d(x_k, \alpha_{x_k}) > d(y_{n-2}, \alpha_{y_{n-2}}) \geq \frac{D+1}{2} + 1$ which implies that $d(x_k, \alpha_{x_k}) > \frac{D+1}{2} + t_{\alpha_{x_k}}$. Therefore, by the contrapositive of Proposition 9, the associated radio labeling f requires jumps which is a contradiction to our assumption.

Now consider when both of the inequalities of (6) are equalities. Then $d(x_k, \alpha_{x_k}) = d(y_{n-2}, \alpha_{y_{n-2}})$. Since $\alpha_{x_k} \notin \mathscr{A}$, α_{x_k} is either the same vertex as $\alpha_{y_{n-2}}$ or is entered in Column 3 of Table 1 and in a cell above the cell for $y_{n-3} = \alpha_{y_{n-2}}$.

Note that if $\alpha_{x_k} = \alpha_{y_{n-2}}$, then since $d(x_k, \alpha_{x_k}) = d(y_{n-2}, \alpha_{y_{n-2}}) > \frac{D+1}{2} + t_{\alpha_{y_{n-2}}} = \frac{D+1}{2} + t_{\alpha_{x_k}}$, by the contrapositive of Proposition 9, f requires jumps, which is a contradiction to the assumption.

Now, suppose $\alpha_{x_k} \neq \alpha_{y_{n-2}}$. Since $d(\alpha_{y_{n-2}}, u_{c_b}) = d(\alpha_{x_k}, u_{c_b})$, the vertices $\alpha_{y_{n-2}}$ and α_{x_k} have the same subscript in the original edge-balanced caterpillar notation. Also, since α_{x_k} is in Column 3 of Table 1 in a cell above the cell for $\alpha_{y_{n-2}}$, this means that either both α_{x_k} and $\alpha_{y_{n-2}}$ are leg vertices or $\alpha_{y_{n-2}}$ is a leg vertex and α_{x_k} is on the spine of G. Since $\alpha_{y_{n-2}}$ is a leg vertex, $t_{\alpha_{y_{n-2}}} = 1$ and thus $d(y_{n-2}, \alpha_{y_{n-2}}) > \frac{D+1}{2} + 1$. Since $d(y_{n-2}, \alpha_{y_{n-2}}) = d(x_k, \alpha_{x_k})$, it follows that $d(x_k, \alpha_{x_k}) > \frac{D+1}{2} + 1 \geq \frac{D+1}{2} + t_{\alpha_{x_k}}$. Therefore, by the contrapositive of Proposition 9, the associated radio labeling f for the ordering $x_1, \ldots, x_n$ requires jumps, which is a contradiction to the assumption.

2. y_{n-2} is in Column 4 of Table 1.

To use similar arguments as in the previous cases, we notice that $y_{n-2} = \alpha_{y_{n-3}}$.

We now consider the following two cases.

a) Suppose $\alpha_{y_{n-2}}$ is on the spine of G, y_{n-2} is a leg vertex, and $d(y_{n-2}, \alpha_{y_{n-2}}) = \frac{D+1}{2} + 1$.

To use similar arguments as those found in Case I, we want to find another pair of vertices y_i, α_{y_i} with y_i in Column 1 or 3 such that $d(y_i, \alpha_{y_i}) > \frac{D+1}{2} + t_{\alpha_{y_i}}$. Note that in this particular case, this strict inequality is not satisfied for $i = n-3$.

Claim: $d(y_{n-7}, \alpha_{y_{n-7}}) > \frac{D+1}{2} + t_{\alpha_{y_{n-7}}}$.

Proof of Claim: Notice that y_{n-7} is a vertex entered in the cell directly above y_{n-3} in Table 1. Also, y_{n-8} is the vertex in the cell directly above y_{n-2} in Table 1. By Lemma 2, $y_{n-8} = \alpha_{y_{n-7}}$.

Recall that Algorithm 1 enters vertices into Column 3 of Table 1 so that the subscripts of the vertices are non-increasing and leg vertices are inserted after spine vertices with the same subscript. Since y_{n-3} is on the spine of

G and the last entry in Column 3, it follows that the subscript for y_{n-7} is exactly one more than the subscript of y_{n-3}. This means $d(u_{c_b}, y_{n-7}) = d(u_{c_b}, y_{n-3}) + 1$.

Also, Algorithm 1 enters vertices into Column 4 of Table 1 so that the subscripts of the vertices are non-increasing and leg vertices are inserted before spine vertices with the same subscript. Since y_{n-2} is a leg vertex and is the last entry in Column 4, either

- y_{n-8} is a leg vertex with the same subscript as y_{n-2} which means that $d(y_{n-8}, u_{c_a}) = d(y_{n-2}, u_{c_a})$, or
- y_{n-8} is on the spine of G where its subscript is exactly one more than the subscript of y_{n-2} which means that $d(y_{n-8}, u_{c_a}) = d(y_{n-2}, u_{c_a}) - 1$.

Then we get the following bounds for $d(y_{n-8}, y_{n-7})$.

If y_{n-8} is a leg vertex, $t_{\alpha_{y_{n-7}}} = 1$ and

$$\begin{aligned} d(y_{n-8}, y_{n-7}) &= d(y_{n-8}, u_{c_a}) + d(u_{c_a}, u_{c_b}) + d(u_{c_b}, y_{n-7}) \\ &= d(y_{n-2}, u_{c_a}) + d(u_{c_a}, u_{c_b}) + d(u_{c_b}, y_{n-3}) + 1 \\ &= d(y_{n-2}, y_{n-3}) + 1 \\ &= \tfrac{D+1}{2} + 2 \\ &> \tfrac{D+1}{2} + t_{\alpha_{y_{n-7}}}. \end{aligned}$$

If y_{n-8} is on the spine of G, $t_{\alpha_{y_{n-7}}} = 0$ and

$$\begin{aligned} d(y_{n-8}, y_{n-7}) &= d(y_{n-8}, u_{c_a}) + d(u_{c_a}, u_{c_b}) + d(u_{c_b}, y_{n-7}) \\ &= d(y_{n-2}, u_{c_a}) - 1 + d(u_{c_a}, u_{c_b}) + d(u_{c_b}, y_{n-3}) + 1 \\ &= d(y_{n-2}, y_{n-3}) \\ &= \tfrac{D+1}{2} + 1 \\ &> \tfrac{D+1}{2} + t_{\alpha_{y_{n-7}}}. \end{aligned}$$

Therefore, $d(y_{n-7}, \alpha_{y_{n-7}}) > \frac{D+1}{2} + t_{\alpha_{y_{n-7}}}$ and the claim has been proven.

Let x_m be the same vertex as y_{n-7}. By the assumption that f does not require jumps and Proposition 9, $d(x_m, \alpha_{x_m}) \leq \frac{D+1}{2} + t_{\alpha_{x_m}}$.

If y_{n-8} is a leg vertex, $d(x_m, \alpha_{x_m}) \leq \frac{D+1}{2}+1 < \frac{D+1}{2}+2 = d(y_{n-7}, y_{\alpha_{n-7}})$. The proof now follows the proof of Case I: Subcase A:2 where $n-7 = i$ and $m = j$.

If y_{n-8} is on the spine of G, $d(x_m, \alpha_{x_m}) \leq \frac{D+1}{2} + 1 = d(y_{n-7}, y_{\alpha_{n-7}})$. When this is a strict inequality, the proof now follows the proof of Case I: Subcase A:2 with $n-7 = i$ and $m = j$. If this is an equality, the proof now follows the proof of Case I: Subcase B with $n-7 = i$ and $m = j$.

b) Suppose it is not the case that all of the following conditions are true:

- $\alpha_{y_{n-2}}$ is on the spine of G
- y_{n-2} is a leg vertex, and
- $d(y_{n-2}, \alpha_{y_{n-2}}) = \frac{D+1}{2} + 1$.

In this case, $d(y_{n-3}, \alpha_{y_{n-3}}) > \frac{D+1}{2} + t_{\alpha_{y_{n-3}}}$. Let x_m be the same vertex as y_{n-3}. By assumption, f does not require jumps and therefore $d(x_m, \alpha_{x_m}) \leq \frac{D+1}{2} + t_{\alpha_{x_m}}$. We can now use an analogous argument to that of Case I: Subcase A:1 if $d(x_m, \alpha_{x_m}) < d(y_{n-3}, \alpha_{y_{n-3}})$ by considering the original y_{n-3} as y_i in a triple of vertices. If $d(x_m, \alpha_{x_m}) = d(y_{n-3}, \alpha_{y_{n-3}})$, the proof is analogous to the proof of Case I: Subcase B. From these arguments, we conclude that f would also require jumps, which is a contradiction.

Subcase B: $d(x_j, \alpha_{x_j}) = d(y_{n-2}, \alpha_{y_{n-2}})$.

Note that the only way this can happen is when $d(x_j, \alpha_{x_j}) = \frac{D+1}{2} + 1 = d(y_{n-2}, \alpha_{y_{n-2}})$ where $\alpha_{y_{n-2}} = y_{n-3}$ is on the spine of G and α_{x_j} is a leg vertex.

1. y_{n-2} is in Column 1 of Table 1.

a) y_{n-3} is in Column 2 of Table 1.

The proof of this case is the same as that of Case I: Subcase B with $i = n - 2$.

b) y_{n-3} is in Column 3 of Table 1.

Let $\mathscr{A}$ be the set of all vertices entered into cells in Column 2 of Table 1 by Algorithm 1. Let $\mathscr{B}$ be the set of all vertices entered into cells above the cell for y_{n-2} in Column 1 of Table 1. Note that $|\mathscr{B}| = 2|\mathscr{A}| - 1$.

<u>Claim:</u> α_{x_j} is in $\mathscr{A}$.

Proof of Claim: Algorithm 1 inserts leg vertices into Column 3 after spine vertices with the same subscript. Since y_{n-3} is on the spine of G and is the last vertex entered in Column 3 of Table 1, it follows that since α_{x_j} is a leg vertex with the same subscript as y_{n-3}, it is in Column 2 of Table 1. Therefore, α_{x_j} is in $\mathscr{A}$ and the claim has been proven.

By the same argument as in the proof of Case II: Subcase A:1b, we conclude that there exists a vertex $x_k \in \mathscr{B}$ such that $\alpha_{x_k} \notin \mathscr{A}$. The same argument holds when at least one of the inequalities of (6) is strict to show that f require jumps.

<u>Claim:</u> In this case, it is not possible for both inequalities of (6) to be equal.

Proof of Claim: Suppose by contradiction that both inequalities of (6) are equalities. Algorithm 1 inserts leg vertices into Column 3 of Table 1 after spine vertices with the same subscript. The last vertex entered into Column 3 is y_{n-3} which is on the spine of G. Since y_{n-3} and α_{x_k} have the same subscript in the edge-balanced caterpillar notation, α_{x_k} is in $\mathscr{A}$ which is a contradiction and thus the claim has been proven.

2. y_{n-2} is in Column 4 of Table 1.

From Lemma 2, it follows that y_{n-3} is in Column 3. If y_{n-2} is a leg vertex, the proof is the same as the proof of Case II: Subcase A:2b. If y_{n-2} is on the spine of G, then $d(y_{n-3}, \alpha_{y_{n-3}}) > \frac{D+1}{2} + t_{\alpha_{y_{n-3}}}$. Now we reach a contradiction like in the proof of Case II: Subcase A: 2a.

In all of the above cases, we have shown that when G is not a jumpless caterpillar such that condition (ii) above is satisfied, the labeling associated with an arbitrary distance maximizing ordering requires jumps which is a contradiction to the assumption. Therefore, from Propositions 1 and 2 and the definition of a labeling requiring jumps, we have that

$$rn(G) \geq (n-1)(D+1) + 1 - \max\left(\sum_{i=1}^{n-1} d(x_i, x_{i+1})\right) + 1.$$

where the maximum is taken over all possible orderings of the vertices of G.

□

Example 1. To help visualize the idea of some of the arguments found in the proof of Theorem 2, we consider the example graph found in Figure 5.

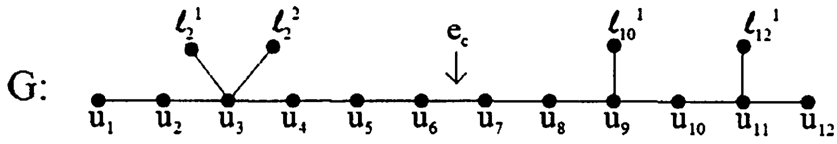

Figure 5: An edge-balanced caterpillar.

The graph G in this figure is edge-balanced. However, it can be checked that both G and H representations of this graph are not jumpless.

Table 4 shows the table given by Algorithm 1 for this graph and Figure 6 gives the graph with the ordering $y_1, \ldots, y_n$ of the vertices given by this algorithm.

Group 1				Group 2			
Column 1		Column 2		Column 3		Column 4	
u_1	2	u_7	1	u_{12}	15	u_6	16
u_2	6	u_8	7	l_{12}^1	3	u_5	4
l_2^1	8			u_{11}	5		
l_2^2	12	u_9	13	u_{10}	9	u_4	10
u_3	14			l_{10}^1	11		

Table 4: Table given by Algorithm 1 for the graph G of Figure 5.

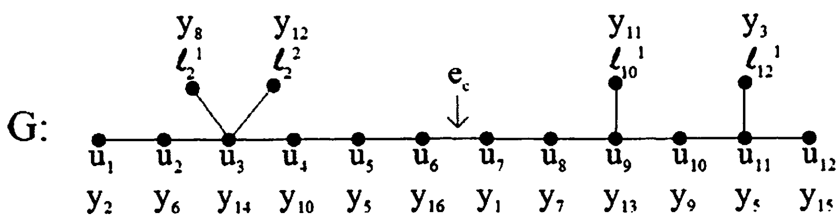

Figure 6: An edge-balanced caterpillar with the ordering given by Algorithm 1.

Notice that $D = 11$ and $y_{13} = u_9$ is on the spine of G. Thus, for the pair of vertices y_{12} and y_{13} in horizontally adjacent cells of Group 1 of Table 4, $t = 0$ and $d(y_{12}, y_{13}) = 7 > \frac{D+1}{2} + t$. This shows that G is not jumpless such that it falls into Case I of Theorem 2.

Consider the ordering $x_1, \ldots, x_n$ shown in Figure 7.

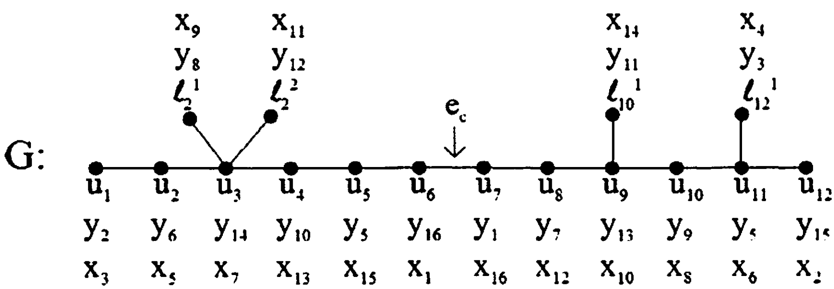

Figure 7: An edge-balanced caterpillar with ordering $y_1, \ldots, y_n$ as given by Algorithm 1 and another ordering $x_1, \ldots, x_n$.

It can be checked that $x_1, \ldots, x_n$ is a distance maximizing ordering. Notice that x_{11} is the same vertex as y_{12}. Also, $x_{12} = \alpha_{x_{11}}$ and $d(x_{11}, x_{12}) <$

$d(y_{12}, y_{13})$. Thus this graph gives an example of Case I Subcase A of the proof of Theorem 2.

From looking at Table 4, we see that $\mathscr{A} = \{u_7, u_8\}$ and $\mathscr{B} = \{u_1, u_2, l_2^1\}$. Notice that one example of the x_k vertex described in the proof is x_5. This is because x_5 is in $\mathscr{B}$, α_{x_5} is in $\mathscr{A}$ and $d(x_5, \alpha_{x_5}) = 9 > \frac{D+1}{2} + t_{\alpha_{x_5}}$.

5 Bounds for the Radio Number of Other Edge-Balanced Caterpillars

In Section 4, we determined a specific labeling that gives the radio number of edge-balanced caterpillars that are jumpless caterpillars. However, not all edge-balanced caterpillars are jumpless caterpillars. In Section 5.1, we establish some definitions and propositions to help improve the lower bound of the radio number of some other edge-balanced caterpillars. Then, in Section 5.2, we determine an improved lower bound for the radio number of the caterpillars discussed in Section 5.1. Finally, Section 5.3 gives conclusions from the results of these sections.

5.1 Preliminaries

To help improve the lower bounds of some edge-balanced caterpillars, we have the following definition:

Definition. In an edge-balanced caterpillar G with n vertices and diameter D, a vertex v_* is a *problem vertex* if one of the following conditions hold:

(i) $v_* \in A$ and $d(v_*, u_{c_b}) \geq \frac{D+2}{2}$ or

(ii) $v_* \in B$ and $d(v_*, u_{c_a}) \geq \frac{D+2}{2}$.

As some of the following results will rely on characteristics of caterpillars based on where legs are located on the caterpillar, we use the following notation for the rest of the paper.

Notation. Let G be an edge-balanced caterpillar. Let a be the number of legs in component A and let b be the number of legs in component B.

Remark 5. *For an edge-balanced caterpillar, $|V(A)| = |V(B)|$. Without loss of generality, let $a \geq b$.*

The next results are useful in categorizing edge-balanced caterpillar graphs based on the the relationship of the values for a and b. This helps to determine which types of edge-balanced caterpillars have an improved lower bound due to a problem vertex.

Proposition 10. *Let G be an edge-balanced caterpillar. If $a > b$, then there exists at least one problem vertex.*

Proof. Since G is an edge-balanced caterpillar, G has one center edge and $N(e_c)$ is odd. Then there are $\frac{N(e_c)+1}{2} =: w$ vertices in A and w vertices in B. This means there are $w - a$ vertices on the spine of G in A and $w - b$ vertices on the spine of G in B. Thus the number of vertices on the spine of G is $w - a + w - b$ and therefore $D = 2w - a - b - 1$. Note that there exists a vertex $u \in B$ such that $d(u_{c_a}, u) = w - b$.

Consider

$$\begin{aligned} \frac{D+2}{2} &= \frac{2w-a-b+1}{2} \\ &< \frac{2w-b-b+1}{2} \\ &= \frac{2w-2b+1}{2} \\ &= w-b+\frac{1}{2}. \end{aligned} \tag{7}$$

Case I: D is odd.

Since D is odd, $D = 2k + 1$ for some integer k. Then (7) becomes $k+1+\frac{1}{2} < w-b+\frac{1}{2}$. Since $k+1 < w-b$ and both of those quantities are integers, it follows that $k+1+\frac{1}{2} < w-b$. Therefore, $\frac{D+2}{2} < w-b = d(u_{c_a}, u)$ and u is a problem vertex.

Case II: D is even.

Since D is even, $D = 2k$ for some integer k. Then (7) becomes $k + 1 < w-b+\frac{1}{2}$. Since $k+1$ and $w-b$ are both integers, it follows that $k+1 \leq w-b$. This means $\frac{D+2}{2} \leq w - b = d(u_{c_a}, u)$ and thus u is a problem vertex. □

Proposition 11. *Let G be an edge-balanced caterpillar with n vertices. If D is even, then $a \neq b$.*

Proof. Recall that G has one center edge with $N(e_c)$ odd and so by Proposition 6 n is even.

Suppose by contradiction that $a = b$. By Proposition 6, $|V(A)| = |V(B)|$. Let $w := \frac{N(e_c)+1}{2} = |V(A)| = |V(B)|$. There are $w - a$ vertices on the spine in A and $w-b$ vertices on the spine in B. So, there are $w-a+w-b$ vertices on the spine of G. Thus, $D = w - a + w - b - 1 = 2w - 2a - 1$ which is odd, a contradiction to the assumption. Therefore, when D is even, $a \neq b$. □

5.2 Improved Bounds

We now use results from Section 5.1 to improve the lower bound for the radio number of certain edge-balanced caterpillars.

Proposition 12. *Let G be an edge-balanced caterpillar with a problem vertex v_*. Then G is not a jumpless caterpillar.*

Proof. Without loss of generality, assume $v_* \in B$. Note that u_{c_b} cannot be a problem vertex so $u_{c_b} \neq v_*$.

Use Algorithm 1 to place the vertices of G into Table 1. From Proposition 8, the corresponding ordering $y_1, \ldots, y_n$ is distance maximizing. Since this is a distance maximizing ordering, by Propositions 5 and 7, e_c is the only edge with $n_y(e)$ odd. By Corollary 1, $\{u_{c_a}, u_{c_b}\} = \{y_1, y_n\}$. Since u_{c_b} is not a problem vertex, v_* is not the first or last labeled vertex. Thus, $v_* = y_i$ for some triple of vertices y_{i-1}, y_i, y_{i+1}.

By definition of being a problem vertex, $d(v_*, u_{c_a}) \geq \frac{D+2}{2}$. Also, by the structure of an edge-balanced caterpillar, $d(u_{c_a}, v_*) \leq d(u_{c_a}, u_s)$. Therefore,

$$d(u_{c_a}, u_s) \geq d(u_{c_a}, v_*) \geq \frac{D+2}{2} > \frac{D+1}{2}$$

$$\Rightarrow d(u_{c_a}, u_s) > \frac{D+1}{2}. \tag{8}$$

The ordering of vertices of G given by Algorithm 1 has $y_{n-1} = u_s$ and $y_n = u_{c_a}$. Also, y_{n-1} is entered into Column 3 and y_n is entered into Column 4 of Table 1 such that y_{n-1} and y_n are in horizontally adjacent cells. Thus, by Lemma 2, $y_n = \alpha_{y_{n-1}}$. Since $u_{c_a} = \alpha_{y_{n-1}}$ is on the spine of G, $t_{\alpha_{y_{n-1}}} = 0$. By (8), $d(y_{n-1}, \alpha_{y_{n-1}}) > \frac{D+1}{2} = \frac{D+1}{2} + t_{\alpha_{y_{n-1}}}$. This contradicts condition (1) of the definition of a jumpless caterpillar. Therefore, G is not a jumpless caterpillar. □

Corollary 4. *Let G be an edge-balanced caterpillar with n vertices. Suppose G is such that $a \neq b$. Then*

$$rn(G) \geq (n-1)(D+1) + 1 - \max\left(\sum_{i=1}^{n-1} d(x_i, x_{i+1})\right) + 1$$

where the maximum is taken over all possible orderings of the vertices of G.

Proof. As in Remark 5, we assume without loss of generality that $a > b$. From Proposition 10, G has a problem vertex. So, by Proposition 12, G is not a jumpless caterpillar. Therefore, the bound follows from Theorem 2. □

5.3 Conclusions about Edge-Balanced Caterpillars

Corollary 4 establishes a way to determine when the bound for the radio number given by Proposition 2 is increased for an edge-balanced caterpillar G based on the structure of G.

The results of Corollary 4 and Proposition 11 indicate that edge-balanced caterpillars with the potential to have radio labelings that require no jumps are such that D is odd and $a = b$.

When there is exactly one leg adjacent to each vertex on the spine expect for u_1 and u_s, this is a thorn graph. The radio number of this particular thorn graph has been determined in [10].

In other cases when D is odd and $a = b$, one can enter the vertices of G into Table 1 using Algorithm 1 to determine if G is a jumpless caterpillar. If it is, then the span of the labeling associated with the ordering given by Algorithm 1 is the radio number of G.

Appendix

Theorem 2 in Section 4 improved the lower bound for the radio number of edge-balanced caterpillars that are not jumpless caterpillars. In some cases, the proof assumed $n \geq 8$. Recall that for an edge-balanced caterpillar, n is even. Thus, we only need to check for edge-balanced caterpillar graphs for $n = 2, 4$, and 6. The following graphs in Figure 8 show all the edge-balanced caterpillars such that $n < 8$. Most of these are jumpless caterpillars and thus would not be considered in Theorem 2. In all the cases shown below, whether the caterpillar is jumpless or not, the radio number of these graphs is known either from previous results or from work in this paper.

The graph (a) in Figure 8 is the path P_2. This is a complete graph whose radio number is known: $rn(P_2) = 2$. The graphs (b) and (c) are paths P_4 and P_6. The radio numbers for these paths were determined in [9]: $rn(P_4) = 6$ and $rn(P_6) = 14$. The graphs (d) and (e) are spire graphs, $S_{6,2}$ and $S_{6,4}$. The radio number of $S_{6,2}$ was determined in [1]: $rn(S_{6,2}) = 12$. The spire $S_{6,4}$ can be redrawn as $S_{6,2}$. Thus, $rn(S_{6,4}) = 12$. Finally, it can be checked that the graph (f) of Figure 8 is a jumpless caterpillar. Thus, using Algorithm 1 from Section 4, the radio number of graph (f) is 8.

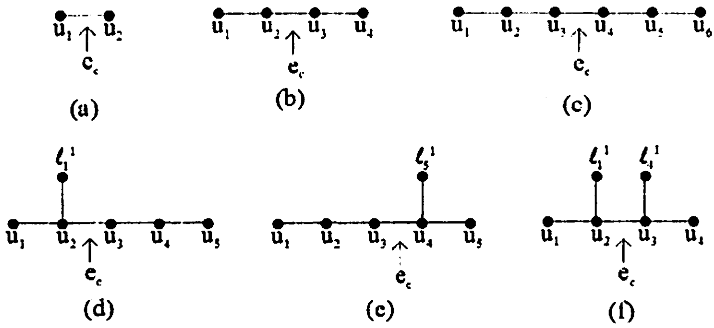

Figure 8: Edge-Balanced Caterpillars with $n < 8$.

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Katherine F. Benson
501 Westminster Ave, Westminster College, Fulton, MO 65251
katie.benson@westminster-mo.edu