Designs from maximal subgroups and conjugacy classes of finite simple groups

J. D. Key
Department of Mathematics and Applied Mathematics
University of the Western Cape
7535 Bellville, South Africa
J. Moori*
School of Mathematical Sciences
North-West University (Mafikeng)
Mmabatho 2735, South Africa
keyj@clemson.edu, Jamshid.Moori@nwu.ac.za

Abstract

Let G be a finite simple group, M be a maximal subgroup of G and $C_g = nX$ be the conjugacy class of G containing g. In this paper we discuss a new method for constructing 1-(v, k, λ) designs $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, where $\mathcal{P} = nX$ and $\mathcal{B} = \{(M \cap nX)^y | y \in G\}$. The parameters v, k, λ and further properties of $\mathcal{D}$ are determined. We also study codes associated with these designs.

1 Introduction

Symmetric 1-designs and binary codes obtained from the primitive permutation representations (from the action on the maximal subgroups) of the sporadic simple groups have been examined in [11], [14] and [15]. In this paper we introduce a new method from which a large number of non-symmetric 1-designs could be constructed. Let G be a finite simple group, M be a maximal subgroup of G and $C_g = [g] = nX$ be the conjugacy

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class of G containing g. We construct 1-(v, k, λ) designs $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, where $\mathcal{P} = nX$ and $\mathcal{B} = \{(M \cap nX)^y | y \in G\}$. The parameters v, k, λ and further properties of $\mathcal{D}$ are determined. We also study codes associated with these designs. In Sections 5, 6, and 7 we apply our method to the groups A_7, $PSL_2(q)$, and the Janko group J_1, respectively.

2 Terminology and notation

Our notation will be standard, and it is as in [2] for designs and ATLAS [5] for groups. For the structure of groups and their maximal subgroups we follow the ATLAS notation. Computations have been done with Magma [3, 4].

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is a t-(v, k, λ) design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. The **complement** of $\mathcal{D}$ is the structure $\tilde{\mathcal{D}} = (\mathcal{P}, \mathcal{B}, \tilde{\mathcal{I}})$, where $\tilde{\mathcal{I}} = \mathcal{P} \times \mathcal{B} - \mathcal{I}$. The **dual** structure of $\mathcal{D}$ is $\mathcal{D}^t = (\mathcal{B}, \mathcal{P}, \mathcal{I}^t)$, where $(B, P) \in \mathcal{I}^t$ if and only if $(P, B) \in \mathcal{I}$. Thus the transpose of an incidence matrix for $\mathcal{D}$ is an incidence matrix for $\mathcal{D}^t$. We will say that the design is **symmetric** if it has the same number of points and blocks, and **self dual** if it is isomorphic to its dual. A t-(v, k, λ) design is called **self-orthogonal** if the block intersection numbers have the same parity as the block size.

The code C_F of the design $\mathcal{D}$ over the finite field F is the space spanned by the incidence vectors of the blocks over F. We take F to be a prime field F_p, in which case we write also C_p for C_F, and refer to the dimension of C_p as the **p-rank** of $\mathcal{D}$. If the point set of $\mathcal{D}$ is denoted by $\mathcal{P}$ and the block set by $\mathcal{B}$, and if $\mathcal{Q}$ is any subset of $\mathcal{P}$, then we will denote the incidence vector of $\mathcal{Q}$ by $v^{\mathcal{Q}}$. Thus $C_F = \langle v^B \,|\, B \in \mathcal{B} \rangle$, and is a subspace of $F^{\mathcal{P}}$, the full vector space of functions from $\mathcal{P}$ to F. For any code C, the **dual** code $C^{\perp}$ is the orthogonal subspace under the standard inner product. The **hull** of a design's code over some field is the intersection $C \cap C^{\perp}$. If a linear code over the finite field F of order q is of length n, dimension k, and minimum weight d, then we write $[n, k, d]_q$ to represent this information. If c is a codeword then the **support** of c, Supp(c), is the set of non-zero coordinate positions of c and the **weight** of c, written wt(c) to be the size of the support, $|\text{Supp}(c)|$. A **constant word** in the code is a codeword all of whose coordinate entries are either 0 or 1. The all-one vector will be denoted by $\jmath$, and is the constant vector of weight the length of the code. Two linear codes of the same length and over the same field are **equivalent** if each can be obtained from the other by permuting the coordinate positions and multiplying each coordinate position by a non-zero field element. They are **isomorphic** if they can be obtained from one another by permuting the coordinate positions. An **automorphism** of a code is any permutation of the coordinate positions that maps codewords to codewords. An automorphism thus preserves each weight class of C. A binary code with all weights divisible by 4 is said to be a **doubly-even** binary code.

If G is a group and M is a G-module, the **socle** of M, written $\mathrm{Soc}(M)$, is the largest semi-simple G-submodule of M. It is the direct sum of all the irreducible G-submodules of M. In this paper we determine $\mathrm{Soc}(V)$ for each of the relevant full-space G-modules $V = F^n$.

3 Group actions and permutation characters

Suppose that G is a finite group acting on a finite set Ω. For $\alpha \in \Omega$, the *stabilizer* of α in G is given by

$$G_\alpha = \{g \in G | \alpha^g = \alpha\}.$$

Then $G_\alpha \leq G$ and $[G : G_\alpha] = |\Delta|$, where Δ is the orbit containing α.

The action of G on Ω gives a permutation representation π with corresponding permutation character χ_π denoted by $\chi(G|\Omega)$. Then from elementary representation theory we deduce that

Lemma 1 (*i*) *The action of G on Ω is isomorphic to the action of G on G/G_α, that is on the set of all left cosets of G_α in G. Hence $\chi(G|\Omega) = \chi(G|G_\alpha)$.*

(*ii*) *$\chi(G|\Omega) = (I_{G_\alpha})^G$, the trivial character of G_α induced to G.*

(*iii*) *For all $g \in G$, we have $\chi(G|\Omega)(g)$ = number of points in Ω fixed by g.*

Proof: For example see Isaacs [10] or Ali [1]. ■

In fact for any subgroup $H \leq G$ we have

$$\chi(G|H)(g) = \sum_{i=1}^{k} \frac{|C_G(g)|}{|C_H(h_i)|},$$

where $h_1, h_2, ..., h_k$ are representatives of the conjugacy classes of H that fuse to $[g] = C_g$ in G.

Lemma 2 *Let H be a subgroup of G and let Ω be the set of all conjugates of H in G. Then we have:*

(i) $G_H = N_G(H)$ *and* $\chi(G|\Omega) = \chi(G|N_G(H)$.

(ii) For any g in G, the number of conjugates of H in G containing g is given by

$$\chi(G|\Omega)(g) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|} = [N_G(H) : H]^{-1} \sum_{i=1}^{k} \frac{|C_G(g)|}{|C_H(h_i)|},$$

where the x_i's and h_i's are representatives of the conjugacy classes of $N_G(H)$ and H that fuse to $[g] = C_g$ in G, respectively.

Proof:

(i)
$$G_H = \{x \in G | H^x = H\} = \{x \in G | x \in N_G(H)\} = N_G(H).$$
Now the results follows from Lemma 1 (i).

(ii) The proof follows from (i) and Corollary 3.1.3 of Ganief [8] which uses a result of Finkelstein [6]. ■

Remark 1 *Note that*

$$\chi(G|\Omega)(g) = |\{H^x : (H^x)^g = H^x\}| = |\{H^x | H^{x^{-1}gx} = H\} =$$

$$|\{H^x | x^{-1}gx \in N_G(H)\}| = |\{H^x | g \in xN_G(H)x^{-1}\}| = |\{H^x | g \in N_G(H)^x\}|.$$

Corollary 3 *If G is a finite simple group and M is a maximal subgroup of G, then the number λ of conjugates of M in G containing g is given by*

$$\chi(G|M)(g) = \sum_{i=1}^{k} \frac{|C_G(g)|}{|C_M(x_i)|},$$

where $x_1, x_2, ..., x_k$ are representatives of the conjugacy classes of M that fuse to the class $[g] = C_g$ in G.

Proof: It follows from Lemma 2 and the fact that $N_G(M) = M$. It is also a direct of application of Remark 1, since

$$\chi(G|\Omega)(g) = |\{M^x | g \in (N_G(M))^x\}| = |\{M^x | g \in M^x\}|. \blacksquare$$

4 Construction of 1-designs

In this section we assume G is a finite simple group, M a maximal subgroup of G, nX a conjugacy class of elements of order n in G and $g \in nX$. Thus $C_g = [g] = nX$ and $|nX| = |G : C_G(g)|$.

As in Section 3 let $\chi_M = \chi(G|M)$ be the permutation character afforded by the action of G on Ω, the set of all conjugates of M in G. Clearly if g is not conjugate to any element in M, then $\chi_M(g) = 0$.

The construction of our 1-designs is based on the following theorem.

Theorem 4 *Let G be a finite simple group, M a maximal subgroup of G and nX a conjugacy class of elements of order n in G such that $M \cap nX \neq \emptyset$. Let $\mathcal{B} = \{(M \cap nX)^y | y \in G\}$ and $\mathcal{P} = nX$. Then we have a 1-$(|nX|, |M \cap nX|, \chi_M(g))$ design $\mathcal{D}$, where $g \in nX$. The group G acts as an automorphism group on $\mathcal{D}$, primitive on blocks and transitive (not necessarily primitive) on points of $\mathcal{D}$.*

Proof: First note that

$$\mathcal{B} = \{M^y \cap nX | y \in G\}.$$

We claim that $M^y \cap nX = M \cap nX$ if and only if $y \in M$ or $nX = \{1_G\}$. Clearly if $y \in M$ or $nX = \{1_G\}$, then $M^y \cap nX = M \cap nX$. Conversely suppose there exists $y \notin M$ such that $M^y \cap nX = M \cap nX$. Then maximality of M in G implies that $G =< M, y >$ and hence $M^z \cap nX = M \cap nX$ for all $z \in G$. We can deduce that $nX \subseteq M$ and hence $< nX > \leq M$. Since $< nX >$ is a normal subgroup of G and G is simple, we must have $< nX >= \{1_G\}$. Note that maximality of M and the fact that $< nX > \leq M$, excludes the case $< nX >= G$.

From the above we deduce that

$$b = |\mathcal{B}| = |\Omega| = [G : M].$$

If $B \in \mathcal{B}$, then

$$k = |B| = |M \cap nX| = \sum_{i=1}^{k} |[x_i]_M| = |M| \sum_{i=1}^{k} \frac{1}{|C_M(x_i)|},$$

where $x_1, x_2, ..., x_k$ are the representatives of the conjugacy classes of M that fuse to g.

Let $v = |\mathcal{P}| = |nX| = [G : C_G(g)]$. Form the design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ given by $x\mathcal{I}B$ if and only if $x \in B$. Since the number of blocks containing an element x in $\mathcal{P}$ is $\lambda = \chi_M(x) = \chi_M(g)$, we have produced a 1-$(v, k, \lambda)$ design $\mathcal{D}$, where $v = |nX|$, $k = |M \cap nX|$ and $\lambda = \chi_m(g)$.

The action of G on blocks arises from the action of G on Ω and hence the maximality of M in G implies the primitivity. The action of G on nX, that is on points, is equivalent to the action of G on the cosets of $C_G(g)$. So the action on points is primitive if and only if $C_G(g)$ is a maximal subgroup of G. ■

Remark 2 *Since in a* 1-(v, k, λ) *design* $\mathcal{D}$ *we have* $kb = \lambda v$, *we deduce that*

$$k = |M \cap nX| = \frac{\chi_M(g) \times |nX|}{[G : M]}.$$

Also note that $\tilde{\mathcal{D}}$, *the complement of* $\mathcal{D}$, *is a* 1-$(v, v - k, \tilde{\lambda})$ *design, where* $\tilde{\lambda} = \lambda \times \frac{v-k}{k}$.

Remark 3 *If* $\lambda = 1$, *then* $\mathcal{D}$ *is a* 1-$(|nX|, k, 1)$ *design. Since* nX *is the disjoint union of* b *blocks each of size* k, *we have* $\mathrm{Aut}(\mathcal{D}) = S_k \wr S_b = (S_k)^b : S_b$. *Clearly In this case for all* p, *we have* $C = C_p(\mathcal{D})$ *is* $[|nX|, b, k]_p$, *with* $\mathrm{Aut}(C) = \mathrm{Aut}(\mathcal{D})$.

Remark 4 *The designs* $\mathcal{D}$ *constructed by using Theorem* 4 *are not symmetric in general. In fact* $\mathcal{D}$ *is symmetric if and only if* $b = |\mathcal{B}| = v = |\mathcal{P}| \Leftrightarrow [G : M] = |nX| \Leftrightarrow [G : M] = [G : C_G(g)] \Leftrightarrow |M| = |C_G(g)|$.

5 Some 1-designs and codes from A_7

A_7 has five conjugacy classes of maximal subgroups, which are listed in Table 1. It has also nine conjugacy classes of elements, some of which are listed in Table 2.

We apply the Theorem 4 to the above maximal subgroups and a few conjugacy classes of elements of A_7 to construct several non-symmetric 1-designs. The corresponding binary codes are also constructed.

5.1 $G = A_7$, $M = A_6$, $nX = 3A$

5.1.1 1-(70, 40, 4) design

Let $G = A_7$, $M = A_6$ and $nX = 3A$. Then

$$b = [G : M] = 7, v = |3A| = 70, k = |M \cap 3A| = 40.$$

Table 1: Maximal subgroups of A_7

No.	Structure	Index	Order
Max[1]	A_6	7	360
Max[2]	$PSL_2(7)$	15	168
Max[3]	$PSL_2(7)$	15	168
Max[4]	S_5	21	120
Max[5]	$(A_4 \times 3){:}2$	35	72

Table 2: Some of the conjugacy classes of A_7

nX	$\|nX\|$	$C_G(g)$	Maximal Centralizer
$2A$	105	$D_8{:}\ 3$	No
$3A$	70	$A_4 \times 3 \cong (2^2 \times 3){:}\ 3$	No
$3B$	280	3×3	No

Also using the character table of A_7, we have $\chi_M = \chi_1 + \chi_2 = \underline{1a} + \underline{6a}$ and hence $\chi_M(g) = 1 + 3 = 4 = \lambda$, where $g \in 3A$. We produce a non-symmetric 1-$(70, 40, 4)$ design $\mathcal{D}$. A_7 acts primitively on the 7 blocks. Since $C_{A_7}(g) = A_4 \times 3$ is not maximal in A_7 (it sits in the maximal subgroup $(A_4 \times 3){:}2$ with index 2), A_7 acts imprimitvely on the 70 points. The complement of $\mathcal{D}$, $\bar{\mathcal{D}}$, is a 1-$(70, 30, 3)$ design.

Computations with Magma [3, 4] shows that the full automorphism group of $\mathcal{D}$ is

$$\mathrm{Aut}(\mathcal{D}) \cong 2^{35}{:}S_7 \cong 2^5 \wr S_7,$$

with $|\mathrm{Aut}(\mathcal{D})| = 2^{39}.3^2.5.7$.

5.1.2 Codes associated with the 1-(70, 40, 4) design

We used Magma to show that the binary code C of this design is a $[70, 6, 32]_2$ code. The code C is self-orthogonal with the weight distribution

$$< 0, 1 >, < 32, 35 >, < 40, 28 > .$$

Our group A_7 acts irreducibly on C.

If W_i denotes the set of all words in C of weight i, then

$$C =< W_{32} >=< W_{40} >,$$

Table 3: Stabilizer of a word w_l in $\mathrm{Aut}(\mathcal{D})$

l	$\|W_l\|$	$\mathrm{Aut}(\mathcal{D})_{w_l}$
32	35	$2^{35}{:}(A_4 \times 3){:}2$
40(1)	7	$2^{35}{:}S_6$
40(2)	21	$2^{35}{:}(S_5{:}2)$

Table 4: Stabilizer of a word w_l in $\mathrm{Aut}(C)$

l	$\|W_l\|$	$\mathrm{Aut}(\mathcal{D})_{w_l}$
32	35	$2^{35}{:}(S_4 \times S_4){:}2$
40	28	$2^{35}{:}(S_6 \times 2)$

so C is generated by its minimum-weight codewords. The full automorphism group of C is $\mathrm{Aut}(C) \cong 2^{35}{:}S_8$ with $|\mathrm{Aut}(C)| = 2^{42}.3^2.5.7$, and we note that $\mathrm{Aut}(C) \geq \mathrm{Aut}(\mathcal{D})$ and that $\mathrm{Aut}(\mathcal{D})$ is not a normal subgroup of $\mathrm{Aut}(C)$.

Furthermore $C^\perp$ is a $[70, 64, 2]_2$ code and its weight distribution has been computed. Since the blocks of $\mathcal{D}$ are of even size 40, we have that $\jmath$ meets evenly every vector of C and hence $\jmath \in C^\perp$. If $\bar{W}_i$ denotes the set of all codewords in $C^\perp$ of weight i, then $|\bar{W}_2| = 35$,, $|\bar{W}_3| = 840$, $|\bar{W}_4| = 14035$ and

$$C^\perp = < \bar{W}_3 >, \dim(< \bar{W}_2 >= 35, \dim(< \bar{W}_4 >= 63.$$

Let e_{ij} denote the 2-cycle (i, j) in S_7, where $\{i, j\} = \mathrm{Supp}(\bar{w}_2)$ is the support of a codeword $\bar{w}_2 \in \bar{W}_2$. Then $e_{ij}(\bar{w}_2) = \bar{w}_2$, and $< e_{ij} | \{i, j\} = \mathrm{Supp}(\bar{w}_2), \bar{w}_2 \in \bar{W}_2 >= 2^{35}$.

Using Magma we can show that $V = F_2^{70}$ is decomposable into indecomposable G-modules of dimension 40 and 30. We also have $\dim(\mathrm{Soc}(V)) = 21$ and

$$\mathrm{Soc}(V) = < \jmath > \oplus C \oplus C_{14},$$

where C is our 6-dimensional code and C_{14} is an irreducible code of dimension 14.

The structures of the stabilizers of $\mathrm{Aut}(\mathcal{D})_{w_l}$ and $\mathrm{Aut}(C)_{w_l}$, where $l \in \{32, 40\}$, are listed in Tables 3 and 4.

5.2 $G = A_7$, $M = A_6$, $nX = 2A$

5.2.1 1-(105, 45, 3) design

Let $G = A_7$, $M = A_6$ and $nX = 2A$. Then

$$b = [G : M] = 7, v = |2A| = 105, k = |M \cap 2A| = 45.$$

Also using the character table of A_7, we have $\chi_M = \chi_1 + \chi_2 = \underline{1a} + \underline{6a}$ and hence $\chi_M(g) = 1 + 2 = 3 = \lambda$, where $g \in 2A$. We produce a non-symmetric 1-(105, 45, 3) design $\mathcal{D}$. A_7 acts primitively on the seven blocks. Since $C_{A_7}(g) = D_8 : 3$ is not maximal in A_7 (it sits in the maximal subgroup $(A_4 \times 3){:}2$ with index 3), A_7 acts imprimitively on the 105 points. The complement of $\mathcal{D}$, $\tilde{\mathcal{D}}$, is a 1-(105, 60, 4) design.

The full automorphism group of $\mathcal{D}$ is

$$\text{Aut}(\mathcal{D}) \cong {S_3}^{35}{:}S_7 \cong {S_3}^5 \wr S_7,$$

with $|\text{Aut}(\mathcal{D})| = 2^{42}.3^{37}.5.7$.

5.2.2 Codes associated with the 1-(105, 45, 3) design

Magma shows that the binary code C of this design is a $[105, 7, 45]_2$ code. The weight distribution of C is

$$< 0, 1 >, < 45, 28 >, < 48, 35 >, < 57, 35 >, < 60, 28 >, < 105, 1 > .$$

We also have that Hull(C) is a [105, 6, 48] code and has the following weight distribution:

$$< 0, 1 >, < 48, 35 >, < 60, 28 > .$$

Note that $C = \text{Hull}(C) \oplus < \jmath >$, and that our group A_7 acts irreducibly on Hull(C). Also note that this result together with the result obtained in Section 5.2.1 imply that the 6-dimensional irreducible representation of A_7 over $GF(2)$ could be represented by two non-isomorphic codes, namely $[105, 6, 48]_2$ and $[70, 6, 32]_2$ codes.

We also have

$$C =< W_{45} >=< W_{57} >,$$

so C is generated by its minimum-weight codewords. The full automorphism group of C is $\text{Aut}(C) = \text{Aut}(\mathcal{D})$ and its structure was given above in Section 5.2.1.

Using Magma we can easily show that $V = F_2^{105}$ is decomposable into indecomposable G-modules of dimension 1, 14, 20 and 70 (the first three are irreducible). We also have dim(Soc(V) = 55 and that

$$\mathrm{Soc}(V) = < \jmath > \oplus C_{14} \oplus C_{14} \oplus C_{20} \oplus \mathrm{Hull}(C),$$

where $C = \mathrm{Hull}(C) \oplus < \jmath >$ is our 7-dimensional code and C_{14} and C_{20} are irreducible codes of dimension 14 and 20 respectively.

5.3 $G = A_7$, $M = S_5$, $nX = 2A$: 1-(105, 25, 5) design

Let $G = A_7$, $M = S_5$ and $nX = 2A$. Then

$$b = [G : M] = 21, v = |2A| = 105, k = |M \cap 2A| = 25.$$

Note that both conjugacy classes of involutions of S_5 fuse to $2A$. Also using the character table of A_7, we have $\chi_M = \chi_1 + \chi_2 + \chi_5 = \underline{1a} + \underline{6a} + \underline{14a}$ and hence $\chi_M(g) = 1 + 2 + 2 = 5 = \lambda$, where $g \in 2A$. We produce a non-symmetric 1-(105, 25, 5) design $\mathcal{D}$. A_7 acts primitively on the 21 blocks. Since $C_{A_7}(g) = D_8{:}3$ is not maximal in A_7 (it sits in the maximal subgroup $(A_4 \times 3){:}2$ with index 3), A_7 acts imprimitively on the 105 points. The complement of $\mathcal{D}$, $\tilde{\mathcal{D}}$, is a 1-(105, 80, 16) design.

5.4 $G = A_7$, $M = PSL_2(7)$, $nX = 2A$: 1-(105, 21, 3) design

Let $G = A_7$, $M = PSL_2(7)$ and $nX = 2A$. Then

$$b = [G : M] = 15, v = |2A| = 105, k = |M \cap 2A| = 21.$$

Also using the character table of A_7, we have $\chi_M = \chi_1 + \chi_6 = \underline{1a} + \underline{14b}$ and hence $\chi_M(g) = 1 + 2 = 3 = \lambda$, where $g \in 2A$. We produce a non-symmetric 1-(105, 21, 3) design $\mathcal{D}$. A_7 acts primitively on the 15 blocks. Since $C_{A_7}(g) = D_8 : 3$ is not maximal in A_7 (it sits in the maximal subgroup $(A_4 \times 3){:}2$ with index 3), A_7 acts imprimitively on the 105 points. The complement of $\mathcal{D}$, $\tilde{\mathcal{D}}$, is a 1-(105, 84, 12) design.

5.5 $G = A_7$, $M = PSL_2(7)$, $nX = 3B$: 1-(280, 56, 3) design

Let $G = A_7$, $M = PSL_2(7)$ and $nX = 3B$. Then

$$b = [G : M] = 15, v = |3B| = 280, k = |M \cap 2A| = 56.$$

Also using the character table of A_7, we have $\chi_M = \chi_1 + \chi_6 = \underline{1a} + \underline{14b}$ and hence $\chi_M(g) = 1 + 2 = 3 = \lambda$, where $g \in 3B$. We produce a non-symmetric 1-$(280, 56, 3)$ design $\mathcal{D}$. A_7 acts primitively on the 15 blocks. Since $C_{A_7}(g) = 3 \times 3 \in \mathrm{Syl}_3(A_7)$ is not maximal in A_7 (it sits in the maximal subgroups A_6 and $(A_4 \times 3){:}2$ with indices 40 and 8 respectively), A_7 acts imprimitively on the 280 points. The complement of $\mathcal{D}$, $\tilde{\mathcal{D}}$, is a 1-$(280, 224, 12)$ design.

6 Design and codes from $PSL_2(q)$

The main aim of this section to develop a general approach to $G = PSL_2(q)$, where M is the maximal subgroup that is the stabilizer of a point in the natural action of degree $q + 1$ on the set Ω. This is fully discussed in Section 6.1. But we start this section by applying the results discussed in previous sections, particularly Theorem 4, to all maximal subgroups and conjugacy classes of elements of $PSL_2(11)$ to construct 1- designs and their corresponding binary codes. These are itemized below after Tables 5 and 6. The group $PSL_2(11)$ has order $660 = 2^2 \times 3 \times 5 \times 11$, it has four conjugacy classes of maximal subgroups, which are listed in the Table 5. It has also eight conjugacy classes of elements which we list in Table 6.

Table 5: Maximal subgroups of $PSL_2(11)$

No.	Order	Index	Structure
Max[1]	55	12	$F_{55} = 11:5$
Max[2]	60	11	A_5
Max[3]	60	11	A_5
Max[4]	12	55	D_{12}

Table 6: Conjugacy classes of $PSL_2(11)$

nX	$\lvert nX\rvert$	$C_G(g)$	Maximal Centralizer
$2A$	55	D_{12}	Yes
$3A$	110	$\mathbb{Z}_6$	No
$5A$	132	$\mathbb{Z}_5$	No
$5B$	132	$\mathbb{Z}_5$	No
$6A$	110	$\mathbb{Z}_6$	No
$11A$	60	$\mathbb{Z}_{11}$	No
$11B$	60	$\mathbb{Z}_{11}$	No

Max[1]

5A: $\mathcal{D}$ is 1-$(132, 22, 2)$, $b = 12$; C is $[132, 11, 22]_2$, $C^\perp$ is $[132, 121, 2]_2$; $\text{Aut}(\mathcal{D}) = \text{Aut}(C) = 2^{66} : S_{12}$.

5B: As for 5A.

11A: $\mathcal{D}$ is 1-$(60, 5, 1)$, $b = 12$; C is $[60, 12, 5]_2$, $C^\perp$ is $[60, 48, 2]_2$; $\text{Aut}(\mathcal{D}) = \text{Aut}(C) = (S_5)^{12} : S_{12}$.

11B: As for 11A.

Max[2]

2A : $\mathcal{D}$ is 1-$(55, 15, 3)$, $b = 11$; C is $[55, 11, 15]_2$, $C^\perp$ is $[55, 44, 4]_2$; $\text{Aut}(\mathcal{D}) = PSL_2(11)$, $\text{Aut}(C) = PSL_2(11) : 2$.

3A : $\mathcal{D}$ is 1-$(110, 20, 2)$, $b = 11$; C is $[110, 10, 20]_2$, $C^\perp$ is $[110, 100, 2]_2$; $\text{Aut}(\mathcal{D}) = \text{Aut}(C) = 2^{55} : S_{11}$.

5A : $\mathcal{D}$ is 1-$(132, 12, 1)$, $b = 11$; C is $[132, 11, 12]_2$, $C^\perp$ is $[132, 121, 2]_2$; $\text{Aut}(\mathcal{D}) = \text{Aut}(C) = (S_{12})^{11} : S_{11}$.

5B : As for 5A.

Max[3]

As for Max[2].

Max[4]

2A : $\mathcal{D}$ is 1-$(55, 7, 7)$, $b = 55$; C is $[55, 35, 4]_2$, $C^\perp$ is $[55, 20, 10]_2$; $\text{Aut}(\mathcal{D}) = \text{Aut}(C) = PSL_2(11) : 2$.

3A : $\mathcal{D}$ is 1-$(110, 2, 1)$, $b = 55$; C is $[110, 55, 2]_2$, $C^\perp$ is $[110, 55, 2]_2$; $\text{Aut}(\mathcal{D}) = \text{Aut}(C) = 2^{55} : S_{55}$.

6A : As for 3A.

6.1 $G = PSL_2(q)$ of degree $q + 1$, $M = G_1$

Let $G = PSL_2(q)$, and let M be the stabilizer of a point in the natural action of degree $q + 1$ on the set Ω. Let $M = G_1$. Then it is well known that G acts sharply 2-transitive on Ω and $M = \mathbb{F}_q : \mathbb{F}_q^* = \mathbb{F}_q : \mathbb{Z}_{q-1}$, if q is even, and $M = \mathbb{F}_q : \mathbb{Z}_{\frac{q-1}{2}}$, if q is odd. Since G acts 2-transitively on Ω, we have $\chi = 1 + \psi$ where χ is the permutation character of the action and ψ is an irreducible character of G of degree q. Also since the action is sharply 2-transitive, only 1_G fixes three distinct elements of Ω. Hence for all $1_G \neq g \in G$ we have $\lambda = \chi(g) \in \{0, 1, 2\}$.

Proposition 5 *For $G = PSL_2(q)$, let M be the stabilizer of a point in the natural action of degree $q+1$ on the set Ω. Let $M = G_1$. Suppose $g \in nX \subseteq G$ is an element fixing exactly one point, and without loss of generality, assume $g \in M$. Then the replication number for the associated design is $r = \lambda = 1$. We also have*

(i) *If q is odd then $|g^G| = \frac{1}{2}(q^2-1)$, $|M \cap g^G| = \frac{1}{2}(q-1)$, and $\mathcal{D}$ is a 1-$(\frac{1}{2}(q^2-1), \frac{1}{2}(q-1), 1)$ design with $q+1$ blocks and*

$$\mathrm{Aut}(\mathcal{D}) = S_{\frac{1}{2}(q-1)} \wr S_{q+1} = (S_{\frac{1}{2}(q-1)})^{q+1} : S_{q+1}.$$

For all p, $C = C_p(\mathcal{D})$ is $[\frac{1}{2}(q^2-1), q+1, \frac{1}{2}(q-1)]_p$, with $\mathrm{Aut}(C) = \mathrm{Aut}(\mathcal{D})$.

(ii) *If q is even then $|g^G| = (q^2-1)$, $|M \cap g^G| = (q-1)$, and $\mathcal{D}$ is a 1-$((q^2-1), (q-1), 1)$ design with $q+1$ blocks and*

$$\mathrm{Aut}(\mathcal{D}) = S_{(q-1)} \wr S_{q+1} = (S_{(q-1)})^{q+1} : S_{q+1}.$$

For all p, $C = C_p(\mathcal{D})$ is $[(q^2-1), q+1, q-1)]_p$, with $\mathrm{Aut}(C) = \mathrm{Aut}(\mathcal{D})$.

Proof: Since $\chi(g) = 1$, we deduce that $\psi(g) = 0$. We now use the character table and conjugacy classes of $PSL_2(q)$ (for example see [9]):

(i) For q odd, there are two types of conjugacy classes with $\psi(g) = 0$. In both cases we have $|C_G(g)| = q$ and hence $|nX| = |g^G| = |PSL_2(q)|/q = (q^2-1)/2$. Since $b = [G:M] = q+1$ and

$$k = \frac{\chi(g) \times |nX|}{[G:M]} = \frac{1 \times (q^2-1)/2}{q+1} = (q-1)/2,$$

the results follow from Remark 3.

(ii) For q even, $PSL_2(q) = SL_2(q)$ and there is only one conjugacy class with $\psi(g) = 0$. A class representative is the matrix $g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ with $|C_G(g)| = q$ and hence $|nX| = |g^G| = |PSL_2(q)|/q = (q^2-1)$. Since $b = [G:M] = q+1$ and

$$k = \frac{\chi(g) \times |nX|}{[G:M]} = \frac{1 \times (q^2-1)}{q+1} = q-1,$$

the results follow from Remark 3. ■

If we have $\lambda = r = 2$, and if blocks meet in at most one point, then a graph can be defined on b vertices, where b is the number of blocks, i.e. the index of M in G, by stipulating that the vertices labelled by the blocks b_i and b_j are adjacent if b_i and b_j meet. Then the incidence matrix for the design is an incidence matrix for the graph.

In the case where we get a graph, the following result from [7, Lemma] can be used.

Result 1 ([7]) *Let $\Gamma = (V, E)$ be a regular graph with $|V| = N$, $|E| = e$ and valency v. Let $\mathcal{G}$ be the 1-$(e, v, 2)$ incidence design from an incidence matrix A for Γ. Then $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\mathcal{G})$.*

Note: If the graph Γ is also connected, then it is an easy induction to show that $\mathrm{rank}_p(A) \geq |V| - 1$ for all p with obvious equality when $p = 2$. If in addition (as happens for some classes of graphs, see [7, 13, 12]) the minimum weight is the valency and the words of this weight are the scalar multiples of the rows of the incidence matrix, then we also have $\mathrm{Aut}(C_p(\mathcal{G})) = \mathrm{Aut}(\mathcal{G})$.

Proposition 6 *For $G = PSL_2(q)$, let M be the stabilizer of a point in the natural action of degree $q+1$ on the set Ω. Let $M = G_1$. Suppose $g \in nX \subseteq G$ is an element fixing exactly two points, and without loss of generality, assume $g \in M = G_1$ and that $g \in G_2$. Then the replication number for the associated design is $r = \lambda = 2$. We also have*

(*i*) *If g is an involution, so that $q \equiv 1 \pmod 4$, the design $\mathcal{D}$ is a 1-$(\frac{1}{2}q(q+1), q, 2)$ design with $q+1$ blocks and $\mathrm{Aut}(\mathcal{D}) = S_{q+1}$. Furthermore $C_2(\mathcal{D})$ is $[\frac{1}{2}q(q+1), q, q]_2$, $C_p(\mathcal{D})$ is $[\frac{1}{2}q(q+1), q+1, q]_p$ if p is an odd prime, and $\mathrm{Aut}(C_p(\mathcal{D})) = \mathrm{Aut}(\mathcal{D}) = S_{q+1}$ for all p.*

(*ii*) *If g is not an involution, the design $\mathcal{D}$ is a 1-$(q(q+1), 2q, 2)$ design with $q+1$ blocks and $\mathrm{Aut}(\mathcal{D}) = 2^{\frac{1}{2}q(q+1)} : S_{q+1}$. Furthermore $C_2(\mathcal{D})$ is $[q(q+1), q, 2q]_2$, $C_p(\mathcal{D})$ is $[q(q+1), q+1, 2q]_p$ if p is an odd prime, and $\mathrm{Aut}(C_p(\mathcal{D})) = \mathrm{Aut}(\mathcal{D}) = 2^{\frac{1}{2}q(q+1)} : S_{q+1}$ for all p.*

Proof: A block of the design constructed will be $M \cap g^G$. Notice that from elementary considerations or using group characters we have that the only powers of g that are conjugate to g in G are g and g^{-1}. Since M is transitive on $\Omega \setminus \{1\}$, g^M and $(g^{-1})^M$ give $2q$ elements in $M \cap g^G$ if $o(g) \neq 2$, and q if $o(g) = 2$. These are all the elements in $M \cap g^G$ since M_j is cyclic so if $h_1, h_2 \in M_j$ and $h_1 = g_1^x, h_2 = g_2^x$ for some $x_1, x_2 \in G$, then h_1 is a power of h_2, so they can only be equal or inverses of one another.

(i) In this case by the above $k = |M \cap g^G| = q$ and hence

$$|nX| = \frac{k \times [G:M]}{\chi(g)} = \frac{q \times (q+1)}{2}.$$

So $\mathcal{D}$ is a 1-$(\frac{1}{2}q(q+1), q, 2)$ design with $q+1$ blocks. An incidence matrix of the design is an incidence matrix of a graph on $q+1$ points labelled by the rows of the matrix, with the vertices corresponding to rows r_i and r_j being adjacent if there is a conjugate of g that fixes both i and j, giving an edge $[i,j]$. Since G is 2-transitive, the graph we obtain is the complete graph K_{q+1}.

The automorphism group of the design is the same as that of the graph (see [7]), which is S_{q+1}. By [12], $C_2(\mathcal{D}) = [\frac{1}{2}q(q+1), q, q]_2$ and $C_p(\mathcal{D}) = [\frac{1}{2}q(q+1), q+1, q]_p$ if p is an odd prime. Further, the words of the minimum weight q are the scalar multiples of the rows of the incidence matrix, so $\mathrm{Aut}(C_p(\mathcal{D})) = \mathrm{Aut}(\mathcal{D}) = S_{q+1}$ for all p.

(ii) If g is not an involution, then $k = |M \cap g^G| = 2q$ and hence

$$|nX| = \frac{k \times [G:M]}{\chi(g)} = \frac{2q \times (q+1)}{2} = q(q+1).$$

So $\mathcal{D}$ is a 1-$(q(q+1), 2q, 2)$ design with $q+1$ blocks. In the same way we define a graph from the rows of the incidence matrix, but in this case we have the complete directed graph.

The automorphism group of the graph and of the design is $2^{\frac{1}{2}q(q+1)} : S_{q+1}$. Similarly to the previous case, $C_2(\mathcal{D})$ is $[q(q+1), q, 2q]_2$ and $C_p(\mathcal{D})$ is $[q(q+1), q+1, 2q]_p$ if p is an odd prime. Further, the words of the minimum weight $2q$ are the scalar multiples of the rows of the incidence matrix, so $\mathrm{Aut}(C_p(\mathcal{D})) = \mathrm{Aut}(\mathcal{D}) = 2^{\frac{1}{2}q(q+1)} : S_{q+1}$ for all p. ■

We end this subsection by giving few examples of designs and codes constructed, using Propositions 5 and 6, from $PSL_2(q)$ for $q \in \{16, 17, 19\}$, where M is the stabilizer of a point in the natural action of degree $q+1$ and $g \in nX \subseteq G$ is an element fixing exactly one or two points.

Example 1 ($PSL_2(16)$)

1. *g is an involution having cycle type $1^1 2^8$, $r = \lambda = 1$: $\mathcal{D}$ is a 1-$(255, 15, 1)$ design with 17 blocks. Here $C = C_p(\mathcal{D})$ is $[255, 17, 15]_p$ for all p, and* $\mathrm{Aut}(C) = \mathrm{Aut}(\mathcal{D}) = S_{15} \wr S_{17} = (S_{15})^{17} : S_{17}$.

2. *g is an element of order 3 having cycle type $1^2 3^5$, $r = \lambda = 2$: $\mathcal{D}$ is a 1-$(272, 32, 2)$ design with 17 blocks. $C_2(\mathcal{D})$ is $[272, 16, 32]_2$ and $C_p(\mathcal{D})$ is $[272, 17, 32]_p$ for odd p. Also for all p we have* $\mathrm{Aut}(C_p(\mathcal{D})) = \mathrm{Aut}(\mathcal{D}) = 2^{136} : S_{17}$.

Example 2 $(PSL_2(17))$ *(Note that* $17 \equiv 1 \pmod 4$.*)*

1. *g is an element of order 17 having cycle type $1^1 17^1$, $r = \lambda = 1$: $\mathcal{D}$ is a 1-$(144, 8, 1)$ design with 18 blocks. For all p, $C = C_p(\mathcal{D})$ is $[144, 18, 8]_p$, with* $\mathrm{Aut}(C) = \mathrm{Aut}(\mathcal{D}) = S_8 \wr S_{18} = (S_8)^{18} : S_{18}$.

2. *g is an involution having cycle type $1^2 2^8$, $r = \lambda = 2$: $\mathcal{D}$ is a 1-$(153, 17, 2)$ design with 18 blocks. $C_2(\mathcal{D})$ is $[153, 17, 17]_2$ and $C_p(\mathcal{D})$ is $[153, 18, 17]_p$ for odd p. Also for all p we have* $\mathrm{Aut}(C_p(\mathcal{D})) = \mathrm{Aut}(\mathcal{D}) = S_{18}$.

3. *g is an element of order 4 having cycle type $1^2 4^4$, $r = \lambda = 2$: $\mathcal{D}$ is a 1-$(306, 34, 2)$ design with 18 blocks. $C_2(\mathcal{D})$ is $[306, 17, 34]_2$ and $C_p(\mathcal{D})$ is $[306, 18, 34]_p$ for odd p. Also for all p we have* $\mathrm{Aut}(C_p(\mathcal{D})) = \mathrm{Aut}(\mathcal{D}) = 2^{153} : S_{18}$.

4. *g is an element of order 8 having cycle type $1^2 8^2$, $r = \lambda = 2$: $\mathcal{D}$ is a 1-$(306, 34, 2)$ design with 18 blocks. $C_2(\mathcal{D})$ is $[306, 17, 34]_2$ and $C_p(\mathcal{D})$ is $[306, 18, 34]_p$ for odd p. Also for all p we have* $\mathrm{Aut}(C_p(\mathcal{D})) = \mathrm{Aut}(\mathcal{D}) = 2^{153} : S_{18}$.

Example 3 $(PSL_2(19))$

1. *g is an element of order 19 having cycle type $1^1 19^1$, $r = \lambda = 1$: $\mathcal{D}$ is a 1-$(180, 9, 1)$ design with 20 blocks. For all p, $C = C_p(\mathcal{D})$ is $[180, 20, 9]_p$, with* $\mathrm{Aut}(C) = \mathrm{Aut}(\mathcal{D}) = S_9 \wr S_{20} = (S_9)^{20} : S_{20}$.

2. *g is an element of order 3 having cycle type $1^2 3^6$, $r = \lambda = 2$: $\mathcal{D}$ is a 1-$(380, 38, 2)$ design with 20 blocks. $C_2(\mathcal{D})$ is $[360, 19, 38]_2$ and $C_p(\mathcal{D})$ is $[360, 20, 38]_p$ for odd p. Also for all p we have* $\mathrm{Aut}(C_p(\mathcal{D})) = \mathrm{Aut}(\mathcal{D}) = 2^{190} : S_{20}$.

7 Some 1-designs from the Janko group J_1

The Janko group J_1 of order $2^3 \times 3 \times 5 \times 7 \times 11 \times 19$ has seven conjugacy classes of maximal subgroups, which are listed in Table 7. It has also 15 conjugacy classes of elements some of which are listed in Table 8.

Table 7: Maximal subgroups of J_1

No.	Order	Index	Structure
Max[1]	660	266	$PSL(2,11)$
Max[2]	168	1045	2^3:7:3
Max[3]	120	1463	$2 \times A_5$
Max[4]	114	1540	19:6
Max[5]	110	1596	11:10
Max[6]	60	2926	$D_6 \times D_{10}$
Max[7]	42	4180	7:6

Table 8: Some of the conjugacy classes of J_1

nX	$\|nX\|$	$C_G(g)$	Maximal Centralizer
$2A$	1463	$2 \times A_5$	Yes
$3A$	5852	$D_6 \times 5$	No

We apply Theorem 4 to the above maximal subgroups and a few conjugacy classes of elements of J_1 to construct several symmetric 1- designs.

7.1 $G = J_1$, $M = PSL_2(11)$, $nX = 2A$: 1-(1463, 55, 10) design

Let $G = J_1$, $M = PSL_2(11)$ and $nX = 2A$. Then

$$b = [G : M] = 266, v = |2A| = 1463, k = |M \cap 2A| = 55.$$

Also using the character table of J_1, we have

$$\chi_M = \chi_1 + \chi_2 + \chi_4 + \chi_6 = \underline{1a} + \underline{56a} + \underline{56b} + \underline{76a} + \underline{77a}$$

and hence $\chi_M(g) = 1 + 0 + 0 + 4 + 5 = 10 = \lambda$, where $g \in 2A$. We produce a non-symmetric 1-(1463, 55, 10) design $\mathcal{D}$. Since $C_G(g) = 2 \times A_5$ is also a maximal subgroup of J_1, J_1 acts primitively on blocks and points. The complement of $\mathcal{D}$, $\tilde{\mathcal{D}}$, is a 1-(1463, 1408, 256) design.

7.2 $G = J_1$, $M = 2 \times A_5$, $nX = 2A$: 1-(1463, 31, 31) design

Let $G = J_1$, $M = 2 \times A_5$ and $nX = 2A$. Then

$$b = [G : M] = 1463, v = |2A| = 1463.$$

It is easy to see that $M = 2 \times A_5$ has three conjugacy classes of order 2, namely $x_1 = z$, $x_2 = \alpha$ and $x_3 = z\alpha$, that fuse to $2A$ with corresponding centralizer orders 120, 8 and 8. Now by using Corollary 3 we have

$$\lambda = \chi_M(g) = \sum_{i=1}^{3} \frac{|C_G(g)|}{|C_M(x_i)|} = \frac{120}{120} + \frac{120}{8} + \frac{120}{8} = 31,$$

where $g \in 2A$. Alternatively we can use the character table of J_1 to find that

$$\chi_M = \chi_1 + \chi_2 + \chi_3 + 2\chi_4 + 2\chi_6 + \chi_9 + \chi_{10} + \chi_{11} + 2\chi_{12} + 2\chi_{15},$$

and

$$\chi_M(g) = 1 + 0 + 0 + 8 + 10 + 0 + 0 + 0 + 10 + 2 = 31 = \lambda.$$

In this case clearly $k = |M \cap 2A| = \lambda = 31$, and we produce a symmetric 1-$(1463, 31, 31)$ design $\mathcal{D}$. Obviously J_1 acts primitively on blocks and points. The complement of $\mathcal{D}$, $\tilde{\mathcal{D}}$, is a 1-$(1463, 1432, 1432)$ design.

7.3 $G = J_1$, $M = PSL_2(11)$, $nX = 3A$: 1-$(5852, 110, 5)$ design

Let $G = J_1$, $M = PSL_2(11)$ and $nX = 3A$. Then

$$b = [G : M] = 266, v = |3A| = 5852, k = |M \cap 3A| = 110.$$

Also using the character table of J_1, we have

$$\chi_M = \chi_1 + \chi_2 + \chi_4 + \chi_6 = \underline{1a} + \underline{56a} + \underline{56b} + \underline{76a} + \underline{77a}$$

and hence $\chi_M(g) = 1+4+1-1 = 5 = \lambda$, where $g \in 3A$. We produce a non-symmetric 1-$(5852, 110, 5)$ design $\mathcal{D}$. Since $C_G(g) = D_6 \times 5$ is not a maximal subgroup of J_1, J_1 acts primitively on 266 blocks but imprimitively on 5852 points. The complement of $\mathcal{D}$, $\tilde{\mathcal{D}}$, is a 1-$(5852, 5742, 261)$ design.

7.4 $G = J_1$, $M = PSL_2(11)$, $nX = 3A$: 1-$(5852, 20, 5)$ design

Let $G = J_1$, $M = 2 \times A_5$ and $nX = 3A$. Then

$$b = [G : M] = 1463, v = |3A| = 5852, k = |M \cap 3A| = 20.$$

It is easy to see that $M = 2 \times A_5$ has only one conjugacy class of elements of order 3, which fuses to $3A$, with the corresponding centralizer order 6. Now by using Corollary 3 we have

$$\lambda = \chi_M(g) = \frac{|C_G(g)|}{|C_M(x)|} = \frac{30}{6} = 5,$$

where $g \in 3A$. Alternatively we can use the character χ_M as in Subsection 7.2 to find that

$$\chi_M(g) = 1 + 2 + 2 + 2 - 2 + 0 + 0 + 0 + 2 - 2 = 5 = \lambda,$$

where $g \in 3A$. We produce a non-symmetric 1-(5852, 20, 5) design $\mathcal{D}$. Since $C_G(g) = D_6 \times 5$ is not a maximal subgroup of J_1, J_1 acts primitively on the 1463 blocks but imprimitively on the 5852 points. The complement of $\mathcal{D}$, $\tilde{\mathcal{D}}$, is a 1-(5852, 5832, 1458) design.

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