On the Ramsey number for cycle with respect to identical copies of complete graphs

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Abstract

For graphs G and H, *Ramsey number* $R(G, H)$ is the smallest natural number n such that no (G, H)-free graph on n vertices exists. In 1981, Burr [5] proved the general lower bound $R(G, H) \geq (n-1)(\chi(H)-1)+\sigma(H)$, where G is a connected graph of order n, $\chi(H)$ denotes the chromatic number of H and $\sigma(H)$ is its *chromatic surplus*, namely, the minimum cardinality of a color class taken over all proper colorings of H with $\chi(H)$ colors. A connected graph G of order n is called good with respect to H, *H-good*, if $R(G, H) = (n-1)(\chi(H)-1)+\sigma(H)$. The notation tK_m represents a graph with t identical copies of complete graph K_m. In this note, we discuss the goodness of cycle C_n with respect to tK_m for $m, t \geq 2$ and sufficiently large n. Furthermore, it is also provided the Ramsey number $R(G, tK_m)$, where G is a disjoint union of cycles.

Keywords: *(G, H)-free, H-good, complete graph, cycle, Ramsey number.*

1 Introduction

We consider that all graphs in this paper are finite, undirected and simple. Let G and H be two graphs, where H is a subgraph of G, we define $G - H$ as a graph obtained from G by deleting the vertices of H and all edges incident to them. Let t be a natural number and G_i be a connected graph with the vertex set V_i and the edge set E_i for every $i = 1, 2, ..., t$. *The disjoint union of graphs*, $\bigcup_{i=1}^{t} G_i$, has the vertex set $\bigcup_{i=1}^{t} V_i$ and the edge set $\bigcup_{i=1}^{t} E_i$. Furthermore, if each G_i is isomorphic to a connected graph G then we denote by tG the disjoint union of t copies of G.

For graphs G and H, *Ramsey number* $R(G,H)$ is the minimum n such that in every coloring of the edges of the complete graph K_n with two colors, say red and blue, there is a red copy of G or a blue copy of H. A graph F is called *(G,H)-free* if F contains no subgraph isomorphic to G and its complement $\overline{F}$ contains no subgraph isomorphic to H. The Ramsey number $R(G,H)$ can be equivalently defined as the smallest natural number n such that no (G,H)-free graph on n vertices exists.

Determining $R(G,H)$ is a notoriously hard problem. Burr [6] showed that the problem of determining whether $R(G,H) \leq n$ for a given n is NP-hard. Furthermore in Shaeffer [14] one can find a rare natural example of a problem higher than NP-hard in the polynomial hierarchy of computational complexity theory, that is, Ramsey arrowing is $\prod_2^p$-complete. The few known values of $R(G,H)$ are collected in the dynamic survey of Radziszowski [13].

Burr [5] proved the general lower bound

$$R(G,H) \geq (n-1)(\chi(H)-1)+\sigma(H), \tag{1}$$

where G is a connected graph on n vertices, $\chi(H)$ denotes the chromatic number of H and $\sigma(H)$ is its *chromatic surplus*, namely, the minimum cardinality of a color class taken over all proper colorings of H with $\chi(H)$ colors. Motivated by this inequality, the graph G is said to be *H-good* if equality holds in (1). In particular, Chvátal [9] proved that trees are K_m-good and Sudarsana et al. [18] showed that path P_n is good with respect to $2K_m$ and more recent result can be found in [20].

Faudree and Schelp [10] conjectured that C_n is K_m-good for $n \geq m \geq 3$, except for $n = m = 3$. The conjecture has been verified for $n \geq m^2 - 2$ (Bondy and Erdős [4]), for $m = 3$ (Chartrand and Schuster [7]), $m = 4$ (Yang, Huang and Zhang [16]), $m = 5$ (Bollobás, Jayawardene, Yang, Huang, Rousseau and Zhang [3]), $m = 6$ (Schiermeyer [15]) and $m = 7$ (Chen, Cheng and Zhang [8]). More recently, Nikiforov [12] proved the conjecture for all $m \geq 3$ and $n \geq 4m+2$. Other result concerning the goodness of graphs with the chromatic surplus one can be found in Lin et al. [11] and Sudarsana et al. [22]. However, the goodness of cycle C_n with respect to tK_m is still open.

In this paper, we establish the goodness of cycle C_n with respect to tK_m for $m,t \geq 2$ and sufficiently large n. This is an extending result of Sudarsana in [23].

2 Known Results

For the proof of our new result, Theorem 2, we use the following results.

Lemma 1 (Sudarsana et al. [18]) *Let n and t be positive integers. Then,*

$$R(P_n, tK_2) = \begin{cases} n+t-1, & t \leq \lfloor \frac{n}{2} \rfloor; \\ 2t + \lceil \frac{n}{2} \rceil - 1, & t > \lfloor \frac{n}{2} \rfloor. \end{cases}$$

Theorem 1 (Nikiforov [12]) *Let $m \geq 3$ be an integer. If $n \geq 4m+2$ then $R(C_n, K_m) = (n-1)(m-1)+1$.*

3 The Main Result

The following theorem deals with the goodness of cycle C_n with respect to t identical copies of complete graphs, tK_m.

Theorem 2 *Let $t, m \geq 2$ be integers and $g(t,m) = t((tm-2)(m-1)+1)+1$. If $n \geq g(t,m)$ then $R(C_n, tK_m) = (n-1)(m-1)+t$.*

By extending previous results of Baskoro et al. [1], Stahl [17], Bielak [2] and Sudarsana et al. [19], Sudarsana et al. [21] recently proved a formula for $R(G, H)$ when every connected component of G is not necessary an H–good graph. This result motivates the study of general families of H-good graphs. In particular, Theorem 2 provides the following computation of $R(G, tK_m)$ when G is a set of disjoint cycles.

Corollary 1 *Let $m, t \geq 2$ be integers and $g(t,m) = t((tm-2)(m-1)+1)+1$. Let $G \simeq \bigcup_{i=1}^{k} l_i C_{n_i}$, where $l_i \geq 1$ and each C_{n_i} is a cycle of order n_i.*

If $n_1 \geq n_2 \geq \ldots \geq n_k \geq g(t,m)$ then

$$R(G, tK_m) = \max_{1 \leq i \leq k} \left\{ (n_i - 1)(m-2) + \sum_{j=1}^{i} l_j n_j \right\} + t - 1. \qquad (2)$$

Firstly, by similar way with the proof of Lemma 1 in [18] we obtain the main result for case $m = 2$ as follows.

Lemma 2 *Let $n \geq 3$ and $t \geq 1$ be integers. Then,*

$$R(C_n, tK_2) = \begin{cases} n+t-1, & t \leq \lfloor \frac{n}{2} \rfloor; \\ 2t + \lceil \frac{n}{2} \rceil - 1, & t > \lfloor \frac{n}{2} \rfloor. \end{cases}$$

Next we show the following weaker form of our main result for case $t = 2$.

Lemma 3 *Let $m \geq 2$ be an integer. If $n \geq 4(m-1)^2+3$ then $R(C_n, 2K_m) = (n-1)(m-1)+2$.*

Proof. We obtain the lower bound $R(C_n, 2K_m) \geq (n-1)(m-1)+2$ from the fact that $(m-1)K_{n-1} \cup K_1$ is a $(C_n, 2K_m)$-free graph on $(n-1)(m-1)+1$ vertices. Let us next show the upper bound.

For $m = 2$ the statement follows from Lemma 2. Assume that the lemma is true for $m-1$, that is, if F is a graph of order $n \geq (n-1)(m-2)+2$, then F contains C_n or $\overline{F}$ contains $2K_{m-1}$. We shall show that the lemma is also valid for m.

Set $m \geq 3$ and let F be an arbitrary graph on $(n-1)(m-1)+2$ vertices. We will show that F contains C_n or $\overline{F}$ contains $2K_m$. Note that if $m \geq 3$ then $n \geq 4(m-1)^2+3 > 4m+2$. By Theorem 1, we have a copy of C_n in F or a copy of K_m in $\overline{F}$. If F contains C_n then we are done. Thus we may assume that $\overline{F}$ contains K_m. Observe that the subgraph $F - \overline{K}_m$ of F has $(n-2)(m-1)+1$ vertices. Since $m \geq 3$, we have $n-1 \geq 4(m-1)^2+2 > 4m+2$. By Theorem 1, $F - \overline{K}_m$ contains C_{n-1} or the complement of $F - \overline{K}_m$ contains K_m. If the complement of $F - \overline{K}_m$ contains K_m then we obtain a copy of $2K_m$ in $\overline{F}$ and hence we are done. Therefore, F has a cycle C_{n-1}. Thus the subgraph $F - C_{n-1}$ of F has $(n-1)(m-2)+2$ vertices. By the induction hypothesis, $F - C_{n-1}$ contains C_n or the complement of $F - C_{n-1}$ contains $2K_{m-1}$. If $F - C_{n-1}$ contains C_n then we are done. Hence we may assume that F contains a cycle C_{n-1} with vertex set, say $c_1, c_2, \ldots, c_{n-1}$ and edges $c_i c_{i+1}$ (subscripts modulo $(n-1)$), and that $\overline{F}$ contains $2K_{m-1}$. It is clear that the subgraphs C_{n-1} and $2K_{m-1}$ have no vertices in common.

Assume that F contains no C_n. We will show that $\overline{F}$ contains $2K_m$. Let us consider the relation between the vertices in $A = \{c_1, c_2, ..., c_{n-1}\}$ and in $B = V(2K_{m-1})$. Suppose that the neighborhood $N_A(u)$ in A of a vertex $u \in B$ satisfies $|N_A(u) \cap V(C_{n-1})| \geq 2m-1$. Let $c_i, c_j \in N_A(u) \cap V(C_{n-1})$ with $i < j$. Note that $j - i > 1$ since otherwise we can extend C_{n-1} to a cycle of length n containing u. If c_{i+1} and c_{j+1} are adjacent in F then we also have the cycle $\{c_i u c_j c_{j-1} \ldots c_{i+1} c_{j+1} c_{j+2} \ldots c_{n-1} c_1 c_2 \ldots c_i\}$ of length

n in F. If $c_{i+1}c_{j+1}$ is not an edge for every pair $c_i, c_j \in N_A(u) \cap V(C_{n-1})$ then $\{c_{i+1} : c_i \in N_A(u) \cap V(C_{n-1})\} \cup \{u\}$ is a set of $2m$ independent vertices in F so that $\overline{F}$ contains $2K_m$. Hence, for each $u \in B$ we have $|N_A(u) \cap V(C_{n-1})| \leq 2m - 2$. Therefore,

$$\left| A \setminus \bigcup_{u \in B} N_A(u) \right| \geq (n-1) - 4(m-1)^2. \tag{3}$$

Since $n \geq 4(m-1)^2 + 3$, it follows that there are at least 2 vertices in A which are adjacent to no vertex in B and hence $\overline{F}$ contains $2K_m$. This completes the proof of lemma. □

We are now ready to prove our main result.

Proof of Theorem 2. The lower bound $R(C_n, tK_m) \geq (n-1)(m-1)+t$ follows from the fact that $(m-1)K_{n-1} \cup K_{t-1}$ is a (C_n, tK_m)–free graph of order $(n-1)(m-1)+t-1$.

To prove the upper bound $R(C_n, tK_m) \leq (n-1)(m-1)+t$ we use inductions on t and m. For $t = 2$, we have $R(C_n, 2K_m) = (n-1)(m-1)+2$ from Lemma 3. Hence, the assertion holds for $n \geq g(2, m) = 4(m-1)^2+3$. In what follows we assume that the theorem is true for $n \geq g(t-1, m)$, that is $R(C_n, (t-1)K_m) \leq (n-1)(m-1)+t-1$.

From Lemma 2, we have $R(C_n, tK_2) = n+t-1$ for $n \geq 2t$. Note that if $t \geq 2$ then $n \geq g(t, 2) > 2t$. Therefore, the theorem holds for $m = 2$. In what follows we also assume $m \geq 3$ and the theorem is true for $n \geq g(t, m-1)$, that is $R(C_n, tK_{m-1}) \leq (n-1)(m-2)+t$.

We will show that the theorem is also valid for $n \geq g(t, m)$. Let F be an arbitrary graph on $(n-1)(m-1)+t$ vertices. We shall show that F contains C_n or $\overline{F}$ contains tK_m. Note that if $t \geq 2$ and $m \geq 3$ then $n \geq g(t, m) > 4m+2$. So Theorem 1 now guarantees that F contains C_n or $\overline{F}$ contains K_m. If F contains C_n then we are done. Thus we may assume that $\overline{F}$ contains K_m. Since the subgraph $F - \overline{K}_m$ of F has $(n-2)(m-1)+t-1$ vertices and $n-1 \geq g(t, m)-1 > g(t-1, m)$, by the induction hypothesis on t we know that $F - \overline{K}_m$ contains C_{n-1} or the complement of $F - \overline{K}_m$ contains $(t-1)K_m$. If the complement of $F - \overline{K}_m$ contains $(t-1)K_m$ then we have a tK_m in $\overline{F}$ and hence the proof is done. Therefore, F has a cycle C_{n-1}. Thus the subgraph $F - C_{n-1}$ of F has $(n-1)(m-2)+t$ vertices. Note that, since $t \geq 2$, we have $n \geq g(t, m) > g(t, m-1)$. By the induction hypothesis on m, we know that $F - C_{n-1}$ contains C_n or the complement of $F - C_{n-1}$ contains tK_{m-1}. If $F - C_{n-1}$ contains C_n then we are done. Hence we may assume that

F contains a cycle C_{n-1} with vertex set, say $c_1, c_2, \ldots, c_{n-1}$ and edges $c_i c_{i+1}$ (subscripts modulo $(n-1)$), and that $\overline{F}$ contains t disjoint copies $K^1_{m-1}, K^2_{m-1}, \ldots, K^t_{m-1}$ of the complete graph with $m-1$ vertices. It is clear that the subgraphs C_{n-1} and tK_{m-1} have no vertices in common.

Assume that F contains no C_n. We will show that $\overline{F}$ contains tK_m. Let us consider the relation between the vertices in $A = \{c_1, c_2, \ldots, c_{n-1}\}$ and in $B = V(K^1_{m-1}) \cup V(K^2_{m-1}) \cup \ldots \cup V(K^t_{m-1})$. Suppose that the neighborhood $N_A(u)$ in A of a vertex $u \in B$ satisfies $|N_A(u) \cap V(C_{n-1})| \geq tm-1$. Let $c_i, c_j \in N_A(u) \cap V(C_{n-1})$ with $i < j$. Note that $j - i > 1$ since otherwise we can extend C_{n-1} to a cycle of length n containing u. If $c_{i+1}c_{j+1}$ is an edge in F then we also have the cycle $\{c_i u c_j c_{j-1} \ldots c_{i+1} c_{j+1} c_{j+2} \ldots c_{n-1} c_1 c_2 \ldots c_i\}$ of length n in F. If $c_{i+1}c_{j+1}$ is not an edge for every pair $c_i, c_j \in N_A(u) \cap V(C_{n-1})$ then $\{c_{i+1} : c_i \in N_A(u) \cap V(C_{n-1})\} \cup \{u\}$ is a set of tm independent vertices in F so that $\overline{F}$ contains tK_m. Hence, for each $u \in B$ we have $|N_A(u) \cap V(C_{n-1})| \leq tm-2$. Therefore,

$$\left| A \setminus \bigcup_{u \in B} N_A(u) \right| \geq (n-1) - t(tm-2)(m-1). \qquad (4)$$

Since $n \geq g(t,m)$, it follows that there are at least t vertices in A which are adjacent to no vertex in B and hence $\overline{F}$ contains tK_m. The proof of Theorem 2 is now complete. □

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