COHESION AND NON-SEPARATING TREES IN CONNECTED GRAPHS

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ABSTRACT. If T is a tree on n vertices, $n \geqslant 3$, and if G is a connected graph such that $d(u) + d(v) + d(u,v) \geqslant 2n$ for every pair of distinct vertices of G, it has been conjectured that G must have a non-separating copy of T. In this note, we prove this result for the special case in which $d(u) + d(v) + d(u,v) \geqslant 2n + 2$ for every pair of distinct vertices of G, and improve this slightly for trees of diameter at least four and for some trees of diameter three.

Introduction and Definitions. All graphs in the article are finite and have no loops and no multiple edges. An *n-tree* is a tree with n vertices. The vertex set of G is $V(G)$ and the edge set is $E(G)$. If G is a graph, and $X \subseteq V(G)$, the *subgraph of G induced by X*, written $G[X]$, is the subgraph H with $V(H) = X$ and $E(H) = \{uv \in E(G) : u, v \in X\}$. The *complete graph* on m vertices is K_m. The *complete bipartite graph* with k vertices in one color class and m in the other is $K_{k,m}$. The *distance* between two vertices $x, y \in V(G)$ is the length (number of edges) of a shortest path from x to y and is denoted $d_G(x,y)$. The diameter of G is the maximum distance taken over all pairs of vertices of G. The number of edges of G incident with the vertex x is the *degree* of x, $d_G(x)$. An *end-vertex* or *leaf* of a graph is a vertex of degree one. We assume the reader has some familiarity with separable graphs, and decomposition of graphs into blocks and cut-vertices. For graph theoretic terms not defined in this paper see [10].

A connected graph G with at least two vertices is *k-cohesive* if for every pair of distinct vertices u, v, $d(u) + d(v) + d(u,v) \geqslant k$. Suppose, for example, that G is connected, with cut vertex b, and that G has exactly two blocks H_1 and H_2 with H_1 and H_2 isomorphic as rooted graphs (rooted at

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b). Then, H_1 (and, of course, H_2) would satisfy somewhat weaker conditions. Specifically, for any pair of distinct vertices u and v in $V(H_1) - \{b\}$, $d(u) + d(v) + d(u,v) \geqslant k$, and for any vertex u, $u \in V(H_1) - \{b\}$, $2d(u) + 2d(u,b) \geqslant k$. (The second condition follows from looking at the vertex $u' \in V(H_2) - \{b\}$, with u' corresponding to u under an isomorphism).

This discussion motivates the following definitions. For a graph G and a vertex $b \in V(G)$, the pair (G,b) will be called a rooted graph, and b will be called the root of G. We call the rooted graph (G,b) *weakly k-cohesive* if G is a connected graph, $b \in V(G)$, G has at least three vertices and for every pair of distinct vertices u, v, with $u, v \neq b$, $d(u) + d(v) + d(u,v) \geqslant k$. Note that if G is k-cohesive, $k \geqslant 4$, then (G,b) is weakly k-cohesive for any $b \in V(G)$. On the other hand, if (G,b) is a weakly k-cohesive graph, it is possible for G to have a vertex w of low degree near b. Now, if H is a separable graph with exactly one cut-vertex b' and each block (H_j, b') isomorphic (as a rooted graph) to (G,b), it is possible that H is not k-cohesive, since the needed inequality may not be satisfied by two images of w. However, at this point, our definition of weakly k-cohesive seems reasonable and a little less complicated than also requiring the condition $2d(u) + 2d(u,b) \geqslant k$, for any vertex u, $u \in V(H_1) - \{b\}$.

Lovász [7] proved that if G is 5-cohesive, then G has a non-separating copy of the 2-tree. Locke [5] conjectured that if G is $(2n)$-cohesive, $n \geqslant 3$, then G has a non-separating copy of the path with n vertices. This was proven by Locke, Tracy, Voss [6]. In [3], this conjecture is extended to n-trees, and a slightly weaker version is proven for trees of diameter at most four. Mader [8, 9] demonstrates conditions under which a k-connected graph has a copy of a specified tree whose deletion results in a k-connected graph. Diwan and Tholiya [4] prove that if T is an n-tree, $n \geqslant 1$, and if the minimum degree of a connected graph G is at least n, then G has a non-separating copy of T. This latter result leads one to believe that the following conjecture may hold.

Conjecture 1. If (G,b) is weakly $(2n)$-cohesive, $n \geqslant 4$, and if T_0 is a tree with n vertices, then $G - b$ contains a copy T of T_0 such that $G - V(T)$ is connected.

We begin with some preliminary results on weakly-k-cohesive graphs. Let T be an n-tree and (G,b) be a weakly-$(2n)$-cohesive graph. We say that T *separates* (G,b) if for every copy T_0 of T with $T_0 \subseteq G - b$, $G - V(T_0)$ is disconnected.

Let $f(T)$ denote the minimum positive integer m such that every weakly-m-cohesive graph (G, b) contains a non-separating copy of T. We consider the following variant of Conjecture 1.

Conjecture 2. For any n-tree, $n \geqslant 3$, $f(T) = 2n$.

In [1] it is recorded that this author has shown $f(T) < 4n$ for any n-tree. Using [4], we shall show that $f(T) \leqslant 2n + 2$, for all n-trees, and improve this slightly to show that $f(T) \leqslant 2n+1$, for all n-trees of diameter at least four.

We assume throughout that $m \geqslant 2n$ and $n \geqslant 4$. We shall also assume that T is an n-tree which separates the weakly-m-cohesive graph (G, b) and, subject to this, that G is as small as possible.

Lemma 1. G has a copy of T avoiding b.

Proof. Label the vertices of $T = \{q_1, q_2, \ldots, q_n\}$ with q_1 an end-vertex of T and $T[\{q_1, q_2, \ldots, q_s\}]$ connected. Let $u \in V(G) - \{b\}$ with $d(u)$ minimum. Define $\alpha(q_1) = u$. We proceed by induction. Suppose that $\alpha(q_1)$, $\alpha(q_2)$, ..., $\alpha(q_s)$ have been chosen, where $1 \leqslant s < n$. The vertex q_{s+1} is adjacent to some vertex q_j in T, with $1 \leqslant j \leqslant s$. Let ℓ denote the distance in T from q_1 to q_j, let P denote the path in T from q_1 to q_j, and let β denote the number of neighbors of $\alpha(q_j)$ in the set $\{\alpha(q) : q \in V(P)\}$. Then, $d_G(\alpha(q_1), \alpha(q_j)) \leqslant \ell + 1 - \beta$.

If $d(u) \geqslant n$, then $d(\alpha(q_j)) \geqslant n$, and

$$|N(\alpha(q_j)) - \{\alpha(q_1), \alpha(q_2), \ldots, \alpha(q_s), b\}| \geqslant 1,$$

and we may choose $\alpha(q_{s+1}) \in N(\alpha(q_j)) - \{\alpha(q_1), \alpha(q_2), \ldots, \alpha(q_s), b\}$.

If $d(u) \leqslant n - 1$, then

$$\begin{aligned} d(\alpha(q_j)) &\geqslant 2n - d_G(\alpha(q_1), \alpha(q_j)) - d_G(\alpha(q_1)) \\ &\geqslant 2n - (\ell + 1 - \beta) - d_G(\alpha(q_1)) \\ &\geqslant n - \ell + \beta. \end{aligned}$$

But, $\alpha(q_j)$ is joined to at most $(s-1) - (\ell - \beta)$ vertices which are of $\{\alpha(q_1), \alpha(q_2), \ldots, \alpha(q_s)\}$. Thus, $\alpha(q_j)$ is joined to at least $(n - \ell + \beta) - ((s-1) - (\ell - \beta)) = n-s+1 > 1$ vertices not in $\{\alpha(q_1), \alpha(q_2), \ldots, \alpha(q_s)\}$. Thus, we may choose $\alpha(q_{s+1}) \in N(\alpha(q_j)) - \{\alpha(q_1), \alpha(q_2), \ldots, \alpha(q_s), b\}$. ■

Lemma 2. If $u \in V(G) - \{b\}$, then $d_G(u) \geqslant \frac{1}{2}(m-5)$.

Proof. Let u be a vertex of minimum degree in G, subject to $u \neq b$, and let $v \in N(u)$, with $v \neq b$ if possible. Suppose that $d(u) \leqslant \frac{1}{2}(m-6)$.

Now, if $v = b$, then $d(u) = 1$. Let w be chosen with $d(u,w) = 2$. Then, $d(w) \geqslant m - d(u) - d(u,w) = m - 3 \geqslant 5$. Thus, $G - u$ has at least three vertices and $(G-u, b)$ is weakly $(2n)$-cohesive, and by our assumption that (G, b) is minimal, $(G-u, b)$ must have a non-separating copy of T. But then (G, b) must have a non-separating copy of T. Thus, we may assume that $v \neq b$. Since $v \neq b$,
$d(v) \geqslant m - d(u,v) - d(u) \geqslant m - 1 - \frac{1}{2}(m-6) = \frac{1}{2}(m+4)$.

Let H be a component of $G - \{u,v\}$. If there is more than one choice for H, select H so that $b \notin V(H)$. If $b \in V(H)$, set $b' = b$, and note that $|N(v) \cap V(H)| = d(v) - 1 \geqslant \frac{1}{2}(m+2) > n$. If $b \notin V(H)$, select any vertex $b' \in N(v) \cap V(H)$. If $b \in V(H)$, there is only one component H of $G - \{u,v\}$, and $|V(H)| \geqslant d(v) - 1 > n$. If $b \notin V(H)$, then $d(u,b') \leqslant 2$, $d(b') \geqslant m - d(u) - d(u,b') \geqslant \frac{1}{2}(m+2)$, and $d_H(b') \geqslant n - 1$. Thus, in either case, H has at least three vertices.

We now establish that (H, b') is weakly m-cohesive. For any pair of vertices $x, y \in V(H)$, $d_H(x,y) \geqslant d_G(x,y)$.

If $x, y \in V(H) - N(\{u,v\}) - \{b'\}$, then

$$\begin{aligned} d_H(x) + d_H(y) + d_H(x,y) &= d_G(x) + d_G(y) + d_H(x,y) \\ &\geqslant d_G(x) + d_G(y) + d_G(x,y) \\ &\geqslant m. \end{aligned}$$

If $x \in V(H) \cap N(\{u,v\}) - \{b'\}$, then

$$\begin{aligned} d_H(x) &= d_G(x) - |N(x) \cap \{u,v\}| \\ &\geqslant (m - d_G(u) - d_G(u,x)) - |N(x) \cap \{u,v\}| \\ &\geqslant m - d_G(u) - 3 \\ &\geqslant m - \frac{1}{2}(m-6) - 3 \\ &= \frac{1}{2}m \\ &\geqslant d_G(u) + 3. \end{aligned}$$

Thus, if $x, y \in V(H) \cap N(\{u,v\}) - \{b'\}$, then $d_H(x) + d_H(y) + d_H(x,y) \geqslant m+1$.

Finally, suppose that $x \in V(H) \cap N(\{u,v\}) - \{b'\}$ and $y \in V(H) - N(\{u,v\}) - \{b'\}$. Let $\beta = |N(x) \cap \{u,v\}|$. Then, $d_G(y,u) \leqslant d_H(y,x) + 3 - \beta$, and $d_H(x) = d_G(x) - \beta$. Thus

$$\begin{aligned}
d_H(x) + d_H(y) + d_H(y,x) &= (d_G(x) - \beta) + d_G(y) + d_H(y,x) \\
&\geqslant d_G(x) - \beta + d_G(y) + d_G(y,u) - 3 + \beta \\
&= d_G(x) + d_G(y) + d_G(y,u) - 3 \\
&\geqslant (d_G(u) + 3) + d_G(y) + d_G(y,u) - 3 \\
&= d_G(u) + d_G(y) + d_G(y,u) \\
&\geqslant m.
\end{aligned}$$

Therefore, (H, b') is weakly m-cohesive, and by minimality of (G, b), H has non-separating copy T_0 of T.

Since T_0 avoids b, we need only show that T_0 is non-separating in G. But, $G - V(H)$ is connected, and $H - V(T_0)$ is connected. We need only show that there is an edge from v to $V(H) - V(T_0)$. If $b \notin V(H)$, the edge vb' is an edge from v to $V(H) - V(T_0)$. If $b \in V(H)$, there are $d(v) - 1 \geqslant \frac{1}{2}(m+2) \geqslant n+1$ edges from v to $V(H)$. At most n of these edges are from v to $V(T_0)$, leaving at least one from v to $V(H) - V(T_0)$. Thus T_0 is non-separating copy of T, avoiding b, contradicting the choice of (G, b). Therefore, $d(u) \geqslant \frac{1}{2}(m-5)$. ■

The result of Diwan and Tholiya [4] allows us to show that we may restrict our attention to $m \leqslant 2n+3$.

Lemma 3. For some vertex $u \in V(G) - \{b\}$, $d_G(u) < n$.

Proof. Suppose that for every vertex $u \in V(G) - \{b\}$, $d_G(u) \geqslant n$. We construct a new graph H from n disjoint copies (G_k, b_k) of (G, b), by identifying the vertices $b_1, b_2, \ldots, b_n$, calling the new vertex c. Then H has minimum degree at least n, and therefore, by [4], has a non-separating copy T_0 of T. But, then, $c \notin V(T_0)$, and T_0 must lie completely in $G_k - b_k$ for some k. Now, T_0 is a non-separating copy of T in (G_k, b_k), contradicting the choice of (G, b). Therefore, $d_G(u) < n$, for some vertex $u \in V(G) - \{b\}$. ■

Corollary 1. If $u \in V(G) - \{b\}$, with $d_G(u)$ minimum, then $n - 2 \leqslant \frac{1}{2}(m-5) \leqslant d_G(u) \leqslant n-1$.

Proof. By Lemma 3, $d_G(u) \leqslant n-1$, and by Lemma 2, $d_G(u) \geqslant n-2$. ■

An immediate consequence of Corollary 1 is that Conjecture 1 holds under the slight strengthening of the cohesiveness condition, since $m \geqslant 2n+4$ would violate Corollary 1.

Corollary 2. For any n-tree T_1, $f(T_1) \leqslant 2n+4$.

The next Lemma [1, 3] is not used in establishing the results for the case $2n+1 \leqslant m \leqslant 2n+3$, but may be useful in narrowing the possibilities for the case $m = 2n$.

Lemma 4 [1, 3]. G is 2-connected.

Proof. Suppose that G is separable. Let B be any end-block of G, with cut-vertex b', and with $b \notin V(B) - \{b'\}$. By **Lemma 2**, every vertex of $B - b'$ has degree at least $n - 2 \geqslant 2$ in G, and thus B has at least three vertices. For vertices $x, y \in V(B) - \{b'\}$,

$$d_B(x) + d_B(y) + d_B(y,x) = d_G(x) + d_G(y) + d_G(y,x) \geqslant 2n.$$

Thus, (B, b') is weakly $(2n)$-cohesive, with $\nu(B) < \nu(G)$, contradicting the minimality of (G, b). Therefore, G must be 2-connected. ■

The next lemma is obvious, but perhaps useful.

Lemma 5. Let uv be an edge of $G - b$. Then, $\max\{d_G(u), d_G(v)\} \geqslant n$ and $\min\{d_G(u), d_G(v)\} \geqslant n-2$. ■

In the next series of lemmas, we examine the possibilities of vertices of degree $n-2$ or $n-1$ in G. The proof of the Lemma 6 closely follows a portion of the proof of Lemma 2.

Lemma 6. Suppose that $\xi \in \{0,1\}$, $m \geqslant 2n+\xi$, and $u \in V(G) - \{b\}$, with $d_G(u) = n - 2 + \xi$. Let $H = G - u$. Then, for any pair of distinct vertices $x, y \in V(H) - \{b\}$, $d_H(x) + d_H(y) + d_H(x,y) \geqslant m$. Also, any $v \in N_G(u) - \{b\}$ has $d_H(v) \geqslant n$.

Proof. Let $x, y \in V(H) - \{b\}$, with $x \neq y$.

If $x,y \notin N_G(u)$, then $d_H(x)+d_H(y)+d_H(x,y) \geqslant d_G(x)+d_G(y)+d_G(x,y) \geqslant m$.

If $x \in N_G(u)$, then $d_G(x) \geqslant m-(n-2+\xi)-1 = m-n+1-\xi \geqslant n+1 \geqslant d_G(u)+3-\xi$. Note that $d_H(x) \geqslant n$ as claimed. Now, if $x,y \in N_G(u)$, then $d_H(x)+d_H(y)+d_H(x,y) \geqslant 2(m-n-\xi)+1 \geqslant m+1-\xi \geqslant m$.

Finally, if $x \in N_G(u)$ and $y \notin N_G(u)$, then

$$\begin{aligned} d_H(x)+d_H(y)+d_H(x,y) &= (d_G(x)-1)+d_G(y)+d_H(x,y) \\ &\geqslant (d_G(u)+2-\xi)+d_G(y)+d_G(x,y) \\ &\geqslant d_G(u)+2-\xi+d_G(y)+(d_G(u,y)-1) \\ &\geqslant m+1-\xi \geqslant m. \end{aligned}$$

Thus, in all cases, $d_H(x)+d_H(y)+d_H(x,y) \geqslant m$.

Lemma 7. Suppose that $m \geqslant 2n+\xi$, $\xi \in \{0,1\}$, and that $u \in V(G)-\{b\}$, with $d_G(u) = n-2+\xi$. Then, $H = G-u$ has exactly one component, (H,b) is weakly-m-cohesive and has a non-separating copy of T. Furthermore, for any non-separating copy T_0 of T in (H,b), $N_G(u) \subseteq V(T_0)$. In particular, $b \notin N_G(u)$.

Proof. By Lemma 4, $H = G-u$ is connected and by Lemma 6 some vertex of H has degree at least $n \geqslant 4$. Thus, by Lemma 6, (H,b) is weakly-m-cohesive. Furthermore, if $w \in N_G(u) - V(T_0)$, then $G - V(H) = \{u\}$ is connected, $H - V(T_0)$ is connected, and $uw \in E(G - V(T_0))$. Hence, T_0 is a non-separating copy of T in (G,b). Therefore, $N_G(u) \subseteq V(T_0)$, and since $b \notin V(T_0)$, $b \notin N_G(u)$ ■

Lemma 8. Suppose $m \geqslant 2n+1$. Then, there is no vertex $u \in V(G)-\{b\}$ with $d_G(u) = n-1$.

Proof. Suppose that $u \in V(G)-\{b\}$ with $d_G(u) = n-1$. Then, $H = G-u$ is connected, (H,b) is weakly m-cohesive, (H,b) has a non-separating copy T_0 of T, and $N(u) \subseteq V(T_0)$.

Let $w \in V(T_0) - N_G(u)$. Note that $N_G(u) \cup \{w\} = V(T_0)$ and, hence, $d_G(u,w) = 2$. Thus, $d_G(w) \geqslant m-d_G(u)-d_G(u,w) = m-(n-1)-2 \geqslant n$. Therefore, there is an edge from w to $G-V(T_0)-\{u\}$. Let $T_1 = (T_0 - w) \cup \{u\} \cup \{uv : wv \in E(T_0)\} \cong T$ and note that T_1 is a non-separating copy of T in (G,b). Hence, there can be no vertex $u \in V(G)-\{b\}$ with $d_G(u) = n-1$.
■

If $m \geqslant 2n+2$, then Lemma 2 would force every vertex $u \in V(G) - \{b\}$ to have $d_G(u) \geqslant \left\lceil \frac{2n+2-5}{2} \right\rceil = n-1$ and Lemma 3 forces some $u \in V(G) - \{b\}$ to have $d_G(u) \leqslant n-1$, we have the following corollary.

Corollary 3. For any n-tree T_1, $f(T_1) \leqslant 2n+2$. ■

We can do slightly more, if we restrict the possibilities for T. We begin with an observation about the non-neighbors of u in a copy of T separating u from the rest of G.

Lemma 9. Suppose that $u \in V(G) - \{b\}$ with $d_G(u) = n-2$. Let Q be a non-separating copy of T in $(G-u, b)$. Then, for every vertex $y \in V(Q)$, there is some vertex in $N_Q(y) - N_G(u)$ and some edge from y to $V(G) - V(Q) - \{u\}$.

Proof. For any $y \in V(Q) - N_G(u)$, note that $d_G(u,y) \in \{2,3\}$.

If $d_G(u,y) = 2$, then $d_G(y) \geqslant m - d_G(u) - 2 \geqslant m - (n-2) - 2 \geqslant n$. But $u \notin N_G(y)$ and, hence, there is an edge from y to $G - V(Q) - \{u\}$.

If $d_G(u,y) = 3$, then $d_G(y) \geqslant m - d_G(u) - 2 \geqslant m - (n-2) - 3 \geqslant n-1$. But $u \notin N_G(y)$ and $|N_G(u) \cap N_G(y)| = 0$. Hence, there is an edge from y to $G - V(Q) - \{u\}$.

Finally, for any $y \in N_G(u)$, note that $d_G(y) \geqslant m - d_G(u) - 1 \geqslant m - (n-2) - 1 \geqslant n+1$. Again, there is an edge from y to $G - V(Q) - \{u\}$.

Now, suppose that for some vertex $y \in V(Q)$, $N_Q(y) \subseteq N_G(u)$. Then, $Q' = (Q-y) \cup \{u\} \cup \{uv : yv \in E(Q)\}$ is a non-separating copy of T in (G,b). ■

Lemma 10. Suppose $m \geqslant 2n+1$, and suppose that $u \in V(G) - \{b\}$ with $d_G(u) = n-2$. Let Q be a non-separating copy of T in $(G-u, b)$. Then, the diameter of T is at most three. Furthermore, if T has diameter three, then the two central vertices of T each have degree at least three.

Proof. Let $W = V(Q) - N_G(u)$. Note that $|W| = 2$ and $N_G(u) \subset V(Q)$. For each $y \in V(Q)$, there is some vertex in $N_Q(y) - N_G(u) \subseteq W$. In particular, for every leaf y of Q, the sole neighbor of y in Q must be in W.

Now, suppose that T has diameter at least $k \geqslant 4$, and let y_1, y_2 be leaves of Q with $d_Q(y_1, y_2) = k$. Let $z_j \in N_Q(y_j)$ and note that $W = \{z_1, z_2\}$.

But then, there is no vertex in $W \cap N_Q(z_1)$. Hence, the diameter of T must be at most three.

Now, suppose that T has diameter three, and thus Q has diameter three. Let the central vertices of Q be c_1, c_2, where $d_Q(c_1) \geqslant n$ and $d_Q(c_2) = 2$, and note that $W = \{c_1, c_2\}$. Then, $(Q - c_1) \cup \{uv : v \in N_G(u)\}$ is a non-separating copy of T in (G, b), contradicting the choice of (G, b). Hence, $d_Q(c_j) \geqslant 3$, $j = 1, 2$. ■

We gather the above results.

Corollary 4. Let $n \geqslant 4$. For any n-tree T, $f(T) \leqslant 2n + 2$. For any n-tree T with diameter$(T) \geqslant 4$, or with diameter$(T) = 3$ if T has a vertex of degree 2, $f(T) \leqslant 2n + 1$.

Lemma 11. Suppose $m \geqslant 2n + 1$, and suppose that $u \in V(G) - \{b\}$ with $d_G(u) = n - 2$. Then, for each $v \in N_G(u)$, there is some $w \in N_G(u)$ with $vw \in E(G)$. Furthermore, if n is odd, there is some $v \in N_G(u)$ with $|N_G(u) \cap N_G(v)| \geqslant 2$.

Proof. Suppose that there is some $v \in N_G(u)$ such that the edge uv is in no triangle. Let $H = G - \{u, v\}$ and $X = N_G(\{u, v\})$. Then, for $x \in X$, $d_H(x) \geqslant m - n - 1 \geqslant n$. So, for $x, y \in X$ with $x \neq y$, $d_H(x) + d_H(y) + d_H(x, y) \geqslant 2(m - n - 1) + 1 = m + (m - 2n - 1) \geqslant m$. Also, for $x \in X$, $y \in V(H) - X$, $d_H(x) + d_H(y) + d_H(x, y) \geqslant (d_G(u) + 2) + d_G(y) + (d_G(u, y) - 2) \geqslant m$.

As in previous lemmas, we may reduce quickly to the case in which H is connected, and thus (H, b) is weakly m-generated. Thus, (H, b) has a non-separating copy T_0 of T. But, $d_G(v) \geqslant n + 2$, and hence v has a neighbor in $H - V(T_0)$ and T_0 is a non-separating copy of T in (G, b), violating the conditions on (G, b). Therefore, every vertex in $G[N(u)]$ has degree at least one. If $|N(u)| = n - 2$ is odd, some vertex of $G[N(u)]$ has even degree and this degree must be at least two. ■

(The remark on the parity of n in Lemma 11 was provided by [2].)

A tree of diameter three has two centers, and the degrees of these two vertices determine the tree up to isomorphism. Let $\langle t_1, t_2 \rangle$ denote the tree Q of diameter three with centers c_1 and c_2 such that $d_Q(c_1) = t_1$ and $d_Q(c_2) = t_2$.

Lemma 12. [2] Suppose that $m \geqslant 2n+1$, that $T \cong \langle t_1, t_2 \rangle$, with centers c_1 and c_2, and that $u \in V(G) - \{b\}$ with $d_G(u) = n-2$. Then, for each $v \in N_G(u)$, $|N_G(u) \cap N_G(v)| \leqslant \min\{t_1, t_2\} - 3$.

Proof. Suppose that there is some $v \in N_G(u) - \{b\}$ such that there is some $X \subseteq |N_G(u) \cap N_G(v)|$ with $|X| = t_1 - 2$. Let Q be a non-separating copy of T in $(G-u, b)$. Again, $N_G(u) = V(Q) - \{c_1, c_2\}$, where c_1, c_2 are the centers of Q. We know that $vc_j \in E(Q)$, for some $j \in \{c_1, c_2\}$. Then, $(V(Q) - \{c_{3-j}\}) \cup \{u\} \cup \{ vx : x \in X \cup \{u, c_j\}\} \cup \{uy : y \in V(Q) - X - \{c_1, c_2\}\}$ is a non-separating copy of T in (G, b). ■

We also note that by Lemma 11, for $v \in N_G(u)$, $|N_G(u) \cap N_G(v)| \geqslant 1$, and thus $\min\{t_1, t_2\} \geqslant 3 + |N_G(u) \cap N_G(v)|$. Hence, Corollary 4 can be slightly strengthened by adding the following for trees of diameter three.

Corollary 5. Suppose that $m \geqslant 2n+1$, that $T \cong \langle t_1, t_2 \rangle$, with centers c_1 and c_2, and that $u \in V(G) - \{b\}$ with $d_G(u) = n-2$. Then, $\min\{t_1, t_2\} \geqslant 4$, if n is even, and $\min\{t_1, t_2\} \geqslant 5$, if n is odd.

We use $[c, X]$ to represent the tree with vertex set $\{c\} \cup X$, and edge set $\{cx : x \in X\}$. The following observation gives some additional hope that for a tree T_1 of diameter two, we might hope to prove that $f(T_1) \leqslant 2n+1$.

Lemma 13. [2] Suppose that $m \geqslant 2n+1$, that $T \cong K_{1,n-1}$, and that $u \in V(G) - \{b\}$ with $d_G(u) = n-2$. Then, for each $v \in N_G(u)$, $|N_G(u) \cap N_G(v)| \leqslant n-4$.

Proof. Suppose Q is a non-separating copy of $K_{1,n-1}$ in $(G - \{u\}, b)$, and that $v \in N_G(u)$ with $|N_G(u) \cap N_G(v)| \geqslant n-3$. Then, $N_G(u) \cap N_G(v) = N_G(u) - \{v\}$. Note that $W = V(Q) - N_G(u) = \{w, c\}$, where c is the center of Q. Then, $[v, V(Q) \cup \{u, v\} - \{w\}]$ is a non-separating copy of T in (G, b). ■

We summarize the results obtained in the next corollary.

Corollary 6. If Q is a tree on n vertices, with $n \geqslant 3$, then $2n \leqslant f(Q) \leqslant 2n+1$ if any of the following conditions hold:
(*i*) $\operatorname{diameter}(Q) \geqslant 4$;
(*ii*) $\operatorname{diameter}(Q) = 3$, and $Q \cong \langle k, n-k \rangle$, $k \in \{2, 3\}$;
(*iii*) $\operatorname{diameter}(Q) = 3$, and $Q \cong \langle 4, n-4 \rangle$, and n is odd; or
(*iv*) $\operatorname{diameter}(Q) = 2$, and $n \in \{3, 4, 5\}$.

If Q is a path on n vertices, with $n \geqslant 3$, then $f(Q) = 2n$.

In all other cases, if Q is a tree on n vertices, with $n \geqslant 3$, then $2n \leqslant f(Q) \leqslant 2n + 2$. ■

The smallest diameter three trees for which we have not established that $f(Q) \leqslant 2|V(Q)| + 1$ are $\langle 4,4 \rangle$, $\langle 4,6 \rangle$ and $\langle 5,5 \rangle$.

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