Monochromatic connectivity in monochromatic-star-free graphs

Colton Magnant*, Pouria Salehi-Nowbandegani†

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Abstract

We consider edge-colorings of complete graphs in which each color induces a subgraph that does not contain an induced copy of $K_{1,t}$ for some $t \geq 3$. It turns out that such colorings, if the underlying graph is sufficiently large, contain spanning monochromatic k-connected subgraphs. Furthermore, there exists a color, say blue, such that every vertex has very few incident edges in colors other than blue.

1 Introduction

All graphs considered in this work are simple and finite with no multiedges or loops. When it is convenient and unambiguous, we will associate a graph G with either its vertex set $V(G)$ or its edge set $E(G)$. By a coloring of a graph, we mean an edge-coloring. More precisely, for a positive integer m, an *m-coloring* of a graph G is a function $c : E(G) \to \{1, 2, \ldots, m\}$. Informally, an m-coloring is an assignment on the edges of G where each edges gets one of m possible colors. For $1 \leq i \leq m$, the *graph induced on color* i is the subgraph of G on the vertex set $V(G)$ containing only the edges having color i. Note that an m-coloring need not use all m colors.

Several recent works have considered variations on the following conjecture of Bollobás and Gyárfás.

Conjecture 1 (Bollobás and Gyárfás [1]). *If $n > 4(k-1)$, then every 2-coloring of K_n contains a monochromatic k-connected subgraph of order at least $n - 2(k-1)$.*

Such works include [4] which considers the conjecture when more colors are available, [2] which proves the conjecture when $n > 6.5k$, and [3] which considers similar results when rainbow subgraphs are forbidden. In particular, it was

*P.O. Box 8093, Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA, USA. cmagnant@georgiasouthern.edu

†P.O. Box 8093, Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA, USA. pouria salehi-nowbandegani@georgiasouthern.edu

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shown in [4] that such a strong result, namely a monochromatic k-connected subgraph of order n minus a constant, is not possible when more than two colors are allowed unless additional assumptions are introduced.

Consider a particular edge-coloring of a "small" graph. From this small coloring, larger colorings are produced by replacing each vertex with a set of vertices (of varying sizes) such that edges between sets are colored with the same color as the edge between the original pair of vertices, and edges within the sets are colored using some strategy (for example, all one color). For instance, the sharpness example for Conjecture 1 begins with a 2-colored K_5 in which each color induces a triangle with two pendant edges. The four vertices of the K_5 incident with the pendant edges are then each replaced by a copy of K_{k-1}, and the remaining vertex is replaced by a copy of $K_{n-4(k-1)}$, with the edges in all five new cliques given arbitrary 2-colorings.

A colored graph is called *monochromatic-claw-free* if there is no induced $K_{1,3}$ in the graph induced on any color. More generally, we call a colored graph *monochromatic-S_t-free* if there is no induced copy of the star $S_t = K_{1,t}$ in the graph induced on any color for $t \geq 3$.

For two graphs G and H, let $R(G, H)$ denote the minimum order of a complete graph such that any 2-coloring of the edges yields either a copy of G in one color or a copy of H in the other color. In particular, we write $R(s, t)$ for $R(K_s, K_t)$. Similarly, we let $R_m(s)$ denote the minimum order of a complete graph such that any m-coloring of the edges yields a monochromatic K_s. The classical theorem of Ramsey [5] implies that $R(G, H)$, $R(s, t)$ and $R_m(s)$ all exist.

It turns out that, if n is large, a monochromatic-S_t-free coloring of K_n will contain a structure that is much stronger than that provided by Conjecture 1.

Proposition 1. *Let m, k, t be positive integers with $t \geq 3$.*

(a) For $n \geq R_m((t-1)(m-1)+k)$, any monochromatic-S_t-free m-coloring of K_n contains a spanning monochromatic k-connected subgraph.

(b) If $m \leq n$ and every monochromatic-S_t-free m-coloring of K_n contains a spanning monochromatic k-connected subgraph, then every monochromatic-S_t-free m-coloring of $K_{n'}$ contains a spanning monochromatic k-connected subgraph for all $n' \geq n$.

Proof. (a) Let G be a monochromatic-S_t-free m-coloring of K_n. Since $n \geq R_m((t-1)(m-1)+k)$, by Ramsey's Theorem [5], G contains a monochromatic clique of order at least $(t-1)(m-1)+k$. Let H be the clique, say in blue, and let v be a vertex in $G \setminus H$. If v has at least t non-blue edges of one color into H, say red, then we have an induced red S_t. This is a contradiction, so v has at most $t-1$ edges of each color other than blue into H. This means that v must have at least k blue edges into H. Since this is true for all $v \in G \setminus H$, the blue subgraph induces a spanning k-connected graph as desired.

(b) Note that it suffices to prove the result for $n' = n + 1$. Suppose the hypothesis, and consider a monochromatic-S_t-free m-coloring of K_{n+1}. Then for every choice of n vertices, there is a spanning monochromatic k-connected subgraph. Since $m < n+1$, two such subgraphs must have the same color. Since these subgraphs intersect on $n-1 \geq k$ vertices, their union forms a spanning monochromatic k-connected subgraph of K_{n+1} as desired. □

Motivated by this result, we propose the following problem. Given m, k and $t \geq 3$, let $sm(m, k, t)$ be the smallest n such that every monochromatic-S_t-free m-coloring of K_n contains a spanning monochromatic k-connected subgraph.

Problem 1. *Given m, k and $t \geq 3$, find $sm(m, k, t)$.*

We can easily find a lower bound on $sm(m, k, t)$.

Proposition 2. *Given positive integers m, k and $t \geq 3$, $sm(m, k, t) \geq m(t-1)+k$.*

Proof. Let $n = m(t-1)+k-1$ and consider the coloring of K_n defined as follows. Let $G_2, G_3, \ldots, G_m$ each be sets of $t-1$ vertices, let C be a set of $k-1$ vertices and let $G_b = C \cup (\cup_{i=2}^{m} G_i)$. Finally let G_0 be the set of the remaining $t-1$ vertices. Color all edges within G_b, within G_0 and between C and G_0 with color 1. The coloring is completed by coloring all edges between G_0 and G_i with color i for each $2 \leq i \leq m$. This coloring contains no induced monochromatic S_t and has no spanning k-connected subgraph (although color 1 is spanning and $(k-1)$-connected). □

Note that the bound on n given in Proposition 2 appears to be sharp only when m, k and t are very small. In fact, it fails to be sharp even in one of the most trivial cases, when $m = 2$, $k = 3$ and $t = 3$ (see Theorem 1).

In general, we believe that the function $sm(m, k, t)$ is a polynomial in k and in m. Indeed, this is true when $t = 3$ (see Proposition 3). In Section 2, we provide results to this effect. Section 3 contains a result concerning the color degree of vertices in large monochromatic-star-free colorings of complete graphs.

2 Monochromatic Connectivity

Proposition 3. *For all positive integers m and k,*

$$sm(m, k, 3) \leq km + 5m(m-1).$$

Proof. Let G be a monochromatic-claw-free coloring of K_n. Let v be a vertex of G. If $n \geq km + 5m(m-1)$, then v has at least $k + 5(m-1)$ edges in a single color, say blue. Let A be a set of $k + 5(m-1)$ neighbors of v by blue edges.

Claim 1. *Each vertex of A has at most 2 edges in a single color other than blue in $G[A]$.*

Proof. For a contradiction, assume that a vertex in A has three edges in a different color, say red, within $G[A]$. Since G is monochromatic-claw-free, we can find a red triangle T in $G[A]$. This means that there is a blue induced claw centered at v with blue edges to T, a contradiction. □

By Claim 1, each vertex of A has at least $k-1+3(m-1)$ blue edges in $G[A]$. Since any two non-adjacent vertices of A in blue have at least $2(k-1+3(m-1))-(k+5(m-1)-2) \geq k$ common blue neighbors in A, the subgraph of $G[A \cup \{v\}]$ induced on the blue edges is k-connected.

Suppose a vertex $w \in G \setminus G[A \cup \{v\}]$ is connected to at least six vertices in A with a single color other than blue, say in red. Then among these vertices, since there is no induced red claw centered at w, there will be a red triangle, a contradiction. Thus, each vertex of $G \setminus G[A \cup \{v\}]$ has at most 5 edges in a single color other than blue into A. Since $|A| = k+5(m-1)$, each vertex in $G \setminus A$ has at least k blue edges into $G[A]$. This provides the desired spanning k-connected blue subgraph.

Theorem 1.

$$sm(2,2,3) = 6 \quad and \quad sm(2,3,3) = 8.$$

Proof. First we show that $sm(2,2,3) = 6$. The lower bound is provided by Proposition 2.

Consider a coloring of K_6 with at most two colors, say red and blue, such that there is no induced monochromatic $K_{1,3}$. Since $R(C_4, C_4) = 6$, there is a monochromatic C_4, say $C = abcd$ in red. If both vertices outside C have two red edges to C, this induces a red spanning 2-connected subgraph. Hence, one of these two vertices, say u, has at least three blue edges, to C. Since the graph contains no monochromatic induced claw, this implies the edges au, bu, cu, du as well as ac and bd are all blue. If we let v be the remaining vertex, then v must have two red neighbors that are opposite vertices on C, say a and c, since otherwise the blue edges will induce a spanning 2-connected subgraph. If both bv and dv are blue, then there is an induced red claw centered at a so we may assume bv is red. Then, regardless of the color of the edge uv, there is an induced monochromatic claw centered at u (if uv is blue) or at v (if uv is red), for a contradiction. This completes the proof that $sm(2,2,3) = 6$.

Next we show that $sm(2,3,3) = 8$. In this case, the lower bound from Proposition 2 is not enough. Here, the lower bound is provided by the following construction. It is easy to verify that the graphs in Figure 1 are complementary, claw-free and have connectivity 2. Thus, together they provide a 2-coloring of K_n with no monochromatic claw and no 3-connected spanning subgraph.

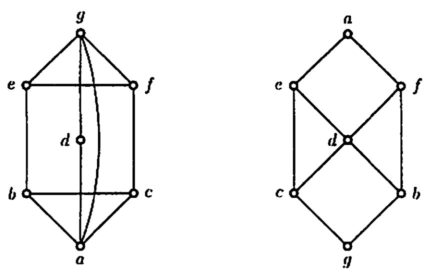


Figure 1: Complementary graphs making a 2-coloring of K_7.

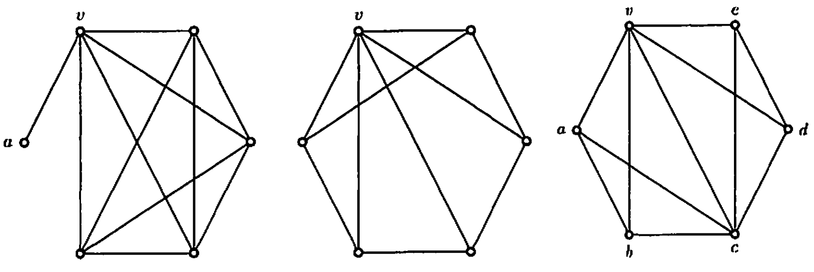
For the upper bound, consider a coloring of K_8 with at most two colors, say blue and red, such that there is no induced monochromatic $K_{1,3}$. Since we are 2-coloring K_8, every vertex must have at least 4 edges in one color. Thus, we get the following two cases.

Case 1. *There exist a vertex v with at least five edges in a single color, say red.*

In this case, the following claim verifies that there are only three possibilities (up to symmetry) shown in Figure 2.

Claim 2. *Among the five neighbors of v by red edges, there exists either a red K_4, a red C_5 or a red bow-tie (i.e. two triangles sharing a vertex).*

Proof. Since there is no induced red claw, there is no blue triangle in $N(v)$. If there is also no red triangle, then the five vertices must induce both a red C_5 and a blue C_5. Otherwise, let abc be a red triangle within $N(v)$. Assume that there is no red K_4 in $N(v)$ and let d and e be the other two vertices. If de is red, then it is easy to see that there is either a red C_5 or a red bow-tie. Otherwise, if de is blue, then it is easy to see that we may assume ad, bd and ce are red. We may then repeat this argument using the red triangle abd and the red edge ce to obtain the desired result. □



(a) $K_4 \subseteq N(v)$. (b) $C_5 \subseteq N(v)$. (c) Bow-tie in $N(v)$.

Figure 2: A vertex v with red degree 5.

In Subcase (a) (see Figure 2a), every vertex outside the $K_5 = K_4 \cup \{v\}$ (including a) has at least three red edges into the K_5. This shows us that the graph induced on the red edges is spanning and 3-connected.

In Subcase (b) (see Figure 2b), if both vertices outside this structure have at least three red edges into $\{v\} \cup N(v)$, the graph induced on the red edges would be spanning and 3-connected. If a vertex outside this structure has a blue edge to v, then it must have at most two blue edges to the cycle $C = C_5 \subseteq N(v)$ and therefore at least three red edges. Thus, we may assume there is a vertex x outside the graph pictured in Figure 2b with a red edge to v. If x has another two red edges to $N(v)$, then we are done as before so suppose x has blue edges to at least four vertices of $N(v)$. This means there is a red claw centered at v with red edges to x and two of the vertices of C for a contradiction.

In Subcase (c) (see Figure 2c), we may assume that $adbea$ is a blue C_4 (since otherwise we are in Subcase (b)). Let x be a vertex outside the structure pictured in the figure. If x has blue edges to two consecutive vertices of the C_4, then this would imply that there is either a blue claw centered at x, or a red claw centered at c or v. Otherwise, if say ax is blue, then dx and ex are red, but then we have a blue claw centered at a. It follows that ax, bx, dx, ex must all be red edges. Repeating the whole argument with the remaining vertex, we obtain a 3-connected spanning subgraph on the red edges.

Case 2. *All vertices have four edges in one color and three edges in the other color.*

Suppose a vertex v has four red edges. Let a, b, c, d be the red neighbors of v and let e, f, g be the remaining (blue) neighbors. Since there is no induced red claw, it is easy to see that there are two subcases, either there is a red triangle or two independent red edges within $\{a, b, c, d\}$.

Subcase (a) *Assume that bcd is a red triangle.*

To avoid an induced blue claw from a to $\{b, c, d\}$, we may assume that ab is a red edge. Since b has red degree 4, all edges between $\{b, v\}$ and $\{e, f, g\}$ must be blue. To avoid a blue induced claw centered at v, the triangle efg must not be entirely red, so assume ef is blue. Then ae is red since otherwise there would be an induced blue claw from e to $\{a, b, v\}$. Similarly, af is red. Then we have an induced red claw from a to $\{e, f, v\}$.

Subcase (b) *Assume that ab and cd are red edges.*

Note first that there is a red edge from e to $\{a, b\}$ to avoid an induced blue claw from e to $\{a, b, v\}$. This is also true replacing e with f or g or replacing $\{a, b\}$ with $\{c, d\}$, or both. Hence, up to symmetry, there are two possible cases to consider.

Subcase (b)(i) *Assume that ae, ag, cf, cg are all red edges.*

The edges fg and eg must be red to avoid red induced claws centered at c and a respectively. Since a, c, g all have red degree four, we get that $ac, ad, bc, af, ce, bg, dg$ are all blue edges. Then bd and ef are also blue to avoid induced blue claws centered at g and v. Then the blue edges contain a spanning 3-connected subgraph.

Subcase (b)(ii) *Assume that ae, af, ce, cf are all red edges.*

Since a and c have red degree four, we have acg is a blue triangle. This implies that eg and fg are blue edges to avoid induced red claws centered at e and f respectively. Then g has blue degree at least five, a contradiction. □

3 Color Degrees

Theorem 2. *Let $t \geq 3$ and $m \geq 2$ be integers and let G be a monochromatic-S_t-free m-coloring of K_n with $n \geq R_m((m-1)(t-1)t+1)$. Then there exists a color, say blue, such that every vertex has degree at most $R(t-1,t)-1$ in every color other than blue. Furthermore, the bound on the degree is the best possible.*

Proof. Suppose $n \geq R_m((m-1)(t-1)t+1)$. Much like the proof of Proposition 1, by Ramsey's Theorem [5], there exists a monochromatic clique K in G of order at least $(m-1)(t-1)t+1$, say in blue. Since G is monochromatic-S_t-free, every vertex in $G \setminus K$ has at most $t-1$ edges in each other color into K. This means that every vertex in $G \setminus K$ has a total of at most $(m-1)(t-1)$ non-blue edges into K. Since $|K| \geq (m-1)(t-1)t+1$, every set of t vertices in $G \setminus K$ shares at least one blue neighbor in K. Furthermore, this also means that every set of t vertices in G shares at least one blue neighbor in K.

Now suppose there exists a vertex $v \in G$ of degree at least $R(t-1,t)$ in some color other than blue, say red. In the red neighborhood of v, there must either be a red clique of order $t-1$ or a set of t vertices inducing no red edge. The latter cannot occur since otherwise there would be a red induced S_t centered at v. This means there must be a red clique K' of order $t-1$ within the red neighborhood of v. Finally since the set of vertices $K' \cup \{v\}$ has a common blue neighbor w in K, but this means that w is the center of a blue induced S_t for a contradiction.

For sharpness, consider the following construction. Let G_t be the sharpness example for $R(t-1,t) = n_t$. This means that G_t is a 2-colored K_{n_t-1} containing no red K_{t-1} and no blue K_t. To this, we add a single vertex with all incident edges colored in red. Repeating this with every other non-blue color and blue, we obtain $m-1$ copies of K_{n_t}, with each 2-colored with blue and another color. Finally, add $n-(m-1)n_t$ vertices with all incident edges colored in blue to produce the m-colored complete graph G on n vertices. For a non-blue color, say red, this coloring has a vertex of red degree $R(t-1,t)-1$ but clearly there is no induced red S_t since the red edges span n_t vertices containing only red and blue edges and with no blue K_t. Also, it is clear that G contains no K_t using non-blue colors, and hence there is also no induced blue S_t. □

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References

[1] B. Bollobás and A. Gyárfás. Highly connected monochromatic subgraphs. *Discrete Mathematics*, 308(9):1722–1725, 2008.

[2] S. Fujita and C. Magnant. Note on highly connected monochromatic subgraphs in 2-colored complete graphs. *Electron. J. Combin.*, 18(1):Paper 15, 5, 2011.

[3] S. Fujita and C. Magnant. Forbidden rainbow subgraphs that force large highly connected monochromatic subgraphs. *SIAM J. Discrete Math.*, 27(3):1625–1637, 2013.

[4] H. Liu, R. Morris, and N. Prince. Highly connected monochromatic subgraphs of multicolored graphs. *J. Graph Theory*, 61(1):22–44, 2009.

[5] F. P. Ramsey. On a Problem of Formal Logic. *Proc. London Math. Soc.*, 30(1):264–286, 1929.