A Note on the Fair Domination in Trees

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Abstract

A dominating set in a graph G is a subset S of vertices such that any vertex not in S is adjacent to some vertex of S. The domination number, $\gamma(G)$, of G is the minimum cardinality of a dominating set. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. A fair dominating set in a graph G (or FD-set) is a dominating set S such that all vertices not in S are dominated by the same number of vertices from S; that is, every two vertices not in S have the same number of neighbors in S. The fair domination number, $fd(G)$, of G is the minimum cardinality of an FD-set. A fair dominating set of G of cardinality $fd(G)$ is called an $fd(G)$-set. We say that $fd(G)$ and $\gamma(G)$ are *strongly equal* and denote by $fd(G) \equiv \gamma(G)$, if every $\gamma(G)$-set is an $fd(G)$-set. In this paper we provide a constructive characterization of trees T with $fd(T) \equiv \gamma(T)$.

Keywords: Domination number, Fair domination number, Strong equality, Tree.
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1 Introduction

Let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. If $S \subseteq V(G)$, then $G[S]$ is the subgraph induced by S. A vertex of degree one in a tree is called a *leaf*, and its neighbor is called a *support vertex*. If v is a support vertex in a tree, then L_v will denote the set of all leaves adjacent to v. A support vertex v is called *strong support vertex* if $|L_v| > 1$. For $r, s \geq 1$, a double star $S(r, s)$ is a tree with exactly two vertices that are not leaves, called the *central* vertices, with one adjacent to r leaves and the other to s leaves. For a vertex v in a rooted tree T, let $C(v)$ denote the set of children of v, $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$, and the depth of v, depth(v), is the largest distance from v to a vertex in $D(v)$. The *maximal subtree* at v is the subtree of T induced by $D(v) \cup \{v\}$, and is denoted by T_v. The *distance* $d(u, v)$ between two vertices u and v in a graph G is the minimum number of edges of a path from u to v. The *diameter* diam(G) of G, is $\max_{u,v \in V(G)} d(u, v)$. For terminology and notation on graph theory not given here, the reader is referred to [7].

A dominating set in a graph G is a set D of vertices such that every vertex $v \in V$ is either in D or adjacent to a vertex of D. A vertex in D is said to dominate a vertex outside D if they are adjacent in G. The domination number of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set of G of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. For a comprehensive study of domination parameters the reader is referred to [6].

Caro et al. [1] initiated the study of *fair domination* in graphs. For an integer $k \geq 1$, a *k-fair dominating set*, abbreviated kFD-set, in G is a dominating set D such that $|N(v) \cap D| = k$ for every vertex $v \in V - D$. The *k-fair domination number* of G, denoted by $fd_k(G)$, is the minimum cardinality of a kFD-set. A kFD-set of G of cardinality $fd_k(G)$ is called an $fd_k(G)$-set. A *fair dominating set*, abbreviated FD-set, in G is a kFD-set for some integer $k \geq 1$. Thus a dominating set D is an FD-set in G if $D = V$ or if $D \neq V$ and all vertices not in D are dominated by the same number of vertices from D; that is, $|N(u) \cap D| = |N(v) \cap D| > 0$ for every two vertices $u, v \in V - D$. Notice that if $G \neq \overline{K_n}$, then G contains a vertex v that is not isolated in G and the set $V - \{v\}$ is an FD-set in G. Thus a non-empty graph has an FD-set of cardinality less than its order. The *fair domination number*, denoted by $fd(G)$, of a graph G that is not the empty graph is the minimum cardinality of an FD-set in G. By convention, if $G = \overline{K_n}$, it is defined $fd(G) = n$. Hence if G is not the empty graph, then $fd(G) = \min\{fd_k(G)\}$, where the minimum is taken over all integers k where $1 \leq k \leq |V| - 1$. An FD-set of G of cardinality $fd(G)$ is called an $fd(G)$-set.

Clearly if $fd(G) = \gamma(G)$, then every $fd(G)$-set is also a $\gamma(G)$-set. However not every $\gamma(G)$-set is a $fd(G)$-set, even when $fd(G) = \gamma(G)$. For example in the path $P_4 : v_1v_2v_3v_4$, $fd(P_4) = \gamma(P_4) = 2$, but $\{v_1, v_3\}$ is a minimum dominating set which is not an FD-set. We say that $fd(G)$ and $\gamma(G)$ are *strongly equal* and denote by $fd(G) \equiv \gamma(G)$, if every $\gamma(G)$-set is an $fd(G)$-set. Haynes and Slater in [5] were the first to introduce strong equality between two parameters. Also in [2] Chellali and Jafari Rad, and in [3] and [4], Haynes, Henning and Slater gave constructive characterizations of trees with strong equality between some domination parameters. Our purpose in this paper is to present a constructive characterization of trees T with $fd(T) \equiv \gamma(T)$.

2 Characterization of trees T with $fd(T) \equiv \gamma(T)$

Our aim is to present a constructive characterization of trees T with $fd(T) \equiv \gamma(T)$. For this purpose we define a family of trees as follows: Let $\mathcal{T}$ be the class of trees T that can be obtained from a sequence $T_1, T_2, \cdots, T_k = T$ $(k \geq 1)$ of trees, where T_1 is a star $K_{1,t}$ with $t \geq 2$, and if $k \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the following operations for $1 \le i \le k-1$.

Operation $\mathcal{O}_1$ Assume that $w \in V(T_i)$ is a vertex contained in no $\gamma(T_i)$-set. Then T_{i+1} is obtained from T_i by joining a leaf of a star $K_{1,r}(r \ge 2)$ to w.

Operation $\mathcal{O}_2$ Assume that $w \in V(T_i)$ is a support vertex such that either $\deg(w) \ge 3$ or $\deg(w) = 2$ and w belongs to every $\gamma(T_i)$-set. Then T_{i+1} is obtained from T_i by joining the center of a star $k_{1,r}(r \ge 2)$ to w.

Lemma 1 *If* $fd(T_i) \equiv \gamma(T_i)$ *and* T_{i+1} *is obtained from* T_i *by Operation* $\mathcal{O}_1$, *then* $fd(T_{i+1}) \equiv \gamma(T_{i+1})$.

Proof. Let y be the leaf of a star $K_{1,r}$ that is joined to w according to Operation $\mathcal{O}_1$, and let x be the center of $K_{1,r}$. Let S be a $\gamma(T_i)$-set containing all support vertices. Clearly S is an $fd(T_i)$-set. By assumption $w \notin S$. Then $S \cup \{x\}$ is an FD-set for T_{i+1}, and so $fd(T_{i+1}) \le fd(T_i) + 1$. Now assume that D is a $fd(T_{i+1})$-set. Suppose that $x \notin D$. Clearly any leaf adjacent to x belongs to D. If $deg(x) \ge 3$ then $y \in D$, since $\deg(y) = 2$. Now $(D \cap V(T_i)) \cup \{w\}$ is an FD-set for T_i implying that $fd(T_i) \le fd(T_{i+1}) - 2$. This leads to $fd(T_i) + 2 \le fd(T_{i+1}) \le fd(T_i) + 1$, a contradiction. Thus $\deg(x) = 2$. Let x_1 be the leaf of T_{i+1} which is adjacent to x. It is obvious that $x_1 \in D$. Assume that $y \in D$. If $w \in D$ then $D \cap V(T_i)$ is an FD-set for T_i implying that $fd(T_i) \le fd(T_{i+1}) - 2$. This leads to $fd(T_i) + 2 \le fd(T_{i+1}) \le fd(T_i) + 1$, a contradiction. Thus $w \notin D$. Then $D \cap V(T_i)$ is a dominating set for T_i, implying that $fd(T_i) = \gamma(T_i) \le fd(T_{i+1}) - 2$. This leads to $fd(T_i) + 2 \le fd(T_{i+1}) \le fd(T_i) + 1$, a contradiction. We deduce that $y \notin D$. Then $w \in D$, and $D \cap V(T_i)$ is an FD-set for T_i, implying that $fd(T_i) \le fd(T_{i+1}) - 1$. Now $D \cap V(T_i)$ is an $fd(T_i)$-set containing w, a contradiction. We deduce that $x \in D$. If $y \notin D$ then $D \cap V(T_i)$ is an FD-set for T_i implying that $fd(T_i) \le fd(T_{i+1}) - 1$. Thus assume that $y \in D$. If $w \in D$ then $D \cap V(T_i)$ is an FD-set for T_i implying that $fd(T_i) \le fd(T_{i+1}) - 2 \le fd(T_i) - 1$, a contradiction.

Thus $w \notin D$. Then $D \cap V(T_i)$ is an FD-set for $T_i - w$ implying that $fd(T_i - w) \leq fd(T_{i+1}) - 2$. But $fd(T_i - w) \geq \gamma(T_i - w) \geq \gamma(T_i) - 1 = fd(T_i) - 1$. Now $fd(T_i) - 1 \leq fd(T_i - w) \leq fd(T_{i+1}) - 2$, and so $fd(T_i) \leq fd(T_{i+1}) - 1$. We conclude that $fd(T_{i+1}) = fd(T_i) + 1$. Similarly $\gamma(T_{i+1}) = \gamma(T_i) + 1$. Now $fd(T_{i+1}) = fd(T_i) + 1 = \gamma(T_i) + 1 = \gamma(T_{i+1})$.

Next we show that $fd(T_{i+1}) \equiv \gamma(T_{i+1})$. Assume that $fd(T_{i+1}) \not\equiv \gamma(T_{i+1})$. Let S_1 be a $\gamma(T_{i+1})$-set that is not an $fd(T_{i+1})$-set. Assume that $x \in S_1$. Suppose that $y \notin S_1$. If $w \in S_1$ then $S_1 \cap V(T_i)$ is a $\gamma(T_i)$-set containing w, a contradiction. Thus $w \notin S_1$. Then $S_1 \cap V(T_i)$ is a $\gamma(T_i)$-set that is not an $fd(T_i)$-set, a contradiction. So $y \in S_1$, and we observe that $w \notin S_1$. Then $N(w) \cap S_1 = \{y\}$, and $(S_1 \cap V(T_i)) \cup \{w\}$ is a $\gamma(T_i)$-set containing w, a contradiction. Next assume that $x \notin S_1$. Then any leaf adjacent to x belongs to S_1. From $\gamma(T_{i+1}) = \gamma(T_i) + 1$, we obtain that $\deg(x) = 2$. Let x_1 be the leaf of T_{i+1} which is adjacent to x. Clearly $x_1 \in S_1$. If $y \notin S_1$ then $(S_1 \cap V(T_i))$ is a $\gamma(T_i)$-set that is not an $fd(T_i)$-set, a contradiction. Thus $y \in S_1$. If $w \in S_1$ then $S_1 \cap V(T_i)$ is a dominating set for T_i implying that $\gamma(T_i) \leq \gamma(T_{i+1}) - 2$, a contradiction. Thus $w \notin S_1$. Since $|N(w) \cap S_1| = |N(x) \cap S_1|$, we obtain that $S_1 \cap V(T_i)$ is a dominating set for T_i of cardinality less than $\gamma(T_i)$. This is a contradiction. ■

Lemma 2 *If $fd(T_i) \equiv \gamma(T_i)$ and T_{i+1} is obtained from T_i by Operation $\mathcal{O}_2$, then $fd(T_{i+1}) \equiv \gamma(T_{i+1})$.*

Proof. Let x be the center of the star $K_{1,r}$ which is joined to w according to Operation $\mathcal{O}_2$. Let S be a $\gamma(T_i)$-set containing w. By assumption S is a $fd(T_i)$-set. Then $S \cup \{x\}$ is an FD-set for T_{i+1}, and thus $fd(T_{i+1}) \leq fd(T_i) + 1$. Now let D be a $fd(T_{i+1})$-set. Assume that $x \notin D$. Then any leaf adjacent to x belongs to D. In particular D contains all leaves of T_{i+1}. Then $D \cap V(T_i)$ is a dominating set for T_i implying that $fd(T_i) = \gamma(T_i) \leq fd(T_{i+1}) - r$. This leads to $fd(T_i) + r \leq fd(T_{i+1}) \leq fd(T_i) + 1$, a contradiction, since $r \geq 2$. Thus $x \in D$. Since w is a support vertex, and D is an FD-set, we find that $w \in D$. Then $D \cap V(T_i)$ is an FD-set for T_i, and thus $fd(T_i) \leq fd(T_{i+1}) - 1$. Hence $fd(T_{i+1}) = fd(T_i) + 1$. Similarly $\gamma(T_{i+1}) = \gamma(T_i) + 1$. Now $fd(T_{i+1}) = fd(T_i) + 1 = \gamma(T_i) + 1 = \gamma(T_{i+1})$.

Next we show that $fd(T_{i+1}) \equiv \gamma(T_{i+1})$. Assume that $fd(T_{i+1}) \not\equiv \gamma(T_{i+1})$. Let D_1 be a $\gamma(T_{i+1})$-set that is not $fd(T_{i+1})$-set. Clearly we may assume that $x \in D_1$. If $w \in D_1$ then $D_1 \cap V(T_i)$ is a $\gamma(T_i)$-set that is not an $fd(T_i)$-set, a contradiction. Thus $w \notin D_1$. If w is a strong support vertex then $(D_1 \cap V(T_i) - L(w)) \cup \{w\}$ is a dominating set for T_i of cardinality less than $\gamma(T_i)$, a contradiction. Thus w is not a strong support vertex. Let w_1 be the leaf adjacent to w. Clearly $w_1 \in D_1$, and $D_1 \cap V(T_i)$ is a $\gamma(T_i)$-set. Now $D_1 \cap V(T_i)$ or $D_2 = (D_1 \cap V(T_i) - \{w_1\}) \cup \{w\}$ is a $\gamma(T_i)$-set that is not an FD-set, a contradiction. ■

By a simple induction on the number of operations performed to construct a tree T, and Lemmas 1 and 2 we obtain the following.

Lemma 3 *Let T be a tree. If $T \in \mathcal{T}$ then $fd(T) \equiv \gamma(T)$.*

Theorem 4 *Let T be a tree of order $n \geq 3$. Then $fd(T) \equiv \gamma(T)$ if and only if $T \in \mathcal{T}$.*

Proof. Let T be a tree of order $n \geq 3$ with $fd(T) \equiv \gamma(T)$. We employee an induction on n to show that $T \in \mathcal{T}$. If diam$(T) = 2$ then T is a star and so $T \in \mathcal{T}$. If diam$(T) = 3$, then T is a double star. Let x and y be the centers of T. If deg$(x) = 2$ then $\{y, x_1\}$, where x_1 is the leaf adjacent to x, is a $\gamma(T)$-set which is not an FD-set, a contradiction. Thus deg$(x) \geq 3$, and similarly deg$(y) \geq 3$. Let T_0 be the component of $T - xy$ which contains x. Clearly T_0 is a star with a unique minimum dominating set, and $T_0 \in \mathcal{T}$. Then T is obtained from T_0 by Operation $\mathcal{O}_2$. Thus assume that diam$(T) \geq 4$.

Let $d =$ diam(T), and $x_0, x_1, \cdots, x_d$ be a diametrical path of T, where x_0 and x_d are two leaves of T. We root T at x_0. Let S be a

$\gamma(T)$-set containing all support vertices. Clearly S is an $fd(T)$-set. We consider the following cases.

Case 1. $\deg(x_{d-1}) \geq 3$. Assume that $\deg(x_{d-2}) = 2$. If $x_{d-2} \in S$, then we may assume that $x_{d-3} \notin S$, and then $(S - \{x_{d-2}\}) \cup \{x_{d-3}\}$ is a $\gamma(T)$-set that is not an $fd(T)$-set, a contradiction. Thus $x_{d-2} \notin S$. Then $x_{d-3} \notin S$. Let $T_1 = T - T_{x_{d-2}}$. Then $S \cap V(T_1)$ is an FD-set for T_1 and so $fd(T_1) \leq fd(T) - 1$. Furthermore every $fd(T_1)$-set can be extended to a dominating set for T by adding to it the vertex x_{d-1} and so $fd(T) = \gamma(T) \leq fd(T_1) + 1$. Hence $fd(T) = fd(T_1) + 1$. Similarly $\gamma(T) = \gamma(T_1)+1$. Now we obtain that $fd(T_1) = fd(T)-1 = \gamma(T) - 1 = \gamma(T_1)$. Next we show $fd(T_1) \equiv \gamma(T_1)$. Assume that $fd(T_1) \not\equiv \gamma(T_1)$. Let S_1 be a $\gamma(T_1)$-set that is not an $fd(T_1)$-set. Then $S_1 \cup \{x_{d-1}\}$ is a $\gamma(T)$-set that is not $fd(T)$-set, a contradiction. Hence $fd(T_1) \equiv \gamma(T_1)$. Applying the inductive hypothesis, we have that $T_1 \in \mathcal{T}$. If there is some $\gamma(T_1)$-set containing x_{d-3} then adding x_{d-1} to it yields a $\gamma(T)$-set which is not an $fd(T)$-set, a contradiction. Thus no $\gamma(T_1)$-set contains x_{d-3}. Consequently T is obtained from T_1 by Operation $\mathcal{O}_1$.

Now we assume that $\deg(x_{d-2}) \geq 3$. Suppose that x_{d-2} has a child $u \neq x_{d-1}$ with $\deg(u) \geq 2$. Then $u \in S$. Since S is an FD-set and x_d is dominated by precisely one vertex of S, we obtain that $x_{d-2} \in S$. Let u_1 be a leaf adjacent to u. If $\deg(u) = 2$, then $(S-\{u\}) \cup \{u_1\}$ is a $\gamma(T)$-set which is not an FD-set, a contradiction. So $\deg(u) \geq 3$. If x_{d-2} is not a support vertex then we observe that $x_{d-3} \notin S$, and $(S - \{x_{d-2}\}) \cup \{x_{d-3}\}$ is a $\gamma(T)$-set which is not an FD-set, a contradiction. Thus x_{d-2} is a support vertex. Let $T_2 = T - T_{x_{d-1}}$. Then $S \cap V(T_2)$ is an FD-set for T_2, and so $\gamma(T_2) \leq fd(T_2) \leq fd(T) - 1$. Clearly every $\gamma(T_2)$-set can be extended to a dominating set for T by adding x_{d-1} to it, and thus $\gamma(T) \leq \gamma(T_2)+1$. Thus $\gamma(T) = \gamma(T_2) + 1$. Now every $fd(T_2)$-set can be extended to a dominating set for T by adding to it the vertex x_{d-1}, and so $fd(T) = \gamma(T) \leq fd(T_2) + 1$. Hence $fd(T) = fd(T_2) + 1$. Now we obtain $fd(T_2) = fd(T) - 1 = \gamma(T) - 1 = \gamma(T_2)$. Next we show that $fd(T_2) \equiv \gamma(T_2)$. Assume that $fd(T_2) \not\equiv \gamma(T_2)$. Let D_1 be a $\gamma(T_2)$-set that is not an $fd(T_2)$-set. Then $D_1 \cup \{x_{d-1}\}$ is a $\gamma(T)$-set that is not $fd(T)$-set, a contradiction. Hence $fd(T_2) \equiv \gamma(T_2)$. Applying the inductive hypothesis, we have that $T_2 \in \mathcal{T}$. If $\deg_{T_2}(x_{d-2}) \geq 4$, then T is obtained from T_2 by Operation $\mathcal{O}_2$. Assume that $\deg_{T_2}(x_{d-2}) = 3$. If there is a $\gamma(T_2)$-set not containing x_{d-2} then adding to it the vertex x_{d-1} yields a $\gamma(T)$-set which is not an FD-set, a contradiction. Thus every $\gamma(T_2)$-set contains x_{d-2}. Consequently T is obtained from T_2 by Operation $\mathcal{O}_2$. Now suppose that every child of x_{d-2} except x_{d-1} is a leaf. Clearly $x_{d-2} \in S$. As before if $T_2 = T - T_{x_{d-1}}$ then we can easily obtain that $T_2 \in \mathcal{T}$. If $\deg_{T_2}(x_{d-2}) \geq 3$ then x_{d-2} is a strong support vertex of T_2 belonging to every $\gamma(T_2)$-set, and thus T is obtained from T_2 by Operation $\mathcal{O}_2$. Thus assume that $\deg_{T_2}(x_{d-2}) = 2$. If there is a $\gamma(T_2)$-set D_1' not containing x_{d-2} then $D_1' \cup \{x_{d-1}\}$ is a $\gamma(T)$-set that is not an FD-set, a contradiction. Thus every $\gamma(T_2)$-set contains x_{d-2}. Now T is obtained from T_2 by Operation $\mathcal{O}_2$.

Case 2. $\deg(x_{d-1}) = 2$. Assume that $\deg(x_{d-2}) \geq 3$. If x_{d-2} has a child $v \neq x_{d-1}$ that is not a leaf, then according to Case 1 we may assume that $\deg(v) = 2$. Then $v, x_{d-1} \in S$. Since S is an FD-set, we find that $x_{d-2} \in S$. Now $(S - \{x_{d-1}\}) \cup \{x_d\}$ is a $\gamma(T)$-set that is not an FD-set, a contradiction. Thus any child of x_{d-2} except x_{d-1} is a leaf. Since $x_{d-2} \in S$, we obtain that $(S - \{x_{d-1}\}) \cup \{x_d\}$ is a $\gamma(T)$-set that is not an FD-set, a contradiction. We thus assume that $\deg(x_{d-2}) = 2$. If $x_{d-2} \in S$ then $(S - \{x_{d-1}\}) \cup \{x_d\}$ is a $\gamma(T)$-set which is not an FD-set, a contradiction. Thus $x_{d-2} \notin S$. Since $|N(x_d) \cap S| = |N(x_{d-2}) \cap S|$, we find that $x_{d-3} \notin S$. Let $T_3 = T - T_{x_{d-2}}$. Then $S \cap V(T_3)$ is an FD-set for T_3. So $fd(T_3) \leq fd(T) - 1$. Furthermore every $fd(T_3)$-set can be extended to an FD-set for T by adding to it the vertex x_{d-1}. So $fd(T) \leq fd(T_3) + 1$. Hence $fd(T) = fd(T_3) + 1$. Similarly $\gamma(T) = \gamma(T_3) + 1$. Now we obtain $fd(T_3) = \gamma(T_3)$. Next we show $fd(T_3) \equiv \gamma(T_3)$. Assume that $fd(T_3) \not\equiv \gamma(T_3)$. Let R be a $\gamma(T_3)$-set that is not an $fd(T_3)$-set. Then $R \cup \{x_{d-1}\}$ is a $\gamma(T)$-set that is not $fd(T)$-set, a contradiction. Hence $fd(T_3) \equiv \gamma(T_3)$. Applying the inductive hypothesis, we have that $T_3 \in \mathcal{T}$. If there is a $\gamma(T_3)$-set R_1 containing x_{d-3} then $R_1 \cup \{x_{d-1}\}$ is a $\gamma(T)$-set which is not an FD-set, a contradiction.

Thus no $\gamma(T_3)$-set contains x_{d-3}. Consequently T is obtained from T_3 by Operation $\mathcal{O}_1$.

The converse follows from Lemma 3. ∎

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