A WORST CASE OF THE FIBONACCI-SYLVESTER EXPANSION

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Let a/b be a rational number with $(a,b) = 1$ and $0 < a < b$. The Fibonacci-Sylvester expansion of a/b, also called the R-expansion [3], a sorites [5], or the greedy algorithm expansion, is the representation of a/b as a sum of unit fractions

(1) $$a/b = 1/x_1 + 1/x_2 + \dots + 1/x_k$$

for which, for each j, $1 \le j \le k$, $1/x_j$ is the largest unit fraction not exceeding

$$a/b - \sum_{i=1}^{j-1} 1/x_i.$$

This expansion is guaranteed to terminate, and in at most a steps, by the identity

(2) $$a/b = 1/(q+1) + (a-r)/b(q+1),$$

where $b = qa + r$ is the standard division algorithm representation of b as a quotient times a plus a remainder. Note $0 < r < a$, since if $r = 0$ then the fraction a/b was not in reduced form. Each application of the identity gives a fraction which has a smaller numerator than the fraction before, even when the intermediate fractions have to be reduced, and when the numerator is 1 the algorithm terminates.

Campbell [1] has compiled a bibliography of more than 200 articles concerning various techniques for representing numbers as sums of unit fractions.

It is an interesting problem to determine which fractions make the algorithm proceed for as many steps as possible before

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terminating. Table 1 gives the first few fractions, smallest in the sense of having smallest denominators, for which the algorithm first takes exactly n steps before terminating. In Table 1 the size of the numerator is ignored, except that if a/b and $\hat{a}$/b both take n steps and a < $\hat{a}$, then only a/b is shown.

TABLE 1. FRACTIONS WITH SMALL DENOMINATORS WITH PRESCRIBED EXPANSION LENGTH

n	a/b
1	1/2
2	2/3
3	4/5
4	8/11
5	16/17
6	27/29
7	60/67
8	44/53
9	65/131

Already the denominators generated in the expansions of the latter fractions are in the hundreds of digits, and it is at first surprising that the denominators are not monotone. A more tractable problem, one that also sheds some light on the work above, arises by restricting the size of the numerators. In particular, we will investigate those fractions a/b whose greedy algorithm expansion is exactly a terms long. This is the longest the Fibonacci-Sylvester expansion can be, since the numerators of intermediate fractions must be strictly decreasing by the argument in the first paragraph. A short table of these fractions is given by

TABLE 2. FRACTIONS WITH EXPANSION LENGTHS MATCHING THEIR NUMERATORS

a	a/b
1	1/2
2	2/3
3	3/7
4	4/17
5	5/31
6	6/109
7	7/253
8	8/97
9	9/271
10	10/1621
11	11/199

The denominators that work here turn out to be minimal solutions to a set of congruences. The denominators that work in Table 1 can be seen by similar methods to be globally minimal solutions to any of several sets of congruences, these sets indexed by compositions of a.

Any representation of a/b of the form (1) can be used to generate a sequence of fractions. Write, for

$$a/b = 1/x_1 + 1/x_2 + \dots + 1/x_k,$$

with

$$x_1 < x_2 < \dots < x_k,$$

$$
\begin{aligned}
a/b &= a_1/b_1 \\
a_1/b_1 - 1/x_1 &= a_2/b_2 = 1/x_2 + \dots + 1/x_k \\
&\;\vdots \\
a_{k-1}/b_{k-1} &= 1/x_{k-1} + 1/x_k \\
a_k/b_k &= 1/x_k.
\end{aligned}
\tag{3}
$$

We will call the fractions a_i/b_i, $2 \le i \le k-1$, intermediate fractions.

Identity (2) can be applied to give a characterization of the greedy algorithm in terms of the equations (3).

THEOREM 1. The expansion of a/b as a sum of a unit fractions,

$$a/b = 1/x_1 + 1/x_2 + \dots + 1/x_a,$$

has the sequence of numerators in (3) strictly decreasing if and only if the expansion is the greedy algorithm expansion.

PROOF. The greedy algorithm always generates such a sequence of numerators in (3), since if any intermediate fractions given by the identity (2) are not in lowest terms, then reducing them does not lead to a violation of the inequality when (2) is applied to the reduced fraction in the next step. Conversely, if in a given unit fraction expansion of the form (1),

$$a_i/b_i - 1/x_i = a_{i+1}/b_{i+i},$$

then monotonicity of numerators implies that

$$0 < (a_i x_i - b_i)/\gcd(a_i x_i - b_i, b_i x_i) < a_i.$$

But if $\gcd(a_i x_i - b_i, b_i x_i) \ne 1$, the expansion is shortened to fewer than a terms when the numerator is replaced by a smaller numerator in a reduced fraction. Thus the gcd must equal 1, and the inequalities can be solved for x_i to give

$$b_i/a_i < x_i < b_i/a_i + 1.$$

This means the only possible choice for x_i is $[b_i/a_i + 1]$ unless $a_i = 1$, in which case $x_i = b_i$ is appropriate. These choices make the sequence of intermediate fractions coincide with those of the greedy algorithm expansion.

Salzer [3] develops some of these ideas, using the term "R-series" for the expansion (1) given by the greedy algorithm. He notes that it is possible to keep track of the sequence of operations using an iterated division algorithm based on (2) that is reminiscent of Euclid's algorithm for computing greatest common divisors. We suppress the intermediate fractions of (3) to write each step in terms of the original a and b.

$$
\begin{aligned}
b &= q_1 a + r_1,\ 0 < r_1 < a \\
b(q_1+1) &= q_2(a-r_1) + r_2,\ 0 < r_2 < a-r_1, \\
b(q_1+1)(q_2+1) &= q_3(a-r_1-r_2) + r_3,\ 0 < r_3 < a-r_1-r_2, \\
&\ \ \vdots
\end{aligned}
\tag{4}
$$

where the sequence of divisions is iterated until $r_1 + r_2 + \ldots + r_k = a$.

THEOREM 2. The rational number $r = a/b$, with $(a,b) = 1$ and $0 < a < b$, has a Fibonacci-Sylvester expansion of exactly a terms if and only if $P_n(a,b) \equiv Q_n(a) \pmod{Q_n(a)(a-n)}$ for $0 \le n < a$, where the numbers $Q_n(a)$ and polynomials $P_n(a,b)$ are defined by the recurrences

$$Q_n(a) = Q^2_{n-1}(a)(a-n+1), \tag{5}$$

with $Q_0(a) = 1$, and

$$P_n(a,b) = P^2_{n-1}(a,b) + P_{n-1}(a,b)Q_{n-1}(a)(a-n), \tag{6}$$

with $P_0(a,b) = b$.

PROOF. We untangle the greedy Euclid's algorithm of (4), using the extra information that, since the numerators of (3) are decreasing from a to 1 in a-1 steps, each remainder r_i must be 1. Then the first step is

$$b = q_1 a + 1,$$

so $b \equiv 1 \pmod a$, and $q_1 = (b-1)/a$. Since

$$b(q_1+1) = q_2(a-r_1)+1,$$

$b(q_1+1) \equiv 1 \pmod{a-1}$ and $q_2 = (b(q_1+1)-1)/(a-1)$. We keep track of the products b, $b(q_1+1)$, $b(q_1+1)(q_2+1)$, ... as rational functions of a and b by defining

$$b(q_1+1)(q_2+1)\ldots(q_r+1) = P_r(a,b)/Q_r(a)$$

The greedy Euclid step

$$b(q_1+1)(q_2+1)\ldots(q_{n-1}+1) = q_n(a-n+1) + 1$$

gives that

$$q_n + 1 = (b(q_1+1)\ldots(q_{n-1}+1)-1)/(a-n+1) + 1 =$$
$$(P_{n-1}/Q_{n-1} - 1)/(a-n+1) + 1 =$$
$$(P_{n-1} + Q_{n-1}(a-n))/(Q_{n-1}(a-n+1)),$$

and therefore

$$P_n/Q_n = b(q_1+1)\ldots(q_{n-1}+1)(q_n+1) =$$
$$P_{n-1}(P_{n-1}+Q_{n-1}(a-n))/Q_{n-1}(Q_{n-1}(a-n+1)).$$

Numerator and denominator here give (5) and (6).

The point of expressing the successive steps of the greedy Euclid algorithm as congruences is to suppress the quotients q_i.

We remark that though the recurrences (5) and (6) involve a, the appearance of P and Q in congruences modulo $Q_n(a)(a-n)$ mean that there is special interest in the values for a particular a. Table 3 list the Q_n as constants and the P_n as functions of b alone by using a special choice for a. Further specialization is possible in the identity $P_n(a,1) = Q_n(a)$.

TABLE 3. SPECIAL VALUES OF P_n and Q_n

n	$Q_n(n)$	$P_n(n,b)$
0	1	b
1	1	b^2
2	4	$b^2(b+1)^2$
3	324	$b^2(b+2)^2(b^2+2b+3)^2$
4	21233664	$b^2(b+3)^2(b^2+3b+8)^2(b^4+6b^3+17b^2+24b+48)^2$

It is more illuminating to write the sequence $\{Q_n(n)\}$ as

$$1,\ 1,\ 2^2,\ 2^23^4,\ 2^23^44^8,\ 2^23^44^85^{16},\ \ldots,\ 2^23^4\ldots n^{2^{n-1}},\ \ldots$$

Neither this sequence of "superfactorial" numbers nor any of its more obvious variants arises in Sloane's Handbook [4].

COROLLARY 1. If a/b has a greedy algorithm expansion of length a, so does $a/(b+Q_{a-1}(a))$.

PROOF. By Theorem 2, the sequence of congruences that b must satisfy has each modulus dividing the next. Thus any solution is determined only up to the last modulus, $Q_{a-1}(a)$.

Calculations indicate that often much smaller moduli work. Table 4 compares period length with this worst case modulus.

TABLE 4. PERIODICITY OF a/b

a	$Q_{a-1}(a)$	b giving a longest expansion
2	2	1 (mod 2)
3	18	1 (mod 6)
4	4608	1 or 17 (mod 24)
5	1800000000	1 (mod 30)
6	507799783342080000000	1 or 109 (mod 180)
7		1,253,281,or 533 (mod 630)
8		1,97,337,769,1009,or 1441 (mod 1680)

COROLLARY 2. Given a, define $D_a(x) = |\{b \le x : a/b$ has a greedy algorithm expansion of length $a\}|$. Then $\lim_{x \to \infty} D_a(x)/x$ is a positive rational number for any a.

For example, from Table 4 we read off that $D_3(x)/x$ approaches 1/6 and $D_4(x)/x$ approaches 1/12. These comments have a marginal relationship to the conjectures of Erdős and Straus and Sierpiński [2], in that only the residual sets need another algorithm to yield the expansion of shorter length that is conjectured to exist.

The density is non-zero (and in particular some solutions exist) because, though a/1 = 1 + 1 + ... + 1 is ruled out by our assumptions on a and b, that expansion does result if the greedy algorithm is applied to a/1, and hence by Theorem 2 fractions of the form $a/(1+kQ_{a-1}(a))$ have greedy algorithm expansions of the proper length.

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